

## INTRODUCTION TO LEVEL SET METHODS

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### Abstract.

In this talk we present a brief introduction to level set methods. The basic mathematical formulation will be recalled, as well as the most common numerical methods used for the discretization of the equation. Some application to the computation of signed distance function, motion by mean curvature, and crystal growth will be illustrated.

*Keywords:* Level-set methods; Crystal growth; Motion by mean curvature; Implicit schemes

### 1. Level set approach

The level set approach is a powerful tool to represent the evolution of surfaces (lines in 2D), once the normal velocity of the surface at each point is known or can be expressed as a function of the geometrical properties of the surface itself.

The most common representations of an evolving surface are the parametric and the implicit ones. Let us consider 1D first. A parametric curve  $\Sigma(t)$  is identified by the parametric equations of its points  $P \equiv (x, y)$  at each time  $t$ :  $x = X(\xi, t)$ ,  $y = Y(\xi, t)$ . The motion of the curve is determined once the velocity  $\vec{v}$  each point  $P$  is assigned. The velocity  $\vec{v}$  may be a function of the position of point  $P$ , or it may depend on other geometric components, such as orientation of curvature.

A Lagrangian description of the curve is obtained by assigning a parametric representation of the initial curve,  $\Sigma_0 = \{P \in \mathbb{R}^2 | P = P_0(\xi), \xi \in [0, 1]\}$ , and by following each point according to its velocity:

$$\frac{dP}{dt} = \vec{v}(P, \vec{n}, \kappa), \quad P(t=0) = P_0(\xi),$$

where we indicated the dependence of  $\vec{v}$  on the unit normal  $\vec{n}$  and the local curvature  $\kappa$  at each point, and, without loss of generality, we restricted the interval of variation of the parameter on the curve in  $[0, 1]$ .

The velocity of a Lagrangian point  $P$  may be decomposed in its normal and tangential component,  $\vec{v} = v_n \vec{n} + v_\tau \vec{\tau}$ . The tangential component does not influence the evolution of the curve; it has the only effect of changing its parametrization. We shall assume that the curve is a closed curve or a nonintersecting finite set of closed curves, for each of which

we can identify an inner region. We denote by  $\vec{n}$  the unit normal pointing from the inner to the outer region of each curve. For a smooth curve, such a normal will not jump at inflection points, where the curvature changes sign (see Figure 1)

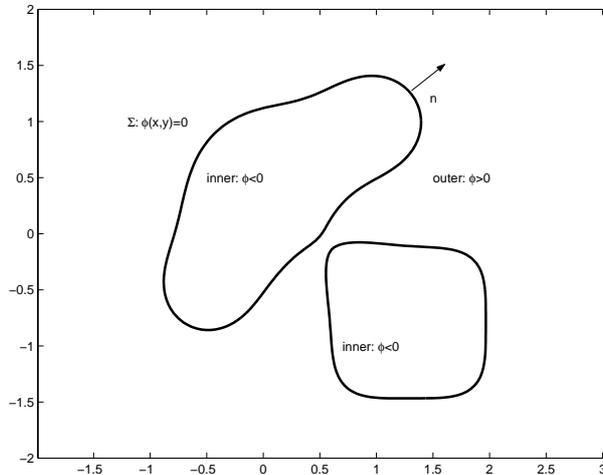


Fig. 1. Surfaces identified by a level set function

Lagrangian schemes are obtained by tracing the position of a finite number  $N$  of selected points  $\{P_i, i = 1, \dots, N\}$  on the interface. At each time, the points can be used to compute the geometric quantities necessary to evaluate the velocity  $\vec{v}$  at each point, and the points are then advanced by solving the set of ordinary differential equations

$$\frac{dP_i}{dt} = \vec{v}_i(P_1, \dots, P_N).$$

One drawback of such an approach is that, as a consequence of the spreading in the velocity field, the points may become too sparse in some regions, and too dense on other regions. This problem can be solved in part by the addition of a suitable transversal velocity that has the effect of maintaining the (line) density of points as desired (e.g. uniform, or higher near points of higher curvature).

