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Informations about planar graphs via Groebner basis

Mariafortuna Paratore
Univ. di Messina, Dipartimento di Matematica
C.da Papardo, Salita Sperone, 31
paratore@dipmat.unime.it

Introduction

Graphs represented a geometric model to solve practical problems of connection. They have some applications in the field of transport and nets of telecommunications. Although the graphs represent a general instrument of analysis of the structure phenomena, the city and territorial analysis uses mostly planar graphs. The street net can be a planar graph: a public square it could be considered a vertex of a graph and the edges are the connection with other roads. Similarly, we can construct a planar graph considering the lines of the city bus.
In this work, we study bipartite planar graphs and two objects associated to G:
1. The edge ideal $I(G)$, $I(G) \subset K[X_1, \dots, X_n]$
2. The associated monomial algebra $K[G]$.
Let K be a field, $R = K[X_1, \dots, X_n]$ a polynomial ring and $K[G]$ the K-subalgebra of R. In [2] $K[G]$ is studied via its presentation ideal, $P(G)$. Let $f_1, \dots, f_q$ be the generators of $K[G]$, $P[G]$ is the kernel of the map $K[T_1, \dots, T_q] \longrightarrow R$ induced by $T_i \mapsto f_i$. If G is bipartite the generators of $P[G]$ correspond to cycles in G. Since the graph G is embedded in the plane and divided it into regions, we consider cycles that bound these regions and polarizations of G linked to polarizations of each cycle. In particular, we can have term orders that arise from a polarization.
Our aim is to compute the Castelnuovo - Munford regularity of $I(G)$ and $P(G)$. We believe that there exists a bound for both the regularities in terms of the number r of regions of G and we are working in this direction. The techniques of the theory of Groebner Basis are used.

1 Planar graphs

Definition 1.1 A graph is said planar if it can be embedded in the plane and its edges are incident only in the common vertices.

Let G be a bipartite planar graph on vertices $x_1, \dots, x_n$ and $R = K[X_1, \dots, X_n]$ a polynomial ring over a field K,with one variable X_i for each vertex x_i.

The *edge ideal* $I(G)$ associated to a graph G is the ideal of R generated by monomials of degree two, X_iX_j, on the $X_1, \ldots, X_n$ variables, such that $\{x_i, x_j\} \in E(G)$ for all $1 \leq i \leq j \leq n$:

$$I(G) = (\{X_iX_j | \{x_i, x_j\} \in E(G)\}).$$

Let $K[G]$ be the K-subalgebra of R generated by $\{X_iX_j|\{x_i, x_j\} \in E(G)\}$, that is called edge subring of G:

$$K[G] = K[\{X_iX_j|\{x_i, x_j\} \in E(G)\}] \subset R.$$

If $f_1, \ldots, f_q$ be the edge generators of G, that are the monomials that correspond to the edges of G, then the presentation ideal $P(G)$ of the $K(G)$ is the kernel of the homomorphism of K-algebras

$$\varphi : B = K[T_1, T_2, \cdots, T_q] \longrightarrow K[G], \qquad T_i \longmapsto f_i.$$

The following results are known:

Proposition 1.1 The toric ideal $P(G)$ of a edge subring $K[G]$ has a Groebner basis consisting of binomials w.r.t. any monomials ordering of the polynomial ring $K[T_1, \ldots, T_q]$.

Proposition 1.2 If G is a bipartite graph and $P(G)$ the toric ideal of $K[G]$), then the set

$$(\{T_w : w \text{ is an even cycle}\})$$

is a universal Groebner basis of $P(G)$.

Example 1.1 Let G be a bipartite planar graph on vertex set $V = \{x_1, \ldots, x_6\}$, $w = \{x_1, \ldots, x_6 = x_1\}$ an even closed walk in G and $f_i = x_ix_{i+1}$ the edge generators of G,then the binomial associated to w is $T_w = T_1T_3T_5 - T_2T_4T_6 \in P(G)$ and this set is a Groebner basis for $P(G)$.

Polarizations

Let G be a bipartite planar graph with vertex set $V(G)$ and $R = k[X_1, \ldots, X_n]$ a polynomial ring over K.
Fix an embedding of the graph G in the plane. Let c be an even closed walk, $c = T_{i_1}, \ldots, T_{i_{2r}}$.

Since G is embedded in the plane, G divides the plane into regions.

In the sequel we will consider graphs connected with no cutvertices (2-connected) and with a fixed embedding in the plane, where n are vertices, r are regions and q the edges.

