MIXED DIRICHLET-ROBIN PROBLEMS IN IRREGULAR DOMAINS

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We study mixed Dirichlet-Robin problems for the Laplacian in irregular domains. Our interest in these problems is motivated by some recent applications in physics, electrochemistry, heterogeneous catalysis and physiology.

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1. Introduction

The Laplacian transport to and across irregular and fractal interfaces are often encountered in nature or in technical processes: properties of rough electrodes in electrochemistry, steady-state diffusion towards irregular membranes in physiological processes, the Eley-Rideal mechanism in heterogeneous catalysis in porous catalysts, and in NMR relaxation in porous media (see Refs. 4, 5, 18 and the references therein). To characterize the mathematical formulation of these type of problems, we consider the current flowing through an electrochemical cell as shown in Fig. 1, where the working electrode presents an irregular boundary. The current density $\vec{J}$ is proportional to the electrostatic field $\vec{\nabla} V$, by classical equations of the type $\vec{J} = -\sigma \vec{\nabla} V$, where $\sigma$ is the electrolyte conductivity. The conservation of this current throughout the bulk yields the Laplace equation for the potential $V$ $\text{div}(-\sigma \vec{\nabla} V) = 0$, that is, $\Delta V = 0$. The working electrode presents a finite faradaic resistance $r$ to the current flow. If we assume that the outside of the irregular boundary is at zero potential, current conservation at the boundary leads to the following relation ($\nu$ is the outward normal vector to the interface) $\vec{J} \cdot \nu = \frac{V}{r}$, that is, $\frac{\partial V}{\partial \nu} = -\frac{V}{\Lambda}$, where $\Lambda = \sigma r$. A constant potential $V_0$ is applied on the counter electrode.
Fig. 1. An electrochemical cell.

The previous problem can be formally stated as

\[
\begin{cases}
\Delta V = 0 & \text{in } \Omega \\
V = V_0 & \text{on } \Gamma_0 \\
\frac{\partial V}{\partial n} = 0 & \text{on } \Gamma_1 \\
\frac{\partial V}{\partial n} + \frac{V}{\lambda} = 0 & \text{on } \Gamma_2 \\
\frac{\partial V}{\partial n} = 0 & \text{on } \Gamma_3
\end{cases}
\]

where \( \Gamma_0 = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, y = -1\} \), \( \Gamma_1 = \{(x, y) \in \mathbb{R}^2 : x = 1, -1 < y < 0\} \), \( \Gamma_2 = \{(x, y) \in \mathbb{R}^2 : x = 0, -1 < y < 0\} \), and \( \Gamma_3 \) is either \( K_n^{(l)} \setminus \{A, B\} \) or \( K_n^{(l)} \setminus \{A, B\} \) (where \( K_n^{(l)} \) denotes the prefractal curve approximating the Koch curve type fractal \( K^{(l)} \), \( A = (0, 0) \) and \( B = (1, 0) \)).

The layout of the paper is as follows. In the second section, we recall the definitions and the properties of the Koch curve type fractals. In the third section, we recall traces and embeddings theorems, and Green’s formulae either on polygonal curve or on d-sets in the form which best fits our aims. In section 4, we state a variational principle for the problem (1) either in the prefractal case or in the fractal one. In the last section, we prove regularity results in terms of ordinary fractional Sobolev spaces and weighted Sobolev spaces when the interfaces are the prefractal curves approximating the Koch curve type fractals.
2. The Koch curve type fractals

