Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 2 (2007)
DOI: 10.1685/CSC06101

COMBINATORIC AND ALGEBRAIC ASPECTS OF A CLASS OF PLANAR GRAPHS

M. LA BARBIERA

Department of Mathematics, University of Messina,
C.da Papardo, salita Sperone, 31, 98166 Messina, (Italy)
E-mail: monicalb@dipmat.unime.it

Abstract.

We consider the class of bipartite planar graphs St_r and some problems connected to them, where r is the number of the regions of the planar graph St_r. We prove that r is linked to algebraic invariants of the graph, in particular to the projective resolution of the edge ideal.

Introduction

A graph is a collection of vertices joined by lines said edges. Therefore a graph is a geometric model for several problems in which there are sets with relations between the elements. For this reason graphs can be used to analyze connection problems (for example street nets, railway nets, telephone nets, infrastructure nets, circuits electrical workers). Urban and territorial analysis uses planar graphs. A planar graph is embedded in the plane such that each pair of edges is intersected alone in common vertices and it is divided in some regions by its edges. A street net is an example of planar graph, also the plant of an house can be represented by a planar graph, in which each space is a vertex and the edges indicate if two spaces are communicating.

In this paper we consider algebraic aspects of planar graphs and we study their properties using computational and commutative algebra methods. Villarreal[6] gave interesting results about monomial ideals of the polynomial ring $R = K[X_1, \dots, X_n]$ over a field K; these ideals can arise from the edges of a graph G.

We are interested in considering bipartite planar graphs and some invariants of the edge ideals. More precisely, we consider a class of bipartite planar graphs studied by Doering and Gunston,[2] that give some informations about the K-algebra $K[G]$ associated to some classes of bipartite planar graphs by studying the geometry of the graph G. In particular they give constrains on Hilbert series of $K[G]$.

Our aim is to study the constrains for some Betti numbers and for the projective dimension of this class of planar graphs using their geometry.

Eliahou and Villarreal[3] study the second Betti number in terms of graph properties and they relate the number of triangles of a graph G to the second graded Betti number in degree 3.

An open problem is to study the third Betti number in relation to graph theoretical terms. Eliahou and Villarreal[3] give a conjecture about the value of the third Betti number in degree four. We are interested to prove this conjecture for bipartite planar graphs, but it is a difficult problem. Some results are obtained for complete graphs. Moreover we are able to formulate the conjecture also for bipartite planar graphs that are not complete. Partial results are obtained for planar graphs with a low number of regions.

We finish writing the contents of our paper. In section 1 we introduce the class of bipartite planar graphs St_r. In section 2 we prove that it is possible to give an explicit formula for the second graded Betti number in degree 3 of these planar graphs linked to the number of their regions and we verify the conjecture of Eliahou and Villarreal about the third Betti number in degree 4 for complete graphs and bipartite complete graphs. In section 3 we give upper bounds for the graded Betti numbers and projective dimension of the edge ideals of the class of graphs St_r.

The author is grateful to Professor Gaetana Restuccia for inspiring discussion on the subject of the paper.

1. Planar graphs

A graph G consists of a finite set $V = \{x_1, \ldots, x_n\}$ of vertices and a collection $E(G)$ of subsets of V, that consists of pairs $\{x_i, x_j\}$, for some $x_i, x_j \in V$, $i \neq j$, called *edges*.

Let G be a graph on vertices $x_1, \ldots, x_n$ and $R = K[X_1, \ldots, X_n]$ be a polynomial ring over a field K, with one variable X_i for each vertex x_i.

Definition 1.1. The *edge ideal* $I(G)$ associated to a graph G is the ideal of R generated by monomials of degree two, X_iX_j, on the $X_1, \ldots, X_n$ variables, such that $\{x_i, x_j\} \in E(G)$ for $1 \leq i, j \leq n$:

$$I(G) = (\{X_iX_j | \{x_i, x_j\} \in E(G)\}).$$

Definition 1.2. A graph G on vertices $x_1, \ldots, x_n$ is said complete if there exists an edge for all pair $\{x_i, x_j\}$ of vertices of G. It is denoted K_n.

Definition 1.3. A graph G is bipartite if its vertex set V can be partitioned into disjoint subsets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_m\}$, and any edge joins a vertex of V_1 with a vertex of V_2.

