An Inverse Problem for Two-Frequency Photon Transport in a Slab

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The subject of this communication is an inverse photon transport problem, motivated by astrophysics, which consists in obtaining the unknown densities of two different kinds of materials present in a dusty medium (say, and interstellar cloud) by flux measurements of photons with two different frequencies $\nu_1 > \nu_2$ (say, UV and IR). Clearly, the description of photon transport in an interstellar cloud requires a three-dimensional transport equation in a rather complicated geometry. Here, for the sake of simplicity, we shall set the problem in a space-homogeneous, slab geometry.

The mathematical model consists into a system of two stationary transport equations for the phase-space densities $f_1(x, \mu)$ and $f_2(x, \mu)$ of of photons with frequencies $\nu_1$ and $\nu_2$, respectively. Here, $x \in [0, l]$ is the position variable ($l$ being the thickness of the slab) and $\mu \in (-1, 1)$ is the direction cosine. The stationary transport equations are assumed to have the following form,

$$
\begin{align*}
\mu & \frac{\partial f_1}{\partial x}(x, \mu) + \Sigma_1 f_1(x, \mu) = \Sigma_{1-1} \int_{-1}^{1} p_{1-1}(\mu' \rightarrow \mu) f_1(x, \mu') d\mu' \\
\mu & \frac{\partial f_2}{\partial x}(x, \mu) + \Sigma_2 f_2(x, \mu) = \Sigma_{1-2} \int_{-1}^{1} p_{1-2}(\mu' \rightarrow \mu) f_1(x, \mu') d\mu' + \Sigma_{2-2} \int_{-1}^{1} p_{2-2}(\mu' \rightarrow \mu) f_2(x, \mu') d\mu',
\end{align*}
$$

(0.1)

where, $\Sigma_{1-1} \geq 0$, $\Sigma_{1-2} \geq 0$, $\Sigma_{2-2} \geq 0$ are the scattering cross-sections, $p_{1-1} \geq 0$, $p_{1-2} \geq 0$, $p_{2-2} \geq 0$ are the scattering probability densities and

$$
\Sigma_1 := \Sigma_{1-1} + \Sigma_{1-2} + \Sigma_{1,c}, \quad \Sigma_2 := \Sigma_{2-2} + \Sigma_{2,c},
$$

are the total cross-sections ($\Sigma_{1,c} \geq 0$ and $\Sigma_{2,c} \geq 0$ are the capture cross-sections). Since we assume the medium to be space-homogeneous, then all the cross-sections and scattering probabilities are independent of $x$. Moreover, since $\nu_1 > \nu_2$, the energy-increasing scattering $2 \rightarrow 1$ is not considered for obvious physical reasons.

Assuming the scattering to be number-conservative, we have

$$
\int_{-1}^{1} p_{1-1}(\mu' \rightarrow \mu) d\mu = 1
$$
for all $\mu' \in [-1,1]$ and for all $i, j = 1,2$ excluding $i = 1, j = 2$, because $p_{2\rightarrow 1} \equiv 0$. Moreover, we assume that the scattering is symmetric, i.e.
\[ p(\mu' \rightarrow \mu) = p(\mu \rightarrow \mu'). \]

Let us assume that the medium is homogeneous and composed by two kinds of dust, with different physical properties. Then, for $i, j = 1,2$ we put
\[ \Sigma_{i\rightarrow j} = \rho_1 \sigma_{i\rightarrow j}^1 + \rho_2 \sigma_{i\rightarrow j}^2, \quad \Sigma_{i,c} = \rho_1 \sigma_{i,c}^1 + \rho_2 \sigma_{i,c}^2, \]
where $\rho_1 \geq 0$ and $\rho_2 \geq 0$ are the (constant) densities of the two dusts and the $\sigma$’s are microscopic cross-sections. If the scattering properties are the same for the two kinds of dust, then the probabilities $p$’s do not depend on the dust index.

The model is completed by the following boundary conditions of assigned inflow:
\begin{equation}
\begin{aligned}
    f_1(0, \mu) &= \varphi_1^+(\mu), & f_2(0, \mu) &= \varphi_2^+(\mu), & \text{for } \mu \in (0, 1) \\
    f_1(1, \mu) &= \varphi_1^-(\mu), & f_2(1, \mu) &= \varphi_2^-(\mu), & \text{for } \mu \in (-1, 0),
\end{aligned}
\end{equation}
where $\varphi_1^+(\mu)$ and $\varphi_2^+(\mu)$ are known incoming photon distributions at both sides of the slab at the two frequencies.

The inverse problem consists in finding the unknown dust densities, $\rho_1$ and $\rho_2$, from the knowledge of the integrated right-outflows:
\[ H_1 := \int_0^1 f_1(l, \mu) \mu \, d\mu, \quad H_2 := \int_0^1 f_2(l, \mu) \mu \, d\mu. \]

Under fairly general conditions we can prove the well-posedness of the direct problem \((0.1)+(0.2)\).

Moreover, in the following assumptions:

**A1.** the frequency-scattering vanishes, i.e. $\Sigma_{1\rightarrow 2} \equiv 0$;

**A2.** the left-inflow data $\varphi_1^+$ and $\varphi_2^+$ are positive on nonzero-measure sets;

**A3.** $\sigma_{i,c}^j > 0$ for $i, j \in \{1, 2\}$ and $\det(\sigma_{i,c}^j) \neq 0$;

we can prove that the mapping densities-to-outflows: $(\rho_1, \rho_2) \mapsto (H_1, H_2)$ is globally invertible and, therefore, that the inverse problem is well-posed.

Numerical experiments show that the two densities $(\rho_1, \rho_2)$ can be computed from assigned outflows $(H_1, H_2)$, by means of a simple bisection-like algorithm, over a range of several orders of magnitude.

**REFERENCES**