Definition 1.2 Let c be an even closed walk of G, $c = T_{i_1}, \ldots T_{i_t}$. A polarization of c is the set

$$\tilde{c} = \{T_{i_1}, T_{i_3}, \ldots, T_{i_{t-1}}\} \qquad \text{or} \qquad \tilde{\tilde{c}} = \{T_{i_2}, T_{i_4}, \ldots, T_{i_t}\}.$$

Let $\{c_1, \dots, c_r\}$ be the atomic cycles of G. A polarization $\mathcal{P}$ of G is a set $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_r\}$, where $\mathcal{P}_i$ is a polarization of c_i, $\forall i$ and for $i \neq j$ we have $\mathcal{P}_i \cap \mathcal{P}_j = 0$ and $\{T_i : T_i \notin \mathcal{P}_j,\ \text{for } j = 1, \dots, r\}$ is a polarization of w, where w is the boundary of the unbounded region.

Example 1.2 Let G be a graph on vertex set $V = \{x_1, \dots, x_4\}$. $c = x_1, \dots, x_4$ an atomic cycle for G.A polarization of c is the set of vertices $\{x_1, x_3\}$ or the set $\{x_2, x_4\}$ and the set $\{T_1T_3 - T_2T_4\}$ is a Groebner basis for the toric ideal of $K[G]$.

By a chosen polarization $\mathcal{P}$ of G, we can define a term order on the monomial of $K[G]$ such that its toric ideal has a Groebner basis consisting of atomic cycles in G [1].

2 Regularity of ideals associated to bipartite planar graphs

The definition of the Castelnuovo -Munford regularity of a graded module M on $R = k[X_1, \dots, X_n]$ is the following: If
$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$ is a graded minimal free resolution of M and if b_i is the maximum degree of the generators of the free module F_I then

$$reg(M) = sup\{b_i - i, i \geq 0\}.$$

In other words, $reg(M)$ is the smallest integer m such that for every j, the j-th syzygy module M is generated in degree $\leq m + j$, hence $reg(M) = sup\{\beta_{i,i+j} \geq 0\}$, $\beta_{i,k}$ are the graded Betti numbers of M.

Theorem 2.1(Green, [5]) Let $R = K[X_1, \dots, X_n]$ be a polynomial ring. For any ideal $I \subset R$, $reg(I) \leq reg(in_< I)$, where $in_< I = in_<(\overline{G})$, with $\overline{G}$ Groebner basis for I.Then $reg(I) \leq reg(\overline{G})$.

The aim is to calculate the Groebner basis $\overline{G}$ of the edge ideal $I(G)$ and of the toric ideal $P(G)$, to find the regularity of $(\overline{G})$ and a bound for the regularity of $I(G)$ and $P(G)$ (for $P(G)$ via results about polarization).

1. Regularity of the edge ideal $I(G)$. We can give a conjecture for two particular classes of graphs, called St_r and $B_{2r'}$. These graphs are planar and divide the plane in r and $2r'$ regions respectively . Next, we consider the minimal free resolution of the edge ideal associated to these classes and we study the regularity of $I(G)$ linked to the number of the regions.

Conjecture 1): Let $G = St_r$ be the graph with vertex set $\{v_1, \dots, v_{2r+1}\}$ and edge set $\{\{v_1, v_i\} : 2 \leq i \leq r+1\} \cup \{\{v_i, v_{i+r}\} : 2 \leq i \leq r+1\} \cup \{\{v_i, v_{i+r-1}\} : 3 \leq i \leq r+1\} \cup \{v_2, v_{2r+1}\}\}$, where r is the number of regions of G. Then $regI(St_r) \leq r$.

Conjecture 2): Let $G = B_{2r'}$ be the graph with vertex set $\{v_1, \dots, v_{3r'+3}\}$ and edge set $\{\{v_1, v_{i+1}\} : 1 \leq i \leq 3r'+2, i \neq r'+1, 2r'+2, 3r'+3\} \cup \{\{v_i, v_{i+r'+1}\} : 1 \leq i \leq 2r'+2\}$, where $2r'$ is the number of regions. Then $regI(B_{2r'}) \leq 2r'+1$.

Results:
The conjecture 1) is true for $r \in \{2, 3, 4, 5, 6, 7, 8\}$.
The conjecture 2) is true for $r' \in \{1, 2, 3, 4\}$.

2. In order to compute the regularity for the toric ideal $P(G)$ of $K[G]$, we recall the result:

Theorem 2.2 Let $K[G]$ be a complete intersection and a domain. Suppose that $P(G)$ is equigenerated in degree a and $htP(G) = t$. Then $regP(G) \leq a(t-1)$.

Proof. If $K[G]$ is a complete intersection and a domain, $P(G)$ is generated by a regular sequence of t elements of degree a. The result follows by the graded minimal Koszul resolution of $P(G)$.

In this moment, we don't know if it is easy to calculate the regularity of $P(G)$ for a graph planar G, but this is certainly possible for the graph St_r and $B_{2r'}$. This is our program.

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