Definition 1.4. A bipartite graph G is complete if all the vertices of V_1 are joined to all the vertices of V_2 and it is denoted by $K_{n,m}$.

Remark 1.1. Bipartite graphs determine monomial ideals in a polynomial ring in two sets of variables $R = K[X_1, \ldots, X_n; Y_1, \ldots, Y_m]$, where n is the number of the vertices $x_1, \ldots, x_n$ and m is the number of the vertices $y_1, \ldots, y_m$.

Definition 1.5. A graph G is planar if it has an embedding in the plane such that each pair of edges is intersected alone in common vertices.

Remark 1.2. A planar graph is divided by its edges in plane regions.

Remark 1.3. The complete graphs K_5 and $K_{3,3}$ are the minimal not planar graphs. In fact it is not possible to represent these graphs in the plane so that the edges are not intersected alone in the vertices.

Theorem 1.1. ***(Kuratowski*[4]*)***
A graph is planar if and only if it has no subgraphs containing K_5 and $K_{3,3}$.

Remark 1.4. ([4])
Let G be a planar graph, N be the number of the vertices and q be the number of the edges. The following conditions are verified:
(1) If $N = 3$ then $q = 3N - 6$;
(2) If $N > 3$ and there are not cycles of length 3, then $q = 2N - 4$.

Example 1.1. The previous conditions are used to prove that a graph is not planar. $K_{3,3}$ has $N = 6$ vertices and no cycle of length 3, but $q = 9$. Hence $K_{3,3}$ does not satisfy the condition (2) of remark 1.4. It follows that $K_{3,3}$ is not a planar graph.

We consider the class of planar bipartite graphs St_r.[2]
Let St_r be the planar graph with $r > 1$ regions on vertex set $V = \{v_1, \ldots, v_{2r+1}\}$ and edge set $E = \{\{v_1, v_i\} | 2 \leq i \leq r+1\} \cup \{\{v_i, v_{i+r}\} | 2 \leq i \leq r+1\} \cup \{\{v_i, v_{i+r-1}\} | 3 \leq i \leq r+1\} \cup \{v_2, v_{2r+1}\}$.
St_r is a planar graph by Theorem 1.1.

Remark 1.5. St_r is a bipartite planar graph. The vertex set of St_r can be partitioned into disjoint subsets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_m\}$, where $n = r + 1$ and $m = r$, $m + n = 2r + 1$, and any edge joins a vertex of V_1 with a vertex of V_2. The edge set can be written:
$E = \{\{x_1, y_i\} | 1 \leq i \leq m\} \cup \{\{x_i, y_{i-1}\} | 2 \leq i \leq n\} \cup \{\{x_i, y_i\} | 2 \leq i \leq m\} \cup \{x_n, y_1\}$.
It follows that St_r is bipartite and it is complete only in the case $r = 2$.

Example 1.2. $r = 2$, $G = St_2$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and
$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_3, v_4\}, \{v_2, v_5\}\}$

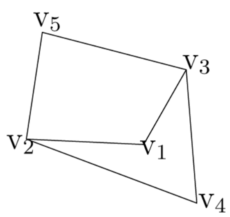

Fig. 1.

V can be partitioned into disjoint subsets: $V = \{x_1, x_2, x_3\} \cup \{y_1, y_2\}$,
where $x_1 = v_1$, $x_2 = v_4$, $x_3 = v_5$, $y_1 = v_2$, $y_2 = v_3$.
Then: $E = \{\{x_1, y_1\}, \{x_1, y_2\}, \{x_2, y_1\}, \{x_3, y_2\}, \{x_2, y_2\}, \{x_3, y_1\}\}$

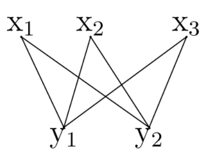

Fig. 2.

The two pictures represent the same graph St_2.

We want to study some aspects of bipartite planar graphs using algebraic methods. We link the number of the regions of St_r to algebraic invariants of its edge ideal, in particular to the graded Betti numbers and to the projective dimension.

2. First and second syzygy modules of $\mathbf{I(St}_r)$

Let G be a graph on vertex set $V = \{x_1, \ldots, x_n\}$ and $E = \{f_1, \ldots, f_q\}$ be its edge set.

