Mixed algebraic methods and local tensor product

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In algebraic numerical grid generation, transfinite interpolation methods are useful means to construct a grid interpolating the boundary of a given domain (see [1], [2], [3]). In spite of their simple definition and low computational cost they have lack of free parameters useful to modify the grid according to specific requirements. In other words, the main drawback of classic transfinite interpolation techniques is that no control of the grid quality apart from the boundary of a given physical domain and possess local free parameters for modeling the grid in the interior has been proposed in [6]. Mixed schemes are obtained from the Control Point Form Method investigated by Eiseman in [4] and [5]. Roughly speaking, a mixed scheme is defined trough a smooth transformation from the parameter domain $[0, 1]^2$ into the physical domain Ω consisting of two parts, say,

$$X := X_{\partial \Omega} + X_{\dot{\Omega}}.$$

The first part, $X_{\partial\Omega}$, takes care of the grid near the boundary and guarantees the transfinite interpolation of $\partial\Omega$. The second part, $X_{\dot{\Omega}}$, allows to model the grid in the interior of the domain Ω . The key mathematical ingredients in defining X are transfinite interpolation and B-spline tensor product approximation. Indeed, X is expressed as

$$X := (P_1 \oplus P_2)(\partial \Omega - \partial T_P) + T_P,$$

where T_P stands for a tensor product of univariate B-splines and P_1 , P_2 for two transfinite interpolant operators applied to the boundary curves of Ω and of the image of T_P (for more details see [5], for example). The Boolean sum is to guarantee boundary conformity, while T_P is to provide more degrees of freedom in generating workable meshes. In fact, T_P is linear combination of functions with control points as coefficients that can be relocated according to some grid quality criteria. This fact is used in [7] where an optimization procedure is proposed to improve the grid quality by moving the control points. Now, classical tensor product functions have many advantages such as easy and cheap computation, simple derivation with respect to any variable, just to mention the more significant ones. Unfortunately, they also present a severe inconvenient. In fact, in case we need more control points in a specific part of the grid to control it locally, we are forced to add control points also in other regions. These are the two strips intersecting each other in the region of interest. In order to shortcut this drawback while keeping most advantage of tensor product functions, we here propose the use of a class of functions whose restriction on subsets are B-spline tensor product functions, while the global function is not a tensor product. For shortness we call it a local tensor product functions. This approach, already used in different contexts from mixed algebraic schemes (see the recent papers [9], [10] and references quoted therein) allows us to define a new mixed scheme particularly suitable for a local grid control. Actually, the use of local tensor product functions appears particularly useful when a *posteriori* improvements of the computed grid are required by *optimization* strategies (see, for example [6] and [7]). Grid optimization means the improvement of an existing grid to achieve the optimal one with respect to given criteria coming from the geometry or the physics of the problem to be solved. This strategy is proposed in [7] and better investigated in [8] where the initially set free parameters are modified following grid quality criteria. Possible quality criteria are based on grid cell angles, areas, edge lengths and aspect ratios. These criteria are used to construct objective functions to be minimized and the optimized algebraic grids via an optimal locations of the control points. Here, by using local tensor product functions, we are able to generate an *a posteriori* grid optimization algorithm working on a small number of variables (the few new introduced control points) to obtain optimal algebraic grids. In other words, we are able to achieve optimal algebraic grids by a cheap generation combined with a cheap optimization. In practice, the physical domain Ω is split in sub-regions and locally modified in order to generate a better grid. This is possible thanks to the use of local tensor product functions that allows us to add control points (and B-splines) only in the area of interest and then relocate them after an *a posteriori* optimization process. The resulting strategy is better summarized by the following procedure.

Local optimization procedure

- Input the boundary curves, the grid size, the number of control points
- Set the initial values for the free parameters
- Generate the mixed algebraic grid \mathcal{X}^0
- Determine the regions where to insert extra control points and extra B-spline bases
- Select an objective function based on some quality criteria
- Move the added control points to minimize the objective function
- Compute the optimized grid \mathcal{X}^*

We conclude by comparing the algebraic grid obtained before and after a local modification of control points (left and right pictures, respectively) by using a particular objective function ([7],page 12). The advantage of the insertion of new control points and their relocation in sub-regions is evident.