Communications to SIMAI Congress,
ISSN 1827-9015, Vol. 1 (2006)

DOI: 10.1685/CSC06025

On the Solution of Indefinite Systems Arising in Nonlinear Optimization [1]

Silvia Bonettini
Dipartimento di Matematica, Università di Modena e Reggio Emilia
via Campi 213/b, 41100 Modena
bonettini.silvia@unimo.it

Valeria Ruggiero
Dipartimento di Matematica, Università di Ferrara
via Saragat 1, Blocco B, 44100 Ferrara
rgv@unife.it

Federica Tinti
Dipartimento di Matematica, Università di Modena e Reggio Emilia
via Campi 213/b, 41100 Modena
tinti.federica@unimo.it

A variety of algorithms for linearly and nonlinearly constrained optimization ([3], [4], [5], [7], [8], [9]) requires, at each step, the solution of an indefinite linear system whose matrix has the following form:

$$\begin{pmatrix} W & A^t \\ A & 0 \end{pmatrix} \tag{0.1}$$

where W is an $\bar{n} \times \bar{n}$ symmetric matrix, while the matrix A is $\bar{m} \times \bar{n}$. The matrix (0.1) represents also the matrix of the Karush–Kuhn–Tucker system for the quadratic programming problem

$$\begin{array}{cc} \min & \frac{1}{2}x^tWx - a^tx \\ Ax = b. & \end{array}$$

It is well known that a sufficient condition for the nonsingularity of (0.1) is that A is full row rank and W is positive definite on the null space of A, which means $p^tWp > 0$ for any $p \in \mathbb{R}^n$ $(p \neq 0)$ such that $Ap = 0$. If the matrix (0.1) is derived from a nonlinear programming problem, it could have that W and A assume the following form

$$W = \begin{pmatrix} Q & C^t \\ C & -F^{-1} \end{pmatrix} \text{ and } A = (B \quad 0) \tag{0.2}$$

where $Q \in \mathbb{R}^{n\times n}$ is the Hessian of the Lagrangian function with respect to the primal variables (or an approximation to it), $C \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{neq\times n}$ are the Jacobian matrices of the inequality and equality constraints respectively, and $F \in \mathbb{R}^{m\times m}$ is a diagonal matrix, with positive diagonal entries, derived from the complementarity conditions. In particular

$$F = S^{-1}\Lambda \tag{0.3}$$

[1] This research was supported by the Italian Ministry for Education, University and Research (MIUR), FIRB Project RBAU01JYPN.

where S and Λ are diagonal matrices with, respectively, the slack variables and the multipliers related to the inequality constraints of the programming problem as diagonal entries.
By applying elimination techniques, the system (0.1)-(0.2) can be reduced to a *condensed form*, where the coefficients matrix is as follows

$$\begin{pmatrix} \bar{Q} & B^t \\ B & 0 \end{pmatrix} \tag{0.4}$$

with $\bar{Q} = Q + C^t F C$.
Following the approach in [4], [8], [9], instead of (0.2) we could also have to solve a system whose coefficient matrix is (0.1) with

$$W = \begin{pmatrix} Q & 0 \\ 0 & F \end{pmatrix} \text{ and } A = \begin{pmatrix} B & 0 \\ C & I_m \end{pmatrix} \tag{0.5}$$

where I_m is the $m \times m$ identity matrix.
Another variant of the same system arises when the equality constraints are treated as range constraints with lower and upper bounds that coincide: in this case the matrix (0.1) could assume the form (0.2) or (0.5) with $neq = 0$.
The first approach for the solution of a system whose matrix is (0.1), which is symmetric but indefinite, consists in the direct factorization. In literature many efforts have been made to propose strategies that allow a Cholesky like factorization LDL^t of (0.1) and many authors (see [7], [9]) proposed regularization techniques which perform a perturbation of the diagonal entries by adding small quantities such that D has n or $n+m$ positive diagonal elements (in the case (0.2) or (0.5) respectively) and $m + neq$ negative diagonal elements .
The other approach is to solve the system by means of the preconditioned conjugate gradient method, as suggested in [3], [5], [6].
Our goal is to compare the effectiveness of the different approaches, in the optimization context, for the solution of the inner linear systems arising at each step of an interior point method: in particular, for the conjugate gradient method, we choose a preconditioner with the same structure of (0.2), (0.4) or (0.5) obtained by approximating the matrix Q (or $\bar{Q}$ in the case (0.4)) with a diagonal $n \times n$ matrix with positive elements derived from the diagonal of Q (or of $\bar{Q}$).
This preconditioner admits a Cholesky like factorization and, in order to obtain a sparsity preserving factorization, it is more convenient to factorize a symmetric permutation of it by means of a suitable routine that provides also a dynamic regularization [2]. We consider also the approach followed in [6], where the matrix of the system has the following form:

$$W = \begin{pmatrix} \bar{\bar{Q}} & C_a^t \\ C_a & -F_a^{-1} \end{pmatrix} \text{ and } A = \begin{pmatrix} B & 0 \end{pmatrix} \tag{0.6}$$

where $\bar{\bar{Q}} = Q + C_I F_I C_I$ and C_a and C_I indicate the jacobian matrices of the active and inactive inequality constraints respectively. Furthermore F_a and F_I have the same meaning that in (0.3), but the slacks and the multipliers are only the ones related to the active and inactive constraints respectively.

Our numerical tests have been made on systems derived from a collection of nonlinear programming problems with both equality and inequality constraints.
From the numerical results we can drawn the following remarks: if we have inequality constraints, the solution of the system in condensed form is not efficient, since the approximation of $\bar{Q}$ by a diagonal matrix in the preconditioner is too inaccurate; the more valid approaches seems to be (0.2) and (0.5) while the method (0.6) provides that the structure of the matrix changes every time that the set of the active constraints changes. Furthermore, since at the first steps of the interior point algorithm almost all the constraints are inactive, the strategy (0.6) has an initial behaviour similar to (0.4) and this may cause the failure of the algorithm.

REFERENCES

1. S.Bonettini, E.Galligani and V. Ruggiero, Inner solvers for interior point methods for large scale nonlinear programming, Technical report n. 64, Dipartimento di Matematica Pura ed Applicata, Università di Modena e Reggio Emilia, 2005. To appear on *Computational Optimization and Applizations*.

2. S.Bonettini and V. Ruggiero, Some iterative methods for the solution of a reduced symmetric indefinite KKT system, Technical report n. 60, Dipartimento di Matematica Pura ed Applicata, Università di Modena e Reggio Emilia, 2005. To appear on *Computational Optimization and Applizations*.

3. L. Bergamaschi, J. Gondzio and G. Zilli, Preconditioning indefinite systems in interior point methods for optimization, *Computational Optimization and Applications*, vol. 28, 2004, 149–171.

4. R. H. Byrd, J. C. Gilbert and J. Nocedal, A trust region method based on interior point techniques for nonlinear programming, *Mathematical Programming*, vol. 89, 2000, 149–185.

5. N.I.M. Gould, M. E. Hribar, and J. Nocedal, On the solution of equality constrained quadratic programming problems arising in optimization, *SIAM J. Sci. Comput.*, vol. 23, 2001, 1376–1395.

6. L. Lukšan and J. Vlček, Indefinitely preconditioned truncated Newton method for large sparse equality constrained nonlinear programming problems, *Numerical Linear Algebra with Applications*, vol. 5, 1998, 219–247.

7. R. J. Vanderbei and D.F. Shanno, An interior-point algorithm for nonconvex nonlinear programming, *Computational Optimization and Applications*, vol. 13, 1999, 231–252.

8. A. Wächter and L. T. Biegler, On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming, Technical Report RC 23149, IBM T.J. Watson Research Center, Yorktown Heights, USA, 2004. To appear in *Mathematical Programming*.

9. R.A.Waltz, J.L. Morales, J. Nocedal and D. Orban, An interior point algorithm for nonlinear optimization that combines line search and trust region steps, Report OTC 6/2003, Optimization Technology Center, Northwestern University, Evanston, IL 60208, USA, 2003. To appear in *Mathematical Programming, Series A*.