Definition 2.1. The edge graph of a graph G, denoted by $L(G)$, has vertex set equal to the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G have one common vertex.

$$V(L(G)) = E = \{f_1, \ldots, f_q\}$$

$$E(L(G)) = \{(f_i, f_j) | f_i = \{x_i, x_j\}, \quad f_j = \{x_j, x_k\}, \ \imath \neq j, \quad \jmath \neq k\}$$

It is showed that the number of edges of the edge graph $L(G)$ is given by

$$|E(L(G))| = -|E(G)| + \sum_{i=1}^{n} \frac{deg^2 x_i}{2},$$

where $degx_i$ is the number of edges incident with x_i ([6]).

Let $I(G) \subset R$ be the edge ideal of G. An interesting problem is to express the second Betti number of $R/I(G)$ in terms of graph theoretical properties. Eliahou and Villarreal[3] give an explicit formula to compute the second graded Betti number in degree 3, that represents the number of the generators of linear syzygies of $I(G)$.

Theorem 2.1. *Let St_r be the bipartite planar graph, r be the number of its regions and $\mathcal{I}$ be the edge ideal. If*

$$\ldots \rightarrow R^c(-4) \oplus R^b(-3) \rightarrow R^q(-2) \rightarrow \mathcal{I} \rightarrow 0$$

is the minimal graded resolution of $I(St_r)$, then
(1) $q = 3r$;
(2) $b = \frac{1}{2}r(r+7)$.

Proof: (1) $q = |E| = |\{\{v_1, v_i\} | 2 \leq i \leq r+1\}| + |\{\{v_i, v_{i+r}\} | 2 \leq i \leq r+1\}| + |\{\{v_i, v_{i+r-1}\} | 3 \leq i \leq r+1\}| + |\{v_2, v_{2r+1}\}| = r + r + (r-1) + 1 = 3r$.
(2) By the formula of Eliahou and Villarreal[3] $b = |E(L(St_r))| - N_3$, where N_3 is the number of the triangles of St_r and $N_3 = 0$ because the graph is bipartite. One has:
$|E(L(St_r))| = -|E(St_r)| + \sum_{i=1}^{N} \frac{deg^2 v_i}{2}$, where $N = 2r+1$.
$\sum_{i=1}^{2r+1} \frac{deg^2 v_i}{2} = \frac{r^2}{2} + r(\frac{3^2}{2}) + r(\frac{2^2}{2}) = \frac{1}{2}(r^2 + 13r)$,
where $degv_1 = r$, $degv_i = 3$ for $2 \leq i \leq r+1$ and $degv_i = 2$ for $r+2 \leq 2r+1$.
Then:
$b = |E(L(St_r))| = -3r + \frac{1}{2}(r^2 + 13r) = \frac{1}{2}r(r+7)$.

It is possible to compute the number of the regions r of particular planar bipartite graphs using the minimal graded resolution of the edge ideal.

Corollary 2.1. *Let G be a bipartite graph on vertex set $V = V_1 \cup V_2$, $V_1 = \{x_1, \ldots, x_n\}$, $V_2 = \{y_1, \ldots, y_{n-1}\}$, and edge set $E(G) = \{\{x_1, y_i\} | 1 \leq i \leq n-1\} \cup \{\{x_i, y_{i-1}\} | 2 \leq i \leq$*

$n\} \cup \{\{x_i, y_i\} | 2 \le i \le n-1\} \cup \{x_n, y_1\}$.
Let $I(G)$ be the edge ideal of G and b the second graded Betti number in degree 3 of $R/I(G)$. Then:
(1) G is a planar graph;
(2) The number of the regions r of G is given by $r(r+7) = 2b$.

Proof: It follows by theorem 2.1 and remark 1.5.

Example 2.1. $G = St_2$, $r = 2$, $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$

$$\ldots \to R^9(-3) \to R^6(-2) \to I(G) \to 0,$$

By theorem 2.1 one has: $q = 3r = 6$ and $b = \frac{1}{2}r(r+7) = 9$.

Eliahou and Villarreal[3] give a conjecture to compute the third Betti number in degree 4 of $R/I(G)$, that is the number of the linear generators of the second syzygy module of $I(G)$.
We prove the conjecture for complete graphs and bipartite complete graphs.

