Evaluation of Insurance Products with Guarantee in Incomplete Markets

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Abstract

The pricing of implicit options embedded in life insurance products is of vital importance for the insurance industry. Risk management, strategic asset allocation, and product design depend on the correct evaluation of the options sold. Also regulators are interested in such issues since they have to be aware of the possible scenarios that the overall industry will face with. Our objective is to use a stochastic programming approach to determine the fair price of the options embedded in policy contracts. We provide extensive numerical experiment to describe the behavior of the insurance policies for different setting of the key parameters, market and regulatory constraints.

1 Insurance products with guarantee

We focus on insurance contracts whose benefits are linked to a reference fund. The value of the reference fund at time t is I_t. The insurer funds his initial investment in the reference fund by raising capitals from both policyholders, which pay $L_0 = \alpha I_0$ as the *premium* for the contracts, and equityholders, which invest the amount $E_0 = (1 - \alpha) I_0$ in the company's asset.

Policies are equipped with a minimum guarantee provision, which entitles policyholders to increment over time their premiums according to a minimum guaranteed interest rate r_G, so that at maturity the guaranteed amount is $L_T = L_0 e^{r_G T}$. Besides the minimum guarantee provision, policyholders are allowed to participate in the firm's profit by a bonus distribution mechanism. Thus, the final payoff of the policyholder is,

$$(1.1) \qquad \Phi(I_T) = \underbrace{\delta \left[\alpha \, I_T - L_T^G \right]^+}_{\text{Bonus option}} + \underbrace{L_T^G - \left[L_T^G - I_T \right]^+}_{\text{Defaultable bond payoff}}$$

The coefficient δ measures the percentage of firm's profit which is corresponded to policyholders, while the payoff of the Put option is embedded to take into account the default possibility of the insurer. We also extended our analysis to comprise regulatory restrictions. These are made explicit by imposing a barrier that forces the option to expire if I_t touches a barrier,

$$(1.2) \qquad I_t \leq \lambda L_0 e^{r_G \, t} \equiv B_t \qquad t \in [0, T[.$$

As explained in Grosen at al. (2002) this is equivalent to the monitoring, on the regulators side, of the insurance assets value. Only in case that I_t is above the barrier, the option will be allowed to expire at maturity.

2 Super-replication via stochastic programming

We describe in this section a stochastic programming model to price European contingent claims. Details about the model and its use for option pricing can be found in King (2002) and King at al. (2005).

We will base our computational machinery on a discrete non-recombining scenario tree. Following King's notation, in the scenario tree, every node $n \in \mathcal{N}_t$, $t = 1, \ldots, T$, has a unique ancestor node $a(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, \ldots, T-1$, has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_t$. The collection of all the nodes is denoted by $\mathcal{N} \equiv \bigcup_{t=0}^{T} \mathcal{N}_t$.

The market consists of $J + 1$ securities with prices $S_t = \left(S_t^0, \ldots, S_t^J\right)$ (J risky assets plus a numeraire asset).

In presence of risk factors others than the traded securities, the process S_t is augmented by K real–valued variables $\xi_t = \left(\xi_t^1, \ldots, \xi_t^K\right)$ whose path histories match the nodes $n \in \mathcal{N}_t$, for each $t = 0, 1, 2, \ldots, T$.

We denote by $\theta_n = \left(\theta_n^0, \ldots, \theta_n^J\right)$ the portfolio of securities held by an investor in state $n \in \mathcal{N}$. The value of the portfolio in state $n \in \mathcal{N}$ is,

$$(2.1) \qquad S_n \cdot \theta_n \equiv \sum_{j=0}^{J} S_n^j \theta_n^j.$$

We say that the portfolio process $\{\theta_n\}_{n \in \mathcal{N}}$ super-replicates the cashflow generated by the contingent claim, if the funds available for investment in each state $n \in \mathcal{N}$ are restricted to those yielded by price changes in the portfolio held at state $a(n)$ (self-financing portfolios), and its value is non-negative. The writer's price of the contingent claim is the smallest amount of current cash, V, needed to start a trading strategy to back the payout process $\{F_n\}_{n \in \mathcal{N}}$ with no risk. The option price is then given by the following stochastic optimization problem,

$$(2.2) \qquad \underset{V, \theta}{\text{Minimize}} \ V$$

$$(2.3) \qquad \text{s.t.} \qquad S_0 \cdot \theta_0 = V - F_0$$

$$(2.4) \qquad (W) \qquad S_n \cdot (\theta_n - \theta_{a(n)}) = -F_n \qquad n \in \mathcal{N}_t, t \geq 1$$

$$(2.5) \qquad S_n \cdot \theta_n \geq 0, \qquad n \in \mathcal{N}_T.$$

This problem can easily be extended to handle the major sources of market incompleteness, such as trading constraints, transaction costs and difference between borrowing and lending. Note that also non-tradeability of the underlying can be considered by assuming that the policy's reference fund in non-tradeable.

3 Implementation notes and results

We ran our experiments assuming a policy horizon $T = 10$ years. The time interval between two periods is set to 1.67 years, for a total of 6 time periods. To run our

experiments in a reasonable time, we assumed that our market is made up by 4 assets plus one risk free ($J = 4 + 1$). Note that, the computational time depends on the number of assets, the discretization adopted and the operational constraints. In the worst case—the problem with transaction costs, that basically doubled the number of variables—the computational time amount to an half an hour (CPU Pentium 4, 2.4 GhZ).

To encompass the more general case, that is when the underlying asset is not tradable, the reference fund, I, is not included among the J assets, therefore, the hedging portfolio is formed by the four risky asset plus the risk free.

We solved the optimization models using the algebraic modelling language GAMS of Brooke at al. (1992).

The price of the contract depends on many parameters. The most important ones are: the minimum guarantee rate (r_G), the participation coefficient (δ), the leverage (α), the barrier buffer parameter (λ). Not all the combinations of these parameters determine a fair value of the insurance contract. In particular, let us denote by $V(0, I_0, r_G, \delta, \alpha, \lambda)$ the value at time 0 of the insurance contract. The latter is said to be *fair* if the initial policyholder's contribution, L_0, is equal to the initial market value of the purchased claim, that is,

$$(3.1) \qquad L_0 \equiv \alpha I_0 = V(0, I_0, r_G, \delta, \alpha, \lambda)$$

We adopt a matching moment method to generate scenario trees. For each asset we fit expected value, variance and covariances of log-returns. The method may also fit skewness, kurtosis and higher-order statistical properties. We include martingale (Klassen, 2002) constraints to avoid arbitrage. The generation is executed by the recursive resolution, from root node to final nodes, of highly non-linear and non-convex optimization problems, solved with GAMS-CPLEX. See Appendix for a description of our results.

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APPENDIX

Figure 3.1: The relationship between leverage and participation rate. The minimum guarantee rate is fixed at 2% per year and the volatility of the reference fund is 10% per year. In a complete market our approach perfectly replicates Montecarlo simulation.

Figure 3.2: Hedging portfolios for different levels of the leverage. Higher values of the leverage imply higher values of the participation rate. In these cases, the bonus provision will prevail on the minimum guarantee, and the hedging portfolios will be more shifted towards those assets with statistical properties more similar to the underlying asset.

Figure 3.3: The relationship between leverage and participation rate when transaction costs and borrowing constraints are introduced. The shaded area represents the feasibility region where both leverage and and participation rate are effective.

Figure 3.4: The effect of market incompleteness on regulatory constraints. The insurance contract evaluated taking into account of the transaction costs lies well below the feasibility area. A possible solution would be to lower the minimum guarantee—from 2% to 1%— so that the curve moves upwards.