Let us give some definitions and notations which will be used later. We consider the family \( \Psi^{(l)} = \{ \psi_{1}^{(l)}, \ldots, \psi_{4}^{(l)} \} \) of contractive similitudes \( \psi_{i}^{(l)} : \mathbb{C} \rightarrow \mathbb{C}, i = 1, \ldots, 4 \), with contraction factor \( l^{-1}, 2 < l \leq 4 \),

\[
\begin{align*}
\psi_{1}^{(l)}(z) &= \frac{z}{l}, \\
\psi_{2}^{(l)}(z) &= \frac{z}{l} e^{i \theta(l)} + \frac{1}{l}, \\
\psi_{3}^{(l)}(z) &= \frac{z}{l} e^{-i \theta(l)} + \frac{1}{2} + i \sqrt{\frac{1}{4} - \frac{1}{l}}, \\
\psi_{4}^{(l)}(z) &= \frac{z - 1}{l} + 1,
\end{align*}
\]

where

\[
\theta(l) = \arcsin \left( \frac{\sqrt{l(4-1)}}{2} \right). \tag{2}
\]

By the general theory of self-similar fractals (see Ref. 8), there exists a unique closed bounded set \( K^{(l)} \), which is invariant with respect to \( \Psi^{(l)} \), that is,

\[
K^{(l)} = \bigcup_{i=1}^{4} \psi_{i}^{(l)}(K^{(l)}). \tag{3}
\]

We remark that for \( l = 3 \) we obtain the usual Koch curve.

Moreover, there exists a unique Borel regular measure \( \mu^{(l)} \), with \( \text{supp} \mu^{(l)} = K^{(l)}, \text{invariant} \) with respect to \( \Psi^{(l)} \), which coincides with the normalized \( d_{f}(l) \)-dimensional Hausdorff measure on \( K^{(l)} \), where \( d_{f}(l) = \frac{\ln 4}{\ln l} \),

\[
\mu^{(l)} = (H^{d_{f}(l)}(K^{(l)}))^{-1}H^{d_{f}(l)}|_{K^{(l)}}. \tag{4}
\]

The measure \( \mu^{(l)} \) has the property that there exists two positive constants \( C_{1}, C_{2} \), such that,

\[
C_{1} \frac{d^{y}}{l} \leq \mu^{(l)}(B(P, r) \cap K^{(l)}) \leq C_{2} r^{d_{f}(l)}, \quad \forall P \in K^{(l)}, \tag{5}
\]

where \( B(P, r) \) denotes the Euclidean ball with center in \( P \) and radius \( 0 < r \leq 1 \). According to Jonsson and Wallin (see Ref. 10), we say that \( K^{(l)} \) is a \( d \)-set, with \( d = d_{f}(l) \).

Let \( K_{0} \) be the line segment of unit length having as endpoints \( A = (0, 0) \) and \( B = (1, 0) \). We set, for each \( n \) in \( \mathbb{N} \), \( K_{1}^{(l)} = \bigcup_{i=1}^{4} \psi_{i}^{(l)}(K_{0}) \), \( K_{n+1}^{(l)} = \bigcup_{i=1}^{4} \psi_{i}^{(l)}(K^{(l)}_{n}) \); \( K^{(l)}_{n} \) is the so-called \( n \)-th prefractal curve. We have that \( K_{n+1}^{(l)} = \bigcup_{M \in F_{n}^{(l)}} \bigcup_{i=1}^{4} \psi_{i}^{(l)}(M) \), where \( F_{n}^{(l)} = \{ M : M \text{ is a segment of } K^{(l)}_{n} \} \) denotes the set of segments of the \( n \)-th iterate \( K^{(l)}_{n} \).
3. Functional spaces

In this section, we recall the definitions of some functional spaces which will be used in the following (we refer to Refs. 6 and 17 for a more complete discussion). Let $D$ be an arbitrary open set of $\mathbb{R}^2$, we denote by $C^0_0(D)$ and by $C_0^\infty(D)$ the usual spaces of continuous or smooth functions with compact support on $D$, respectively. Let $L^2(\mu, \cdot)$ be the Lebesgue space with respect to a measure $\mu$ on subsets of $\mathbb{R}^2$, which will be specified everytime. Moreover, let $H^s(D)$ and $H^s_0(D)$, where $s \in \mathbb{R}^+$, be the Sobolev spaces defined in Ref. 17. We consider the trace Sobolev spaces $H^s(\mathring{K}_n^{(l)})$ on the prefractal curve $K_n^{(l)}$ for each $s \in \mathbb{R}^+$ defined according to Brezzi and Gilardi (see Definition 2.27 in Ref. 2), (see also Ref. 14). From now on, we shall denote $H^s(\mathring{K}_n^{(l)})$ simply by $H^s(\mathring{K}_n)$.