Lagrangian schemes have the advantage of allowing a great accuracy in the location of the interface, and in treatment of very fine details. They are commonly used for lines in 2D. Their use is much more involved in the treatment of surfaces. The main drawback of this approach is the difficulty in treating changes in the topology of the surface. For example, a naive Lagrangian scheme for the evolution by constant normal velocity will cause the appearance of the so called “swallow tails” (see Figure 2).

An alternative representation of a surface is obtained by the implicit form:

$$(1) \quad \Sigma(t) = \{P \in \mathbb{R}^d : \phi(P, t) = 0\},$$

where  $d = 2$  or  $3$  is the dimension of the space. Following the location of a material point  $P(t)$  which lies on the surface, one has:  $\phi(P(t), t) \equiv 0$ , therefore, by differentiating with respect to time, one has:

$$\frac{\partial \phi}{\partial t} + \dot{P} \cdot \nabla \phi = 0.$$

Denoting by  $v_n$  the normal velocity of point  $P$ , this relation becomes

$$(2) \quad \frac{\partial \phi}{\partial t} + v_n |\nabla \phi| = 0,$$

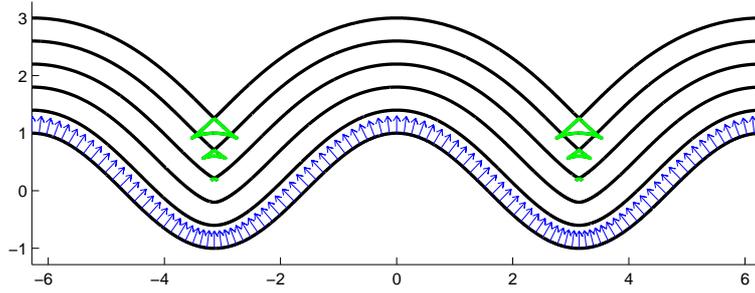


Fig. 2. Motion by constant normal velocity of an initial sine function. Naive Lagrangian method will show the appearance of “swallow tails” in the profile (light part of the lines). Viscosity solution of level set equation will provide the single valued black lines

because the normal  $\vec{n}$  is aligned with the gradient of  $\phi$ :  $\vec{n} = \nabla\phi/|\nabla\phi|$ . This equation is valid for all points of  $\Sigma$ , and it may be used as a tool to describe the evolution of the surface. This is obtained as follows. Suppose a surface  $\Sigma$  is contained in a domain  $\Omega \subset \mathbb{R}^d$ . The normal velocity  $v_n$  is defined only for points on  $\Sigma$ . Let us denote by  $V_n$  an extension of the velocity  $v_n$  on the domain  $\Omega$ , that is  $V_n$  is defined in  $\Omega$ , and  $V_n(P) = v_n(P)$ ,  $\forall P \in \Sigma$ . The evolution of the function  $\phi$  in  $\Omega$  depends on the choice of the extension, but the set  $\phi(P, t) = 0$  will evolve in the same way for any choice of the extension (under mild regularity assumptions).

From the knowledge of the level set function  $\phi$ , all geometric quantities that are needed to compute the normal velocity can be evaluated. For example, the motion by constant normal velocity is obtained by solving the Hamilton-Jacobi equation

$$(3) \quad \frac{\partial\phi}{\partial t} + c|\nabla\phi| = 0,$$

where  $c$  is constant.

In the motion by mean curvature, the normal velocity is given by  $v_n = c\kappa$ , where  $\kappa$  is the mean curvature of the surface (line). The curvature is related to the eigenvalues of the curvature tensor  $\chi_{i,j} = \tilde{\partial}_i n_j$ , where  $\tilde{\partial}_i = \sum_j (\delta_{i,j} - n_i n_j) \partial_j$  is the gradient operator on the surface. Since the curvature tensor is orthogonal to the surface, i.e.  $\sum_j \chi_{i,j} n_j = 0$ , one eigenvalue is always zero. in 2D,  $\kappa$  is the other eigenvalue of the curvature matrix  $\chi$ , while in three dimensions  $\kappa = \kappa_1 + \kappa_2$  is the sum of the two Gauss principal curvatures of the surface, which are eigenvalues of  $\chi$ . In any case,  $\kappa$  is the sum of the eigenvalues of  $\chi$ , and therefore it can be computed as the trace of matrix  $\chi$ , which, in turn, can be computed as

$$\kappa = \nabla \cdot \frac{\nabla\phi}{|\nabla\phi|}$$

since  $\nabla\phi/|\nabla\phi|$  is an extension of  $\vec{n}$  in  $\Omega$ .

