

Thermodynamics for fluid flow in porous structures

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In this paper we construct a geometric model for the thermodynamics of porous media filled by flow using a non-conventional model based on the extended irreversible thermodynamic developed in [1] in which a second order structural permeability tensor à la Kubik [2] (describing a network of thin porous channels in an elastic body filled by fluid flow) and its flux are introduced as internal variables (see Fig. 1, 2 and 3). The models for solids defective by a structure of infinitesimally capillary tubes may have relevance in important advances that have been made during the last decades in the description of problems and phenomena accompanying flows of mass in porous media. They find applications in many fundamental sectors: geology, vulcanology, biology, medical science, technology of materials. The porous channels sometime can self propagate because of changed conditions and surrounding conditions that are favorable and for instance in metallurgy during a process of fabrication this propagation can provoke a premature fracture of these porous media. Following [3], we introduce the concepts of *processes* and *transformations* and we derive the entropy function. To describe a pore structure in [2] Kubik considers a representative elementary sphere volume Ω of a porous skeleton filled with fluid, the diameter D of which is much greater than the characteristic length H of the entire porous medium. An obvious requirement is that Ω is large enough to provide a representation of all the statistical properties of the pore space. Since all pores are considered to be interconnected the *effective volume porosity* is completely defined as $f_v = \frac{\Omega^p}{\Omega}$, where Ω^p represents the pore space of Ω ($\Omega = \Omega^s + \Omega^p$, with Ω^s the solid space). The analysis is restricted to media which are homogeneous with respect to volume porosity f_v , i.e. f_v remains constant in the medium. To avoid confusion all microscopic quantities are described with respect to the ξ_i coordinate system, while macroscopic quantities are assigned to the x_i axes (there is no difference between these systems apart from possible translations). If $f(\boldsymbol{\xi})$ is any scalar, vector or second order tensor which represents some microscopic property of a fluid prescribed at pore space and, by definition, being zero in the solid, the volume and area averaging procedures give

$$\hat{f}(\mathbf{x}) = \frac{1}{f_v \Omega} \int_{\Omega_p} f(\boldsymbol{\xi}) d\Omega, \quad \bar{f}(\mathbf{x}) = \frac{1}{\Omega} \int_{\Omega_p} f(\boldsymbol{\xi}) d\Omega, \quad f^*(\mathbf{x}; \boldsymbol{\mu}) = \frac{1}{\Gamma^p} \int_{\Gamma} f(\boldsymbol{\xi}) d\Gamma, \quad (1)$$

where \hat{f} , \bar{f} and f^* are pore-volume, bulk-volume and pore area average quantities, respectively. Moreover Γ^p represents the pore area of the central sphere section Γ . It should be pointed out that f^* depends on the orientation of Γ . In such a medium Kubik introduces a so called *structural permeability tensor*, responsible for the porous structure, in the following way: $\bar{v}(\mathbf{x})_i = \mathcal{R}_{ij}(\mathbf{x}, \boldsymbol{\mu})^* v_j(\mathbf{x}, \boldsymbol{\mu})$, where

$$\bar{\mathbf{v}}(\mathbf{x}) = \frac{1}{\Omega} \int_{\Omega^p} \mathbf{v}(\boldsymbol{\xi}) d\Omega, \quad \boldsymbol{\xi} \in \Omega^p, \quad \text{and} \quad \bar{\mathbf{v}}^*(\mathbf{x}, \boldsymbol{\mu}) = \frac{1}{\Gamma^p} \int_{\Gamma} \mathbf{v}(\boldsymbol{\xi}) d\Gamma, \quad \boldsymbol{\xi} \in \Gamma^p \quad (2)$$

are the bulk-volume average of fluid velocity and the corresponding pore-area fluid velocity. The above relation gives a linear mapping between the bulk volume average fluid velocity and the local velocity of fluid particles passing through the pore area Γ^p of central sphere section with normal vector $\boldsymbol{\mu}$. In [2] Kubik establishes the geometrical interpretation of \mathcal{R} and such a characterization allows to consider the fluid flow described by the volume average velocity as the superposition of three one-dimensional flows at the area average velocities along mutually perpendicular directions. Only part of the fluid can flow unimpeded while the rest is trapped in the porous skeleton. Now, following Maruszewski in [5], using the previous definitions, for any flux q_i of some physical field transported through a cobweb of lines one postulates that

$$\bar{q}(\mathbf{x})_i = \mathcal{R}_{ij}(\mathbf{x}, \boldsymbol{\mu})^* q_j(\mathbf{x}, \boldsymbol{\mu}), \quad \text{where} \quad \mathcal{R}_{ij}(\mathbf{x}, \boldsymbol{\mu}) = \Gamma r_{ij}(\mathbf{x}, \boldsymbol{\mu}). \quad (3)$$