For $f$ in $H^s(D)$, we define $\gamma_0$ as a trace operator

$$\gamma_0 f(x) := \lim_{r \to 0} \frac{1}{|B(x, r) \cap D|} \int_{B(x, r) \cap D} f(y) \, dy$$

(6)

at every point $x \in \partial D$ where the limit exists (see Ref. 1).

We recall the following trace Theorem on boundaries of polygonal domain (for a more general discussion, see Refs. 2, 6 and 17).

**Theorem 3.1.** Let $D$ be a bounded open subset of $\mathbb{R}^2$ whose boundary $\partial D$ is a polygonal curve. For each $s > 1/2$, $H^{s-\frac{1}{2}}(\Gamma)$ is the trace space on $\Gamma' \subset \partial D$ of $H^s(D)$ in the following sense:

(i) $\gamma_0$ is a continuous linear operator from $H^s(D)$ to $H^{s-\frac{1}{2}}(\Gamma)$;

(ii) there exists a continuous linear operator $\text{Ext}$ from $H^{s-\frac{1}{2}}(\Gamma)$ to $H^s(D)$, such that $(\gamma_0 \circ \text{Ext})$ is the identity operator in $H^{s-\frac{1}{2}}(\Gamma)$.

We recall the definition of the Lions-Magenes trace spaces $H^{\frac{1}{2}}_{0,0}$ and the Green’s formula on Lipschitz domains suitable for the study of the mixed
Theorem 3.2. Let $D$ be a bounded connected open subset of $\mathbb{R}^2$ whose boundary $\partial D$ is Lipschitz. Let $\Gamma$ be a connected open set of $\partial D$. Then, the following Green’s formula

$$\int_D \nabla u \nabla v \, dx \, dy + \int_D v \Delta u \, dx \, dy = \langle \frac{\partial u}{\partial v}, \gamma_0 v \rangle_{(H^1_{0,0}(\partial D \setminus \Gamma))', H^1_{0,0}(\partial D \setminus \Gamma)} \quad (7)$$

holds, whatever $u \in H^1(D)$ such that $\Delta u \in L^2(D)$ and $v \in H^1(D)$ such that $\gamma_0 v = 0$ on $\Gamma$.

Here we denote $H^1_{0,0}(\partial D \setminus \Gamma) := \{ \theta \in L^2(\partial D \setminus \Gamma) : \exists v \in H^1(D) : \gamma_0 v = 0 \text{ on } \Gamma, \gamma_0 v = \theta \text{ on } \partial D \setminus \Gamma \}$, equipped with the quotient norm

$$\| \theta \|_{H^1_{0,0}(\partial D \setminus \Gamma)} := \inf \{ \| v \|_{H^1(D)} : v \in H^1(D), \gamma_0 v = 0 \text{ on } \Gamma, \gamma_0 v = \theta \text{ on } \partial D \setminus \Gamma \}.$$

$(X)'$ denotes the dual of $X$.

In order to consider traces on fractals, we use Besov spaces $B^s_{p,q}$ on $\delta$-sets (for the definition, see, for instance, Refs. 10 and 19) and the following trace Theorem on $\delta$-sets (see Refs. 10, 19 and 20) and Green’s formula on $\delta$-sets (see Ref. 12) specialized to our case.