Theorem 2.2. *Let K_n be the complete graph on vertex set $\{v_1, \ldots, v_n\}$, N_4 be the number of the complete subgraphs of G on 4 vertices. If*

$$\ldots \to \ldots \oplus R^d(-4) \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to I(K_n) \to 0$$

is the minimal graded resolution of $I(K_n)$, then

$$d = \sum_{i=1}^{n} \binom{degv_i}{3} - N_4,$$

where $degv_i$ is the number of the edges incident with the vertex v_i.

Proof: d is the third graded Betti number b_{3_4} of $R/I(K_n)$ and it is given by the formula[5]

$$d = b_{3_4}(K_n) = 3\binom{n}{3+1} = 3\binom{n}{4}.$$

K_n is a complete graph and $I(K_n) = (X_iX_j | 1 \le i < j \le n)$, it follows that $degv_i = n-1$ for all v_i, $1 \le i \le n$.
Hence $\sum_{i=1}^{n} \binom{degv_i}{3} = n\binom{n-1}{3}$ and $N_4 = \binom{n}{4}$. So we have:
$\sum_{i=1}^{n} \binom{degv_i}{3} - N_4 = n\binom{n-1}{3} - \binom{n}{4} = \frac{n!}{3!(n-1)!} - \frac{n!}{4!(n-1)!} = 3\frac{n!}{4!(n-1)!} = 3\binom{n}{4} = d$.

Theorem 2.3. *Let $K_{n,m}$ be the complete bipartite graph on $N = n+m$ vertices, W_4 be the number of squares without chords of G. If*

$$\ldots \to \ldots \oplus R^d(-4) \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to I(K_{n,m}) \to 0$$

is the minimal graded resolution of $I(K_{n,m})$, then

$$d = \sum_{i=1}^{N} \binom{degv_i}{3} + W_4,$$

where $degv_i$ is the number of the edges incident with the vertex v_i.

Proof: d is the third graded Betti number b_{3_4} of $R/I(K_{n,m})$ and it is given by the formula[5]

$$d = b_{3_4}(K_{n,m}) = \binom{n}{1}\binom{m}{3} + \binom{n}{2}\binom{m}{2} + \binom{n}{3}\binom{m}{1}$$

$K_{n,m}$ is a complete bipartite graph on vertex set $V = V_1 \cup V_2 = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$ and $I(K_{n,m}) = (X_i Y_j | 1 \leq i \leq n, \quad 1 \leq j \leq m)$.
It follows that

$$\sum_{i=1}^{N} \binom{degv_i}{3} = \sum_{i=1}^{n} \binom{degx_i}{3} + \sum_{i=1}^{m} \binom{degy_i}{3} = n\binom{m}{3} + m\binom{n}{3}$$

and $W_4 = \binom{n}{2}\binom{m}{2}$ because the squares without chords of $K_{n,m}$ have two vertices in V_1 and two vertices in V_2.
So we have:

$$\sum_{i=1}^{n} \binom{degv_i}{3} + W_4 = n\binom{m}{3} + m\binom{n}{3} + \binom{n}{2}\binom{m}{2} = d.$$

Theorem 2.4. *Let C_n be the cycle on vertex set $\{v_1, \ldots, v_n\}$ with $n > 3$. If*

$$\ldots \to \ldots \oplus R^d(-4) \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to I(C_n) \to 0$$

is the minimal graded resolution of $I(C_n)$, then:
(1) $d = 1$ for $n = 4$;
(2) $d = 0$ for $n > 4$.

Proof: If $n = 4$, it is proved that $d = b_{3_4}(C_n) = 1$([5]). Then we observe that $\sum_{i=1}^{n} \binom{degv_i}{3} = 0$ because $degv_i = 2$ for all v_i , $1 \leq i \leq n$, $N_4 = 0$ and $W_4 = 1$. Hence: $\sum_{i=1}^{n} \binom{degv_i}{3} - N_4 + W_4 = 1 = d$.
If $n > 4$, one has $d = b_{3_4}(C_n) = \frac{n}{n-2}\binom{1}{2}\binom{n-2}{1} = 0$([5]). Hence $\sum_{i=1}^{n} \binom{degv_i}{3} - N_4 + W_4 = 0 = d$.