## 2. Numerical methods

One of the main advantages of level-set methods is that complex geometries can be easily handled by solving the equation for  $\phi$  on a rectangular grid. The interface is visualized as a contour plot of the function  $\phi$  on the grid.

Numerical solution of Eq.(2) can be obtained by a suitable discretization of the equation in space and time. Particular care is needed in the discretization, to avoid formation of

spurious oscillations. For example, in the case of motion by constant normal velocity (or, in general, when the velocity  $v_n$  does not depend on higher order derivatives of  $\phi$ ), the solution of the HJ equation may develop singularities in finite time, even for smooth initial data. In 1D this is easily understood by the analogy with systems of conservation laws: consider the generic (possibly regularized) HJ equation in 1D:

$$(1) \quad \frac{\partial \phi}{\partial t} + f(\partial_x \phi) = \epsilon \partial_x^2 \phi.$$

Differentiating this equation, and denoting by  $u = \partial_x \phi$ , one has:

$$(2) \quad \frac{\partial u}{\partial t} + f(u)_x = \epsilon \partial_x^2 u.$$

If  $\epsilon = 0$ , solutions to Eq.(2) develop jump discontinuities in finite time (shocks), and therefore solutions to Eq.(1) develop discontinuities in the first derivative in finite time. Just as in the case of quasilinear conservation laws, existence and uniqueness theory for the Hamilton-Jacobi equation is based on the concept of *viscosity solution*: the equation is regularized by the addition of a diffusive term on the right hand side; and the solution of the initial value problem for Eq.(1) exists and is unique under quite general assumption on the function  $f$  and on the initial data. This analogy shows that HJ equation of the form (3) is hyperbolic in nature (as is the case of first order PDE's in space and time), with disturbances propagating at finite speed.

There is a vast literature on numerical schemes for conservation laws and for the Hamilton-Jacobi equation. The simplest schemes are the so called first order upwind methods, and are based on the use of a space discretization which takes into account the direction of propagation of disturbances. For example, in the case of Eq.(3) in 1D, first order upwind is obtained by

$$(3) \quad \phi_i^{n+1} = \phi_i^n - c \frac{\Delta t}{\Delta x} \max(|(\phi_i^n - \phi_{i+1}^n)|_+, |(\phi_{i+1}^n - \phi_i^n)|_-),$$

where, for any quantity  $h$ , we denote  $(h)_+ \equiv \max(h, 0)$ , and  $(h)_- \equiv \min(h, 0)$ . This can be easily explained by observing that Eq. (3) in 1D can be written as

$$\begin{aligned} \phi_t + c\phi_x &= 0 & \text{if } \phi_x \geq 0 \\ \phi_t - c\phi_x &= 0 & \text{if } \phi_x < 0 \end{aligned}$$

First order upwind, in the case  $\phi_x \geq 0$ , becomes

$$\phi_i^{n+1} = \phi_i^n - c \frac{\Delta t}{\Delta x} (\phi_i^n - \phi_{i-1}^n),$$

while in the case  $\phi_x < 0$  it becomes

$$\phi_i^{n+1} = \phi_i^n - c \frac{\Delta t}{\Delta x} |\phi_{i+1}^n - \phi_i^n|.$$

Both cases can be summarized in the form (3). Although in one dimension such expression may look unnecessarily complicated, its generalization in several dimensions is almost straightforward, and is performed dimension by dimension.

Several techniques are available in the literature for the construction of high order methods for the evolution of the level set function, see for example the book<sup>2</sup> for a

general discussion or the paper<sup>6</sup> for a second order method for the computation of signed distance function.