Theorem 3.3. Let $D$ be a bounded open subset of $\mathbb{R}^2$. For each $s > 1 - \frac{d}{2}$, $B^{s,2}_{s-1,\frac{d}{s+1}}(K^{(i)})$ is the trace space to $K^{(i)}$ ($K^{(i)} \subset \overline{D}$) of $H^s(D)$ in the following sense:

(i) $\gamma_0$ is a continuous linear operator from $H^s(D)$ to $B^{s,2}_{s-1,\frac{d}{s+1}}(K^{(i)})$,

(ii) there exists a continuous linear operator Ext from $B^{s,2}_{s-1,\frac{d}{s+1}}(K^{(i)})$ to $H^s(D)$, such that $(\gamma_0 \circ \text{Ext})$ is the identity operator in $B^{s,2}_{s-1,\frac{d}{s+1}}(K^{(i)})$.

Theorem 3.4. Let $D$ be a bounded connected open subset of $\mathbb{R}^2$, with boundary $\partial D$. Let $\Gamma$ be a connected open set of $\partial D$ and $\partial D \setminus \overline{\Gamma} = K^{(i)}$.

Then, the following Green’s formula

$$\int_D \nabla u \nabla v \, dx \, dy + \int_D v \Delta u \, dx \, dy = \langle \frac{\partial u}{\partial v}, \gamma_0 v \rangle_{(B^{s,2}_{s-1,\frac{d}{s+1}}(\partial D \setminus \overline{\Gamma}))', B^{s,2}_{s-1,\frac{d}{s+1}}(\partial D \setminus \overline{\Gamma})} \quad (8)$$

holds, whatever $u \in H^1(D)$ such that $\Delta u \in L^2(D)$ and $v \in H^1(D)$ such that $\gamma_0 v = 0$ on $\Gamma$. 

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Here we recall the definition of the space $B^{2,2}_{β,0}$, which is the “fractal” analogue of the Lions-Magenes space $H^{1,0}_{0,0}$:

\[ B^{2,2}_{β,0}(\partial D \setminus \Gamma) := \{ \theta \in L^2(\partial D \setminus \Gamma) : \exists v \in H^1(D) : \gamma_0 v = 0 \text{ on } \Gamma, \gamma_0 v = \theta \text{ on } \partial D \setminus \Gamma \}, \]

equipped with the quotient norm

\[ \| \theta \|_{B^{2,2}_{β,0}(\partial D \setminus \Gamma)} := \inf \{ \| v \|_{H^1(D)} : v \in H^1(D), \gamma_0 v = 0 \text{ on } \Gamma, \gamma_0 v = \theta \text{ on } \partial D \setminus \Gamma \}. \]

For a characterization of the dual of Besov spaces, see also Refs. 11 and 20.

4. Variational formulation

When the interface $\Gamma_2$ is the prefractal curve $K_n^{(l)}$ approximating the Koch curve type fractal $K^{(l)}$, we consider the following - formally stated - problem (which is equivalent to (1))

\[
\begin{aligned}
-\Delta u &= f & \text{in } \Omega^{(l)}_n \\
u &= 0 & \text{on } \Gamma_0 \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \Gamma_1 \\
\frac{\partial u}{\partial n} + c_n u &= d_n & \text{on } K_n^{(l)} \setminus \{A, B\} \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \Gamma_3
\end{aligned}
\]  

(9)

where $f$ is a given function in $L^2(\Omega^{(l)}_n)$, $c_n \geq 0$ and $d_n$ are constant for each $n$ in $\mathbb{N}$. In the present paper, the constants $c_n$ and $d_n$ will be left indetermined since we work with $n$ fixed; in the asymptotic analysis, on the contrary, these constants will be chosen in a suitable way.

We give the weak formulation of the previous problem (9) and, for every $n$ in $\mathbb{N}$, we prove the existence and the uniqueness of its weak solution $u_n$.

