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Graphs of linear type

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Introduction and known results

There are several papers in graph theory in which are studied algebraic properties of a simple graph $\mathcal{G}$. If $R = k[X_1, \dots, X_n]$ is the polynomial ring over a field k and I a finitely generated ideal in R, the symmetric algebra and the Rees algebra of I over R are defined. The ideal I is said of linear type if its symmetric algebra $S(I)$ and Rees algebra $\mathcal{R}(I)$ are isomorphic. When I is the edge ideal associated to a connected graph $\mathcal{G}$ such that $S(I) \simeq \mathcal{R}(I)$, $\mathcal{G}$ is called a graph of linear type.

Our aim is to introduce algebraic tools for finding interesting classes of graphs $\mathcal{G}$ of linear type. In particular, when the symmetric algebra of the edge ideal of $\mathcal{G}$ is an integral domain, then necessarily $\mathcal{G}$ is of linear type. In this way it is proved that graphs of linear type are all the simple graphs with at most three edges, and all the simple graphs with no even cycles and vertex covering number 3.

We need some notations.

Let $\mathcal{G}$ be a graph, $V(\mathcal{G})$ the set of its vertices, $E(\mathcal{G})$ the set of its edges.
$\mathcal{G}$ is said a *simple* graph if, for all $\{v_i, v_j\} \in E(\mathcal{G})$, $i \neq j$, it is $v_i \neq v_j$.
A *cycle* of *length* n, $C_n \subset \mathcal{G}$, is an alternating sequence of $n+1$ distinct vertices and n edges beginning and ending at the same vertex, in which each edge is incident to the two vertices immediately preceding and following it.
A *tree* is a connected graph without cycles.
The *degree* of a vertex $v_\ell \in V(\mathcal{G})$, denoted by $\deg(v_\ell)$, is the number of edges incident with v_ℓ, i.e. $\deg(v_\ell) = \max\{r \mid \exists\, v_1, \dots, v_r \in V(\mathcal{G}) \text{ with } \{v_i, v_\ell\} \in E(\mathcal{G}),\ i = 1, \dots, r\}$.
When $\deg(v_\ell) = 0$, the vertex v_ℓ is said *isolated*.
A subset $A \subset V(\mathcal{G})$ is said a *minimal vertex cover* for $\mathcal{G}$ if every edge of $\mathcal{G}$ is incident with one vertex in A and there is no proper subset of A with such property.
If A satisfies only the incident condition, A is called a *vertex cover* for $\mathcal{G}$.
The smallest number of vertices in any minimal vertex cover for $\mathcal{G}$ is called *vertex covering number* of $\mathcal{G}$ and it is denoted by $\alpha_0(\mathcal{G})$.
If $V(\mathcal{G}) = \{ v_1, \dots, v_n \}$ and $R = k[X_1, \dots, X_n]$ is the polynomial ring over a field k such that each variable X_i corresponds to the vertex v_i, the *edge ideal* $I(\mathcal{G})$ associated to $\mathcal{G}$ is the ideal $(\{X_iX_j \mid \{v_i, v_j\} \in E(\mathcal{G})\}) \subset R$.