In equation (3) the tensor \mathcal{R}_{ij} expresses a structure of microchannels and r_{ij} is a new tensor called *structural permeability core tensor* that refers \mathcal{R}_{ij} to the surface Γ . Its unit is m^{-2} . Now, we recall the model developed in [1], where a model for describing the reciprocal interactions between a fluid flow, a structural permeability field coming from a porous structure in an elastic body and a thermal field was presented. The structural permeability field is described by the state tensor r_{ij} (not necessarily symmetric) and by the flux of this tensor \mathcal{V}_{ijk} . The fluid flow is described by two variables: the concentration of the fluid c and the flux of this fluid j_i . The mass density of the fluid transported through the elastic porous body of density ρ_2 is called ρ_1 . The fluid and the elastic solid form a two-components mixture of density $\rho = \rho_1 + \rho_2$ where $\rho_1 \ll \rho_2$. The concentration of the fluid c is defined as follows $c = \frac{\rho_1}{\rho}$. For the mixture of continua as a whole and for each constituent the continuity equations are satisfied $\dot{\rho} + \rho v_{i,i} = 0$, $\frac{\partial \rho_1}{\partial t} + (\rho_1 v_{1i})_{,i} = r_1$, $\frac{\partial \rho_2}{\partial t} + (\rho_2 v_{2i})_{,i} = r_2$, where a superimposed dot denotes the material derivative, v_{1i} and v_{2i} are the velocities of the fluid particles and the particles of the elastic body respectively, so that the barycentric velocity and the fluid flux are defined in the following form $\rho v_i = \rho_1 v_{1i} + \rho_2 v_{2i}$, $j_i = \rho_1 (v_{1i} - v_i)$ and r_1 and r_2 denote the sources of mass of the constituents. In the following they will be neglected, because it is assumed that there are not chemical reactions between the constituents or coagulations. The mechanical properties of the considered system are described by the total stress tensor σ_{ij} (in general non symmetric) related to the whole body considered as a mixture and the small-strain tensor ε_{ij} describing the deformation of the elastic solid $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ (where u_i are the components of the displacement vector) so that the gradient of the velocity of the body

is given by $v_{i,j} = w_{ij} + \frac{\partial \varepsilon_{ij}}{\partial t}$, where $w_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ is the antisymmetric part of $v_{i,j}$. Finally, the thermal field is governed by the temperature and the heat flux q_i . Thus, the vector space is chosen as follows $C = \{\varepsilon_{ij}, c, T, r_{ij}, j_i, q_i, \mathcal{V}_{ijk}, c_i, T_i, r_{ij,k}\}$, where we have taken into consideration the gradients c_i, T_i and $r_{ij,k}$.

All the processes occurring in the considered body are governed by the following laws: the *balance of mass*: $\rho \dot{c} + j_{i,i} = 0$;

the *momentum balance*: $\rho \dot{v}_i - \sigma_{ji,j} - f_i = 0$, where f_i denotes a body force;

the *moment of momentum balance*: $\varepsilon_{ijk} \sigma_{jk} + c_i = 0$.

In [1] it was demonstrated that the couple per unit volume c_i is vanishing;

the *internal energy balance*: $\rho \dot{e} - \sigma_{ji} v_{i,j} + q_{i,i} = \rho r$,

where e is the internal energy density and r is a heat source distribution, neglected in the following, $v_{i,j} = L_{ij} = L_{(ij)} + L_{[ij]}$ (where $L_{(ij)}$ and $L_{[ij]}$ are respectively the symmetric and antisymmetric part of the velocity gradient and $L_{[ij]} = \Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$).

Introducing the deformation gradient \mathbf{F} , also we have $\mathbf{L} = \nabla \mathbf{v} = \dot{\mathbf{F}} \mathbf{F}^{-1}$.