**Theorem 4.1.** For any $f \in L^2(\Omega^{(l)}_n)$, there exists one and only one solution $u_n$ of the following problem

\[
\begin{aligned}
\text{find } u_n &\in V(\Omega^{(l)}_n) := \{ u_n \in H^1(\Omega^{(l)}_n) : \gamma_0 u_n = 0 \text{ on } \Gamma_0 \} \quad \text{s.t.} \\
\int_{\Omega^{(l)}_n} \nabla u_n \nabla v \, dxdy + c_n \int_{K_n^{(l)}} \gamma_0 u_n \gamma_0 v \, ds &= \int_{\Omega^{(l)}_n} f v \, dxdy + d_n \int_{K_n^{(l)}} \gamma_0 v \, ds \\
\forall v &\in V(\Omega^{(l)}_n).
\end{aligned}
\]

(10)

Moreover, $u_n$ is obtained by

\[
\min_{v \in V(\Omega^{(l)}_n)} \left\{ \int_{\Omega^{(l)}_n} |\nabla v|^2 \, dxdy + c_n \int_{K_n^{(l)}} |\gamma_0 v|^2 \, ds - 2 \int_{\Omega^{(l)}_n} f v \, dxdy - 2d_n \int_{K_n^{(l)}} \gamma_0 v \, ds \right\}.
\]
Proof. The thesis follows by applying Lax-Milgram Theorem to the bilinear form $a_n(u_n, v) = \int_{\Omega_n} \nabla u_n \nabla v \, dx \, dy + c_n \int_{K_n} \gamma_0 u_n \gamma_0 v \, ds$, defined in $V(\Omega_n) \times V(\Omega_n)$ ($ds$ denotes the one-dimensional measure on $K_n$). This form is continuous and coercive in $V(\Omega_n)$ by using the Poincaré inequality and Theorem 3.1. Moreover, the linear functional $F(v) = \int_{\Omega_n} f v \, dx \, dy + d_n \int_{K_n} \gamma_0 v \, ds$ is bounded in $V(\Omega_n)$.

We now consider the problem when the interface $\Gamma_2$ is the Koch curve type fractal $K$: it can be formally stated as

$$
\begin{cases}
- \Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \Gamma_0 \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \\
\frac{\partial u}{\partial \nu} + cu = d & \text{on } K \setminus \{A, B\} \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_3
\end{cases}
$$

(11)

where $f$ is a given function in $L^2(\Omega)$, $c \geq 0$ and $d$ are constant.

Proceeding as above we can prove existence and uniqueness of the weak solution.

Theorem 4.2. For any $f \in L^2(\Omega)$, there exists one and only one solution $u$ of the following problem

$$
\begin{cases}
\text{find } u \in V(\Omega) := \{ u \in H^1(\Omega) : \gamma_0 u = 0 \text{ on } \Gamma_0 \} \ s.t. \\
\int_{\Omega} \nabla u \nabla v \, dx \, dy + c \int_{K} u v \, d\mu = \int_{\Omega} f v \, dx \, dy + d \int_{K} v \, d\mu \\
\forall v \in V(\Omega)
\end{cases}
$$

(12)

Moreover, $u$ is obtained by

$$
\min_{v \in V(\Omega)} \left\{ \int_{\Omega} |\nabla v|^2 \, dx \, dy + c \int_{K} |\gamma_0 v|^2 \, d\mu - 2 \int_{\Omega} f v \, dx \, dy - 2d \int_{K} \gamma_0 v \, d\mu \right\}
$$

Proof. The thesis now follows by applying Lax-Milgram Theorem to the bilinear form $a(u, v) = \int_{\Omega} \nabla u \nabla v \, dx \, dy + c \int_{K} \gamma_0 u \gamma_0 v \, d\mu$, where $\mu$ denotes the normalized $d_f$-dimensional Hausdorff measure on $K$ as in (4) and $\gamma_0$ is the trace operator defined in Theorem 3.3.

This form is continuous and coercive in $V(\Omega)$ by using the Poincaré inequality in $\Omega$ and Theorem 3.3. Moreover, the linear functional $F(v) = \int_{\Omega} f v \, dx \, dy + d_n \int_{K} v \, d\mu$ is bounded in $V(\Omega)$. □
We show in which sense the variational solution of the problem (10) solves the mixed Dirichlet-Robin problem introduced formally above.