In the case of the motion by mean curvature, the normal velocity is given by  $v_n = c\kappa$ . In two dimensions, an initial closed line  $\Sigma$  containing a simply connected domain will modify its shape, until it becomes very similar to a circle, and then it will shrink to a point in finite time. Likewise, in three dimensions, an initial closed surface  $\Sigma$  will become very close to a sphere, and then it will shrink to a point in finite time (see, for example,<sup>1</sup>

The expression of the curvature  $\kappa$  contains second space derivatives, therefore, the nature of the equation changes, and becomes similar to nonlinear diffusion equation. This poses fewer problems at the level of space discretization, since upwinding is no longer necessary, however explicit schemes suffer the restriction on the time step  $\Delta t \leq c^* \Delta x^2$  which is typical of diffusion equation. The use of implicit schemes such as Crank-Nicolson, is not so straightforward in this case, since the equation is strongly nonlinear, and an implicit scheme would require solution of a nonlinear system at each time step. A semi-implicit scheme for overcoming the stiffness still avoiding the solution of a nonlinear system has been recently proposed by Smereka.<sup>7</sup> The method is based on the observation that

$$\kappa|\nabla\phi| = \nabla^2\phi - \vec{n} \cdot \nabla|\nabla\phi|.$$

The first term on the right hand side is linear, and can be treated implicitly with low computational cost, while the nonlinear term can be treated explicitly. Numerical experiments show that the overall scheme is unconditionally stable, although a complete stability analysis of such scheme has not yet been performed. A high order extension of Smereka's method, which makes use of Implicit-Explicit Runge-Kutta time integrators<sup>3</sup> is presented. The results have been obtained in collaboration of Dr. Giuseppe Riccobene.<sup>4</sup>

Alternative approaches, which combines level-set ideas and Lagrangian discretization, exists. They provide very good accuracy, and stability (see for example<sup>9</sup>).

### 3. Applications

Level set methods have been applied in a large number of contexts, ranging from etching in material science, to multi-fluid flow. An account of many applications can be found in recent textbooks on level set methods, such as<sup>1</sup> or.<sup>2</sup>

One interesting application of level set methods, which can be used as a tool by itself or as an intermediate step of long time calculations, is the computation of the signed distance function. After a long time, it is possible that the level set function evolved with a given extension of the normal velocity becomes distorted, and this may in turn affect the accuracy of the location of the interface. It is therefore useful to replace a given level set function, that we shall denote  $\phi^0(P)$ , by the so called *signed distance function*, which has the property that  $\text{sign}(\phi) = \text{sign}(\phi^0)$ , and  $|\nabla\phi(P)| = 1 \forall P \in \Omega$ .

One way of computing the signed distance function, i.e. of imposing that the function  $\phi$  has the same zero level set of  $\phi^0$ , and that  $|\nabla\phi| = 1$ , is obtained by solving the evolutionary problem<sup>10</sup>

$$\frac{\partial\phi}{\partial t} = -\text{sign}(\phi^0)(|\nabla\phi| - 1)$$

using a suitable space and time discretization. The original formulation, however, had the drawback of not maintaining the original zero level set, because in the neighborhood of

$\Sigma$  the discretization was not truly upwinding. An improved over the original technique, which overcomes this drawback, is described in the paper.<sup>6</sup>

Another interesting application of level set methods is the study of crystal growth. A level set method has been developed for the simulation of crystal growth and formation of polycrystal.<sup>5</sup> The method is based on the use of several level set functions, one for each growing seed of a crystal. The normal speed of propagation depends on the orientation. Efficient use of computer time and memory is obtained by using the so called *narrow band level set*, in which the level set function is defined only in a thin region around the interface. This region is in turn evolved in time, making use of the reconstruction of the signed distance function.

The method is then used in<sup>8</sup> to study the surface morphology of diamond growth for various values of the ratio between the growth speed along (111) and (001) orientation of the crystal. The results of the simulation appear in good agreement with experimental results.

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