In general for bipartite graphs the conjecture of Eliahou and Villarreal is the following:
Conjecture
Let G be a bipartite graph on N vertices, W_4 be the number of squares without chords of G. If

$$\ldots \to \ldots \oplus R^d(-4) \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to I(G) \to 0$$

is the minimal graded resolution of $I(G)$, then

$$d = \sum_{i=1}^{N} \binom{degv_i}{3} + W_4,$$

where $degv_i$ is the number of the edges incident with the vertex v_i.

Problem: Prove the conjecture for the bipartite planar graphs St_r.
We have partial results: the conjecture is proved for St_2 because only in this case St_r is complete.

Proposition 2.1. *Let St_2 be the bipartite planar graph with minimal graded resolution*

$$0 \to R(-5) \to R^5(-4) \to R^9(-3) \to R^6(-2) \to I(St_2) \to 0$$

and $d = dim(R^5(-4))$. Then:

$$d = \sum_{i=1}^{N} \binom{degv_i}{3} + W_4.$$

Proof: St_2 has vertex set $V = \{x_1, x_2, x_3\} \cup \{y_1, y_2\}$ and it is a $K_{3,2}$. For bipartite complete graphs,[5] we have:

$$d = b_{3_4}(St_r) = \binom{3}{1}\binom{2}{3} + \binom{3}{2}\binom{2}{2} + \binom{3}{3}\binom{2}{1} = 5$$

$$\sum_{i=1}^{5} \binom{degv_i}{3} = \sum_{i=1}^{3} \binom{degx_i}{3} + \sum_{i=1}^{2} \binom{degy_i}{3} = 3\binom{2}{3} + 2\binom{3}{3} = 2$$

and $W_4 = \binom{3}{2}\binom{2}{2} = 3$. Hence:

$$\sum_{i=1}^{5} \binom{degv_i}{3} + W_4 = 2 + 3 = 5 = d.$$

In general, for not complete bipartite graphs St_r the conjecture is not proved, but it is possible with some computations to verify it for a low number of regions.

Example 2.2. $G = St_{10}$, $r = 10$, $V(G) = \{v_1, \ldots, v_{21}\}$
Computing the resolution of $I(G)$ we find: $d = 140$.
We verify the conjecture: $\sum_{i=1}^{21} \binom{degv_i}{3} + W_4 = \binom{r}{3} + r\binom{3}{3} + r\binom{2}{3} + r = 140 = d$

We suppose that the conjecture is true for $G = St_r$, we have the following result that links d to the number of its regions.

Corollary 2.2. *Let St_r be the bipartite planar graph, r be the number of its regions and $I(St_r)$ be the edge ideal. Let*

$$\ldots \to \ldots \oplus R^d(-4) \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to I(St_r) \to 0$$

be the minimal graded resolution of $I(St_r)$.
Then: $d = \frac{1}{6}r(r^2 - 3r + 14)$ for $r > 2$.

Proof: We assume that the conjecture is true.
Hence one has $d = \sum_{i=1}^{N} \binom{degv_i}{3} + W_4$, where the number of the squares without chords of St_r is $W_4 = r$.
$N = 2r + 1$ is the number of vertices of St_r. It follows:
$d = \sum_{i=1}^{2r+1} \binom{degv_i}{3} = \binom{r}{3} + r\binom{3}{3} + r\binom{2}{3} = \frac{1}{6}r(r-1)(r-2)$,
where $degv_1 = r$, $degv_i = 3$ for $2 \le i \le r+1$ and $degv_i = 2$ for $r+2 \le i \le 2r+1$. Hence:
$d = \frac{1}{6}r(r-1)(r-2) + r = \frac{1}{6}r(r^2 - 3r + 14)$.

Example 2.3. $G = St_3$, $r = 3$, $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

$$\ldots \to R^9(-5) \oplus R^7(-4) \to R^3(-4) \oplus R^{15}(-3) \to R^9(-2) \to I(G) \to 0,$$

By Corollary 2.2 one has: $d = \frac{1}{6}r(r^2 - 3r + 14) = 7$.