**Theorem 4.3.** The weak solution $u_n$ of problem (10) solves in the following sense

$$
\begin{align*}
-\Delta u_n &= f & \text{in } L^2(\Omega_n) \\
u_n &= 0 & \text{in } H^\frac{1}{2}(\Gamma_0) \\
\frac{\partial u_n}{\partial \nu} &= 0 & \text{in } (H^\frac{1}{2}(\Gamma_1))' \\
\frac{\partial u_n}{\partial \nu} + c_n u_n &= d_n & \text{in } (H^\frac{1}{2}(\Gamma_2))' \\
\frac{\partial u_n}{\partial \nu} &= 0 & \text{in } (H^\frac{1}{2}(\Gamma_3))'.
\end{align*}
$$

(13)

**Proof.** The proof is obtained by usual duality arguments. Let us choose $v \in \mathcal{D}(\Omega_n)$ in (10); we obtain $-\Delta u_n = f$ in the sense of distribution of $\Omega_n$, and, then, in $L^2(\Omega_n)$ ($f \in L^2(\Omega_n)$). The homogeneous Dirichlet boundary condition on $\Gamma_0$ - in the sense of the traces of functions belonging to $H^1(\Omega)$ - follows from the fact that $u_n \in V(\Omega)$. Moreover, the other boundary conditions follow from the Green’s formula for Lipschitz domains (Theorem 3.2).

In the fractal case, we obtain that the variational solution of problem (12) solves the mixed problem in the following sense.

**Theorem 4.4.** The weak solution $u$ of problem (12) solves in the following sense

$$
\begin{align*}
-\Delta u &= f & \text{in } L^2(\Omega) \\
u &= 0 & \text{in } H^\frac{1}{2}(\Gamma_0) \\
\frac{\partial u}{\partial \nu} &= 0 & \text{in } (H^\frac{1}{2}(\Gamma_1))' \\
\frac{\partial u}{\partial \nu} + cu &= d & \text{in } (H^\frac{1}{2}(\Gamma_2))' \\
\frac{\partial u}{\partial \nu} &= 0 & \text{in } (H^\frac{1}{2}(\Gamma_3))',
\end{align*}
$$

(14)

with $\beta = \frac{d(\Omega)}{2}$.

**Proof.** As in the proof of the previous Proposition, let us choose $v \in \mathcal{D}(\Omega)$ in (12); we obtain $-\Delta u = f$ in the sense of distribution of $\Omega$, and, then, in $L^2(\Omega)$ ($f \in L^2(\Omega)$). The homogeneous Dirichlet boundary condition on $\Gamma_0$ - in the sense of the traces of functions belonging to $H^1(\Omega)$ - is fulfilled since $u \in V(\Omega)$. The boundary conditions on $\Gamma_1$ and $\Gamma_3$ follow from the Green’s formula for Lipschitz domains. Moreover, on the fractal
5. Regularity results

In this section, we prove some regularity results for the weak solution of the problem (10) in terms of ordinary fractional Sobolev spaces and weighted Sobolev spaces. By the general theory of regularity, three elements are involved in determining the regularity of the solution of a variational problem, associated with a boundary value problem on an open set $D \subset \mathbb{R}^n$:

1) the regularity of the coefficients of the operator;
2) the geometry of the open set $D$ and, in particular, the regularity of its boundary;
3) the type of boundary conditions imposed.

We remark that, since the coefficients are constants, the problem (9) is characterized by the irregular boundary and the boundary mixed Dirichlet-Robin conditions. These two factors are crucial in determining some result of global regularity, that is, up to boundary of the open set $D$.

If we consider a boundary mixed Dirichlet-Robin problem on a regular domain, there are no problems when the Dirichlet condition is “disjoined” from the Robin one. For instance, we obtain $H^2$-regularity of the solution of the Dirichlet-Robin problem on $D = B(0,2) \setminus B(0,1)$ with Dirichlet condition on $\partial B(0,1)$ and Robin condition on $\partial B(0,2)$ (see, for instance, Ref. 3).