Note that the non zero edge ideals are those generated by square-free monomials of degree two. This implies that $I(\mathcal{G})$ is a graded ideal of R of initial degree two, that is $I(\mathcal{G}) = \oplus_{i\geqslant 2}(I(\mathcal{G})_i)$. If $E(\mathcal{G}) = \emptyset$, i.e. $\mathcal{G}$ has only isolated vertices, then $I(\mathcal{G}) = (0)$.
So $I(\mathcal{G})$ has a graded free resolution of length at most n . The length of the (unique) minimal resolution of $I(\mathcal{G})$ is equal to $\mathrm{pd_R(I(\mathcal{G}))}$, the *projective dimension* of $I(\mathcal{G})$.
If $\wp$ is an ideal of R generated by $A = \{X_{i_1}, ..., X_{i_r}\}$, $\wp$ is a minimal prime of $I(\mathcal{G})$ if and only if A is a minimal vertex cover for $\mathcal{G}$.
It is well-known that $\mathrm{ht(I(\mathcal{G}))} = \alpha_0(\mathcal{G})$.
If $\dim(R/I(\mathcal{G}))$ is the Krull dimension of $R/I(\mathcal{G})$, the graph $\mathcal{G}$ is said *Cohen-Macaulay* over k (*C-M graph* for short) if $\mathrm{depth(R/I(\mathcal{G})) = dim(R/I(\mathcal{G}))}$.
So, if $\mathcal{G}$ is a C-M graph, it results $\mathrm{pd_R(I(\mathcal{G}))} = \alpha_0(\mathcal{G}) - 1$.
For an ideal $I \subset R$, the *symmetric algebra* of I over R is defined by $S(I) = \oplus_{t\geq 0} S_t(I)$; moreover, the *Rees algebra* of I over R is given by $\mathcal{R}(I) = \oplus_{t\geq 0} I^t Y^t \subset R[Y]$.
Let $R^p \xrightarrow{\varphi} R^q \xrightarrow{\psi} I \to 0$ be a presentation of I, where $\varphi = (a_{ij})$ is a $q \times p$ matrix with entries in R. Then $S(I) \simeq R[T_1, \ldots, T_q]/J$, where $J = \big(g_j = \sum_{i,j} a_{ij}T_i\big)$ is the *ideal of relations* of $S(I)$.
If $I = (f_1, \ldots, f_q)$, the kernel $P(I)$ of the epimorphism $R[T_1, \ldots, T_q] \to \mathcal{R}(I)$ defined by $T_i \mapsto f_i T$ is said the *toric ideal* of $\mathcal{R}(I)$ with respect to $f_1, \ldots, f_q$.
For each $p = 0, \ldots, n$, let $\partial_p : \wedge^p R^n \to \wedge^{p-1} R^n \otimes I$ be a morphism defined by $\partial_p (e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} e_{i_1} \wedge \ldots \wedge \widehat{e}_{i_j} \wedge \ldots \wedge e_{i_p} \otimes \psi(e_{i_j})$, where $e_1, \ldots, e_n$ is a basis of R^n and $\widehat{e}$ means omission. Denote $Z_p(I) = \ker \partial_p$.
Then one can consider the complex:

$$\mathcal{Z}(I) : 0 \to Z_n(I) \otimes S[-n] \xrightarrow{\partial'_n} Z_{n-1}(I) \otimes S[-n+1] \to \cdots \xrightarrow{\partial'_1} Z_0(I) \otimes S \to S(I) \to 0$$

where ∂'_p is the map induced by ∂_p , $S[-p]_t = S_{t-p}$, $S(I) = \operatorname{coker} \partial'_1$.
According to [2], the study of the integrity of $S(I)$ is joined to acyclicity conditions for the complex $\mathcal{Z}(I)$.

Main results

Definition 0.1. Let R be a polynomial ring over a field k, $\mathcal{G}$ be a simple graph and $I(\mathcal{G})$ be its edge ideal. $\mathcal{G}$ is called a *graph of linear type* if the symmetric algebra and the Rees algebra of $I(\mathcal{G})$ over R are isomorphic. An important characterization of the graphs of linear type is given in [4] .

Property 1. Let $\mathcal{G}$ be a connected graph and $I(\mathcal{G})$ be its edge ideal. Then $\mathcal{G}$ is of linear type if and only if $\mathcal{G}$ is a tree or $\mathcal{G}$ has a unique cycle of odd length. □
Through the graded minimal free resolution of the edge ideal of $\mathcal{G}$, it is possible to establish when the symmetric algebra of $I(\mathcal{G})$ over R is an integral domain, that is when its ideal of relations is prime.
In this case, as showed in [2], it follows that the symmetric algebra of $I(\mathcal{G})$ is isomorphic to the Rees algebra of $I(\mathcal{G})$, and so $\mathcal{G}$ is of linear type.