3. Constrains in the resolution of St_r

Let St_r be the bipartite planar graph with $r > 1$ regions. We are interested to find bounds for the graded Betti numbers that appear in the minimal graded resolution of the edge ideal. These numbers determine the rank of the free modules appearing in the minimal graded resolution and it is not possible to give a generic formula to compute them, except for the second and third graded Betti numbers. But in general, we give upper bounds for them linked to the number of the regions of the planar graph St_r.
In a recent work[5] it is proved that the $i-th$ graded Betti numbers of a subgraph H can not exceed the $i-th$ graded Betti numbers of the larger graph G for all i: $b_{i_j}(H) \leq b_{i_j}(G)$ for $j = i + 1$.

Proposition 3.1. *Let St_r be the bipartite planar graph on $2r+1$ vertices, r be the number of its regions and $\mathcal{I}$ be the edge ideal. Let $b_{i_j}(St_r)$ be the graded Betti numbers in the minimal graded resolution of $R/\mathcal{I}$. Then*

$$b_{i_j}(St_r) \leq \sum_{j+l=i+1, j,l \geq 1} \binom{r+1}{j}\binom{r+1}{l} \quad \forall \ \ i \ \ and \ \ j = i+1.$$

Proof: St_r is a not complete bipartite planar graph on two disjoint vertex set V_1 and V_2, with $|V_1| = r + 1$ and $|V_2| = r$. If follows that St_r is a subgraph of the complete bipartite graph $K_{n,m}$, where $n = m = r + 1$, that has a vertex in more that St_r. Then we have:

$$b_{i_j}(St_r) \leq b_{i_j}(K_{n,m}), \quad \forall \ \ i \ \ and \ \ j = i+1(^5).$$

We have:

$$b_{i_j}(K_{n,m}) = \sum_{j+l=i+1, j,l \geq 1} \binom{n}{j}\binom{m}{l}, \quad for \ \ j = i+1(^5).$$

Hence for $n = m = r + 1$ it follows:

$$b_{i_j}(St_r) \leq \sum_{j+l=i+1, j,l \geq 1} \binom{r+1}{j}\binom{r+1}{l} \quad \forall \ \ i \ \ and \ \ j = i+1.$$

Remark 3.1. Let $\mathcal{I}$ be the edge ideal of St_r. We denote $b_{i_j}(St_r) = b_{i_j}(R/\mathcal{I})$ and $b_{i_j}(K_{n,m}) = b_{i_j}(R/I(K_{n,m}))$. Because $b_{i_j}(I(K_{n,m})) = b_{i+1_j}(R/I(K_{n,m}))$, then

$$b_{i_j}(I(K_{n,m})) = \sum_{j+l=i+2, j,l \geq 1} \binom{n}{j}\binom{m}{l}, \quad for \ \ j = i+2.$$

Hence it follows by the previous theorem:

$$b_{i_j}(\mathcal{I}) \leq \sum_{j+l=i+2, j,l \geq 1} \binom{r+1}{j}\binom{r+1}{l} \quad \forall \ \ i \ \ and \ \ j = i+2.$$

We consider a particular case: we explicit this upper bound for the third Betti number d that appears in the conjecture.

Corollary 3.1. *Let St_r be the bipartite planar graph on $2r + 1$ vertices, r be the number of its regions and $\mathcal{I}$ be the edge ideal. Let*

$$\ldots \to \ldots \oplus R^d(-4) \to R^c(-4) \oplus R^b(-3) \to R^q(-2) \to \mathcal{I} \to 0$$

be the minimal graded resolution of $\mathcal{I}$. Then

$$d \leq \frac{1}{12} r(r+1)(7r-4).$$

Proof: We have:
$d = b_{3_4}(St_r) \leq \sum_{j+l=4} \binom{r+1}{j}\binom{r+1}{l} = \binom{r+1}{1}\binom{r+1}{3} + \binom{r+1}{2}\binom{r+1}{2} + \binom{r+1}{3}\binom{r+1}{1} = \frac{1}{12} r(r+1)(7r-4).$

Now using the geometry of the planar graph St_r, we give bounds for the projective dimension of its edge ideal.

Definition 3.1. Let G be a graph with vertex set V. A subset $\mathcal{A}$ of V is said minimal vertex cover for G if each edge of G is incident with one vertex in $\mathcal{A}$ and there is no proper subset of $\mathcal{A}$ with this property.