The situation is completely different when, for example, the part of boundary where the Dirichlet condition holds meets the part of boundary where another type of condition is satisfied. For instance, if we consider $D = B(0,1) \cap \{0 < \theta < \pi\}$, $\Gamma_1 = \partial D \cap \{\theta = \pi\}$ and $\Gamma_0 = \partial D \setminus \Gamma_1$ the function $u = \rho^{\frac{3}{2}} \sin(\frac{\theta}{2})$ solve the following mixed problem: $\Delta u = 0$, $u = g$ on $\Gamma_0$, $\frac{\partial u}{\partial n} = 0$ on $\Gamma_1$, where $g = \sin(\frac{\theta}{2})$ on $\partial D \cap \partial B(0,1)$ and $g = 0$ elsewhere on $\Gamma_0$. Even if the boundary is smooth near the point $(0,0)$, the solution $u$ does not belong to $H^s$ for $s \geq \frac{3}{2}$ (see Ref. 2).

On the other hand, the regularity of the boundary is involved in a decisive manner. For instance, let $D \subset \mathbb{R}^2$ be a bounded polygon: there exists a function $u$ in $H^1_0(D)$ such that $\Delta u \in C^\infty(D)$ but $u$ does not belong to $H^{1+\frac{\omega}{2}}$, where $\omega$ is the amplitude of the largest of the internal angles of the polygon. In particular, if the polygon is not convex, the solution of the Dirichlet problem cannot be in $H^2$ even if the datum $f$ is very regular (see Refs. 6 and 7).

In the following, we give a result of regularity of the solution of the
problem (10) in terms of ordinary fractional Sobolev spaces.

**Theorem 5.1.** Let \( u_n \) be the weak solution of problem (10), then

\[
u_n \in H^s(\Omega^{(l)}_n), \quad s < 1 + \frac{\pi}{\pi + \theta(l)}, \tag{15}\]

where \( \theta(l) = \arcsin \left( \frac{\sqrt{(4-l)}}{2} \right) \), for \( 2 < l < 4 \).

**Proof.** From Proposition 3.1 of Ref. 16, we deduce that the variational solution of an elliptic problem with mixed Dirichlet-Robin conditions on a polygonal domain has the *Regularity-Decomposition property* with the same number of singular functions as in the case of an elliptic problem with mixed Dirichlet-Neumann conditions. Before recalling this property, we introduce some notations. We denote each segment of the polygonal boundary of \( \Omega^{(l)}_n \) by \( \Lambda_j \), \( \Lambda_j \) open, where the index \( j \) ranges from 1 to \( 3 + 4^n \); these segments are numbered in such a way that \( \Lambda_{j+1} \) follows \( \Lambda_j \) according to the positive orientation. We denote by \( S_j \) the vertex of \( \Omega^{(l)}_n \) which is the right endpoint of \( \Lambda_j \) and by \( \omega_j \) the amplitude of the interior angle at \( S_j \). The polar coordinates with origin at \( S_j \) will be denoted by \( r_j \) and \( \theta_j \). For every \( j \), we introduce a truncation function \( \eta_j \in D(\Omega^{(l)}_n) \) which depends only on the distance \( r_j \) to \( S_j \) such that \( \eta_j \equiv 1 \) near \( S_j \) and \( \eta_j \) vanishes near all \( \Lambda_k \) but for \( k = j \) and \( k = j + 1 \). Moreover, the union of the \( \Lambda_j \) with \( j \in D \) (resp. \( j \in R \)) is going to be the part of the boundary where we consider a Dirichlet (resp. Robin) boundary condition.