The following result establishes an inferior bound for the projective dimension of the edge ideals $I(\mathcal{G})$ of graphs $\mathcal{G}$ with at least four edges. Here it is applied the inequality $\mathrm{pd_R(I(\mathcal{G}))} \geqslant \alpha_0(\mathcal{G}) - 1$.

Proposition 1. Let $I(\mathcal{G})$ be the edge ideal of a graph $\mathcal{G}$ with $|\, E(\mathcal{G})| \geqslant 4$. Then it results $\mathrm{pd_R(I(\mathcal{G}))} > 1$. □

Example 1. The only simple graphs with projective dimension of their edge ideal less or equal to 1 are the following:

$\mathcal{P} := \{v_1, v_2\}$, an edge.

$\mathcal{Q} := \Big\{\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\}\Big\}$, two consecutive edges.

$\mathcal{R} := \Big\{\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_3, v_4\}\}\Big\}$, two non consecutive edges.

$\mathcal{S} := \Big\{\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}\Big\}$, the triangle.

$\mathcal{T} := \Big\{\{v_1, v_2, v_3, v_4\}, \{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}\}\Big\}$, three consecutive edges. Note that, according to Proposition 1, the graphs in the Example 1 have at most three edges. Moreover, by Property 1, such graphs, except $\mathcal{R}$ because not connected, are of linear type.

Next result characterizes the simple graphs with at most three edges.

PROPOSITION 2. Let $I(\mathcal{G})$ be the edge ideal of any simple graph $\mathcal{G}$ with $|\,E(\mathcal{G})| < 4$. Then $\mathcal{G}$ is a graph of linear type. □

COROLLARY 1. Let $\mathcal{G}$ be a simple graph such that $\mathrm{pd_R}(\mathrm{I}(\mathcal{G})) \leqslant 1$. Then $\mathcal{G}$ is a graph of linear type. □ In particular, from Corollary 1, it descends that all the graphs of the Example 1 are of linear type.

In the following let's characterize the simple graphs with no even cycles and vertex covering number 3.

THEOREM 1. Let $\mathcal{G}$ be a C-M graph with no even cycles and $\mathrm{pd_R}(\mathrm{I}(\mathcal{G})) = 2$. Then the symmetric algebra $S(I(\mathcal{G}))$ is an integral domain.

Proof: Remember that for C-M graphs it results $\mathrm{pd_R}(\mathrm{I}(\mathcal{G})) = \alpha_0(\mathcal{G}) - 1$.

Observe that if $\mathcal{G}$ is a C-M graph with q edges, by [4] it is $q \leqslant \big(\,[\alpha_0(\mathcal{G})]^2 + \alpha_0(\mathcal{G})\,\big)/\,2$.

Hence the possible C-M graphs with $\mathrm{pd_R}(\mathrm{I}(\mathcal{G})) = 2$ have at most 6 edges.

By Proposition 2, it follows that the symmetric algebra of the edge ideal of any simple graph having at most 3 edges is integral.

So it remains to examine the symmetric algebra of edge ideals of C-M graphs $\mathcal{G}$ with no even cycles such that $\mathrm{pd_R}(\mathrm{I}(\mathcal{G})) = 2$ and $4 \leqslant |\,E(\mathcal{G})| \leqslant 6$.

Computations on the above C-M graphs, by using softwares `CoCoA 4` (cfr. [3]) and `Macaulay 2` (cfr. [1]), give the assertion. □

COROLLARY 2. Let $\mathcal{G}$ be a Cohen-Macaulay graph with no even cycles and $\alpha_0(\mathcal{G}) = 3$. Then $\mathcal{G}$ is a graph of linear type. □

Last result can be generalized, through a direct proof, to the class of all the graphs with no even cycles and vertex covering number 3.

THEOREM 2. Let $\mathcal{G}$ be any simple graph with no even cycles and $\alpha_0(\mathcal{G}) = 3$. Then $\mathcal{G}$ is a graph of linear type. □

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