Definition 3.2. The smallest number of vertices in any minimal vertex cover of G is said vertex covering number. We denote it $\alpha_0(G)$.

Proposition 3.2. *Let G be a graph and $\mathcal{I}$ be the edge ideal. Then $\alpha_0(G) = ht(\mathcal{I})$.*[6]

Now we find a lower bound for the projective dimension of the edge ideal of St_r using its minimal vertex cover.

Proposition 3.3. *Let St_r be the bipartite planar graph with $r > 1$ regions and $\mathcal{I}$ be the edge ideal. Then $pd_R(\mathcal{I}) \geq r - 1$.*

Proof: It is proved that $pd_R(\mathcal{I}) \geq ht(\mathcal{I}) - 1$([6]), hence by Proposition 3.2 one has $pd_R(\mathcal{I}) \geq \alpha_0(St_r) - 1$.
St_r has vertex set $V = \{v_1, \ldots, v_{2r+1}\}$ and edge set $E = \{\{v_1, v_i\} | 2 \leq i \leq r+1\} \cup \{\{v_i, v_{i+r}\} | 2 \leq i \leq r+1\} \cup \{\{v_i, v_{i+r-1}\} | 3 \leq i \leq r+1\} \cup \{v_2, v_{2r+1}\}$.
By definition of St_r and by its geometry in the plane it follows that the vertices of the minimal vertex cover are all the vertices joined to v_1: $\mathcal{A}(St_r) = \{v_i | 2 \leq i \leq r+1\}$. Each edge of St_r is incident in a vertex of $\mathcal{A}(St_r)$ and this set is minimal as follows by the description of the edge set. Hence $\alpha_0(St_r) = r$ and $pd_R(\mathcal{I}) \geq r - 1$.

Example 3.1. $r = 2$, $G = St_2$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and
$E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_3, v_4\}, \{v_2, v_5\}\}$
$\mathcal{I} = (X_1X_2, X_1X_3, X_2X_4, X_3X_5, X_3X_4, X_2X_5)$
$\mathcal{A}(St_2) = \{v_2, v_3\}$, $\alpha_0(St_2) = 2$ (see Example 1.2, Fig1). Then: $pd_R(\mathcal{I}) \geq 1$.

Now we give an upper bound for the projective dimension of the edge ideal of St_r.

Proposition 3.4. *Let St_r be the bipartite planar graph with $r > 1$ regions and $\mathcal{I}$ be the edge ideal. Then $pd_R(\mathcal{I}) \leq 2r - 1$.*

Proof: We observe that St_r is a subgraph of a bipartite complete graph $K_{n,m}$ such that $V(St_r) = V(K_{n,m})$ with $n + m = 2r + 1$ vertices and $|E(St_r)| < |V(K_{n,m})|$.
Let $p = pd_R(\mathcal{I})$. The projective dimension of a graph is affected by some simple transformations of the graphs, such as deleting some edges. So as a consequence of these results,[5] we have $b_p(I(K_{n,m})) \geq b_p(\mathcal{I}) \neq 0$ and $b_{p+1}(I(K_{n,m})) \geq 0$, where $b_p(I(K_{n,m}))$ (resp.$b_p(\mathcal{I})$) are the total Betti numbers of $K_{n,m}$ (resp. St_r). It follows that $pd_R(I(K_{n,m})) \geq pd_R(\mathcal{I})$, but $pd_R(I(K_{n,m})) = n + m - 2$([5]), with $n + m - 2 = 2r - 1$. Hence: $pd_R(\mathcal{I}) \leq 2r - 1$.

REFERENCES

1. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*. (Cambridge studies in advanced mathematics, 1998).
2. L.R.Doering and T. Gunston, *Algebras arising from bipartite planar graphs*, Communications in Algebra, **24(11)**, 3589 (1996).
3. S. Eliahou and R.H. Villarreal, *The second Betti number of an edge ideal*, Aportaciones Matematicas, Serie Comunicaciones **25**, 101 (1999).
4. F.Harary, *Graph Theory*, (Addison-Wesley, Reading, MA, 1972).
5. S.Jacques, *Betti Numbers of Graph Ideals*, PhD Thesis, University of Sheffield (2004).
6. R.H. Villarreal, *Monomial algebras*, Pure and Applied Mathematics, 2000.