The *evolution equations for the structural permeability tensor and the fluxes of the mass, heat and porous fields*: $\dot{r}_{ij}^* + \mathcal{V}_{ijk,k} = R_{ij}(C)$, $\dot{j}_i^* = J_i(C)$,

$\dot{q}_i^* = Q_i(C)$, $\dot{\mathcal{V}}_{ijk}^* = V_{ijk}(C)$, where $r_{ij}^* = r_{ij} - \Omega_{il} r_{lj} - \Omega_{jl} r_{il}$,

$\mathcal{V}_{ijk}^* = \dot{\mathcal{V}}_{ijk} - \Omega_{il} \mathcal{V}_{ljk} - \Omega_{jl} \mathcal{V}_{ilk} - \Omega_{kl} \mathcal{V}_{ijl}$, $j_i^* = \dot{j}_i - \Omega_{ij} j_j$, $q_i^* = \dot{q}_i - \Omega_{ij} q_j$.

In the above equations the superimposed asterisk indicates the Zaremba-Jaumann derivative. All the admissible solutions of the proposed evolution equations should be restricted by the following *entropy inequality*: $\rho \dot{S} + J_{S_{k,k}} - \frac{\rho r}{T} \geq 0$,

where S denotes the entropy per unit mass and \mathbf{J}_S is the entropy flux associated with the fields of the set C given by $\mathbf{J}_S = \frac{1}{\theta} \mathbf{q} + \mathbf{k}$, with \mathbf{k} an additional term called *extra entropy flux density*.

In [1] all the following constitutive functions $\mathbf{Z} = \tilde{\mathbf{Z}}(C)$ with

$\mathbf{Z} = \{\sigma_{ij}, \mu^c, \nu_{ij}, c_i, e, R_{ij}, J_i, Q_i, V_{ijk}, S, \phi_i\}$, μ^c the chemical potential of mass and ν_{ij} the potential related to the porous field were obtained by analyzing the entropy inequality by Liu's theorem and (using Smith theorem) with the help of the isotropic polynomial representations of the proper constitutive functions, satisfying the objectivity principle. Now, we construct a geometric model for these porous media filled by fluid and we derive the expressions for the existence of an entropy function. We assume that the body \mathcal{B} having a regular boundary $\delta \mathcal{B}$, is regularly embedded into Euclidean space \mathbb{R}^3 by a regular family of instantaneous time-dependent configurations \mathcal{B}_t . We treat the time on equal footing as the other state variables entering explicitly the state functions. Then, we consider a material element and we define the state space at time t as the set B_t of all state variables which "fit" the configuration of the element at time t . B_t is assumed to have the structure of a finite dimensional manifold. The "total state space" is the disjoint union $\tilde{\mathcal{B}} = \bigcup_t \{t\} \times B_t$ with a given natural structure of fibre bundle over \mathbb{R} where time flows [2]; if the instantaneous state space B_t does not vary in time the state space $\tilde{\mathcal{B}}$ reduces to a Cartesian product $\mathbb{R} \times B$. Moreover, we consider an abstract space of *processes* [2], i.e. a set Π of functions $P_t^i : [0, t] \rightarrow \mathcal{G}$, where $[0, t]$ is any time interval, the space \mathcal{G} being a suitable target space defined by the problem under consideration, i a label ranging

in an unspecified index set for all allowed processes and $t \in \mathbb{R}$ the so called *duration* of the process. A continuous function then is defined $\rho : \mathbb{R} \times \Pi \rightarrow C^0(B_0, B_t)$ so that for any instant of time t and for any process $P_t^i \in \Pi$ a continuous mapping called *transformation* (induced by the process) is generated. Now, we assume that the behavior of the porous media filled by fluid is described by the following state variables $\mathbf{C} = \{\mathbf{F}, e, c, \mathbf{r}, \mathbf{j}, \mathbf{q}, \mathbf{V}, \nabla c, \nabla \theta, \nabla \mathbf{r}\}$. The full state space is then $\mathcal{B} = \text{Lin}(\mathcal{V}) \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathcal{W}_1 \oplus \mathcal{V} \oplus \mathcal{V} \oplus \mathcal{W}_2 \oplus \mathcal{V} \oplus \mathcal{V} \oplus \text{Lin}(\mathcal{W}_1)$, where $\mathcal{V} \simeq \mathbb{R}^3$, \mathcal{W}_1 and \mathcal{W}_2 are vector spaces accounting for the internal variables \mathbf{r} e \mathbf{V} respectively. The process \mathbf{P}_t is described by the following functions $P_t^i = [\mathbf{L}, h, \Gamma, \Upsilon, \mathbf{J}, \mathbf{Q}, \Lambda, \mathcal{C}, \gamma, \Xi]$, where $h(\tau) = -\nabla \cdot \mathbf{q}$, $\Gamma(\tau) = \nabla \cdot \mathbf{j}$, $\mathbf{Q}(\tau) = \Omega \mathbf{q} + \mathbf{Q}(C)$, $\mathcal{J}(\tau) = \Omega \mathbf{j} + \mathbf{J}(C)$, $\Lambda_{ijk}(\tau) = \Omega_{il} \mathcal{V}_{ljk} + \Omega_{jl} \mathcal{V}_{ilk} + \Omega_{kl} \mathcal{V}_{ijl} + V_{ijk}(C)$, $\Upsilon_{ij}(\tau) = \Omega_{il} r_{lj} + \Omega_{jl} r_{il} - \mathcal{V}_{ijk,k} + R_{ij}(C)$. Following the method we assume that the transformations induced by the process are governed by the following dynamical system