The *Regularity-Decomposition property* explains the behaviour of the variational solution of these type of problems by showing that the solution can be written as a sum of a regular function that belongs to \( H^2 \) and a finite number of singular functions (not belonging to \( H^2 \)) (see Theorem 2.4.3 in Ref. 7). More precisely, if \( u_n \) is the weak solution of problem (10), there exist unique numbers \( c_{j,m} \) \((m \in N, m \geq 1)\), such that,

\[
u_n - \sum_{1 \leq j \leq 3 + 4^n} \left\{ \sum_{0 < \lambda_j,m < 1} c_{j,m} S_{j,m} \right\} \in H^2(\Omega^{(l)}_n), \tag{16}\]

where

\[ S_{j,m}(r_j, \theta_j) = \eta_j(r_j) r_j^{\lambda_j,m} \varphi_j,m(\theta_j), \]

with

- for \( j \in D \) and \( j + 1 \in D \), \( \varphi_{j,m}(\theta) = \sqrt{\frac{\pi}{\omega_j}} \sin(\theta \lambda_j,m), \lambda_j,m = \frac{m \pi}{\omega_j}, \)

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- for \( j \in \mathbb{R} \) and \( j + 1 \in D \), \( \varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}} \sin(\theta \lambda_{j,m}), \lambda_{j,m} = \frac{(m-\frac{1}{2})\pi}{\omega_j} \),
- for \( j \in D \) and \( j + 1 \in \mathbb{R} \), \( \varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}} \sin((\omega_j - \theta) \lambda_{j,m}), \lambda_{j,m} = \frac{(m-\frac{1}{2})\pi}{\omega_j} \),
- for \( j \in \mathbb{R} \) and \( j + 1 \in \mathbb{R} \), \( \varphi_{j,m}(\theta) = \sqrt{\frac{2}{\omega_j}} \cos(\theta \lambda_{j,m}), \lambda_{j,m} = \frac{(m-1)\pi}{\omega_j} \) when \( m \geq 2 \), \( \varphi_{j,1}(\theta) = \sqrt{\frac{1}{\omega_j}}, \lambda_{j,1} = 0 \) when \( m = 1 \).

From (16), we obtain a precise description of the behavior of the solution near the various corners. In fact, let \( V_j \) be an open neighborhood of \( S_j \) which does not contain any other corner. Then \( u_n \in H^s(V_j) \) for every \( s \leq 2 \) such that

\[
s < 1 + \inf_m \{ \lambda_{j,m}, 0 < \lambda_{j,m} < 1 \}
\]

(see Corollary 2.4.4 in Ref. 7). Globally, the previous estimate gives that \( u_n \in H^s(\Omega_n^{(l)}) \) for every \( s \leq 2 \) such that

\[
s < 1 + \inf_{j,m} \{ \lambda_{j,m}, 0 < \lambda_{j,m} < 1 \}
\]

(see Remark 2.4.6 in Ref. 7). By considering the amplitude of the angles in \( \Omega_n^{(l)} \) and the boundary conditions of problem (9), we obtain estimate (15).

For numerical purposes, the regularity of \( u_n \) in terms of weighted Sobolev spaces (for the definition, see Ref. 6), is more useful than the one in terms of ordinary fractional Sobolev spaces. We note that the regularity of the weak solution \( u_n \) in terms of weighted Sobolev spaces follows from estimate (15) (see Refs. 6, 7 and 9).

**Theorem 5.2.** Let \( u_n \) be the weak solution of the problem (10). Then

\[
\begin{align*}
    u_n & \in H^{2,\alpha}(\Omega_n^{(l)}), & & \alpha > \frac{\theta(l)}{\pi + \theta(l)}, \quad (17) \\
\end{align*}
\]

where \( \theta(l) = \text{arcsin} \left( \sqrt{\frac{l(l-2)}{4}} \right) \), for \( 2 < l < 4 \).

In particular, we remark that when \( K^{(l)} \) is the usual Koch curve we obtain that \( u_n \in H^s(\Omega_n^{(l)}) \), with \( s < \frac{7}{4} \) and \( u_n \in H^{2,\alpha}(\Omega_n^{(l)}) \), with \( \alpha > 1/4 \).

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References