$$\left\{ \begin{array}{l} \dot{\mathbf{F}} = \mathbf{L}(\tau) \mathbf{F}(\tau), \\ \mu \dot{e} = \mathbf{T}(\sigma) \cdot \mathbf{L}(\tau) - h(\tau), \\ \mu \dot{c} = \Gamma(\tau), \\ \dot{\mathbf{r}} = \Upsilon(\tau), \\ \dot{\mathbf{j}} = \mathbf{J}(\tau), \\ \dot{\mathbf{q}} = \mathbf{Q}(\tau), \\ \dot{\mathbf{V}} = \Lambda(\tau), \\ \dot{\nabla} c = \mathcal{C}(\tau), \\ \dot{\nabla} \theta = \gamma(\tau), \\ \dot{\nabla} \mathbf{r} = \Xi(\tau). \end{array} \right. \quad (4)$$

By using this system of differential equations, following standard procedures in this geometrical structure we are able to introduce an "entropy function", which is related to a transformation between the initial and the actual states σ_0 and σ_t , by setting : $s(t) = -\int_0^t \frac{1}{\mu} \nabla \cdot \mathbf{J}_S d\tau$, where \mathbf{J}_S is the extra entropy flux defined above. Finally, we obtain $s = \int_\sigma \Omega$, where $\Omega = -\frac{\mathbf{T}\mathbf{F}^{-T}}{\theta\mu} \cdot d\mathbf{F} + \frac{1}{\theta} de + \frac{1}{\mu\theta^2} \mathbf{q} \cdot \nabla \theta d\tau - \frac{1}{\mu} \nabla \cdot \mathbf{k} d\tau$, where the expression for \mathbf{T} , \mathbf{q} and \mathbf{k} have been calculated in [1] a suitable form. Thus, the entropy function is now calculated as an integral along a path into the space $\mathbb{R} \times B$ of all thermodynamic variables together with the independent time variable. Finally, we derive the closure relations which will give the necessary conditions for the existence of the existence of the entropy function during the analyzed processes.

References

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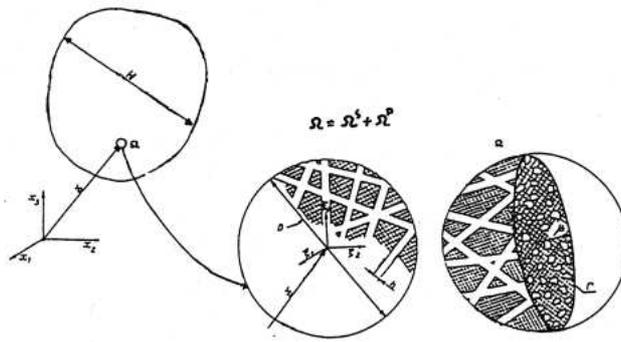


Fig.1 The averaging scheme. Characteristics of the given channel-pore structure (see [1]).

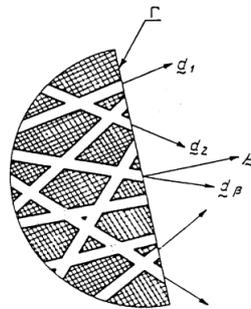


Fig.2 Characteristic of the given channel-pore structure (see [1]).

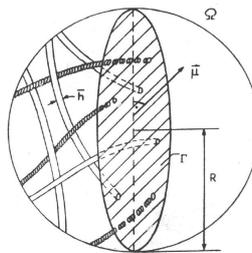


Fig.3 Characteristics of the pore-core structure ($\bar{h} \ll R$) (see [2]).