

A spline collocation method and a special grid of Shishkin type for a singularly perturbed problem

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We consider the two parameter singularly perturbed boundary value problem

$$(0.1) \quad \begin{aligned} Ly := \varepsilon y''(x) + \mu a(x)y'(x) - b(x)y(x) &= f(x), & x \in (0, 1), \\ y(0) = p_0, \quad y(1) &= p_1, \end{aligned}$$

where a, b and f are sufficiently smooth functions, $0 < \varepsilon \ll 1$, $0 < \mu \ll 1$, and

$$a(x) \geq \alpha > 0, \quad b(x) \geq \beta > 0, \quad x \in [0, 1].$$

Under these assumptions the problem (0.1) has a unique solution which exhibits exponential boundary layers at $x = 0$ and $x = 1$. Boundary layers are regions where the solution and its derivatives varies rapidly. Most of the traditional numerical methods fail to catch the rapid change of the solution, and its failure in turn pollutes the numerical approximation on the whole domain. For the construction of any numerical method solving singularly perturbed problem it is crucial to have information about the behavior of derivatives of the exact solution. The bounds on derivatives are required in the mesh refinement strategy as well as in the error analysis. Layer-adapted meshes are usually used to solve singularly perturbed problems. We use a piecewise uniform Shishkin mesh which can be chosen a priori when one has some knowledge of the structure of these layers.

When $\mu = 1$, problem (0.1) becomes convection-diffusion problem with the boundary layer of width $\mathcal{O}(\varepsilon)$ in the neighbourhood of the point $x = 0$. In the case when $\mu = 0$, we have reaction-diffusion problem with boundary layers of width $\mathcal{O}(\sqrt{\varepsilon})$ at $x = 0$ and $x = 1$. We consider problem (0.1) and offer a unified treatment for all possible classes of subproblems. Problem (0.1) satisfies the minimum principle.

Next, we give the solution decomposition from [1] and related estimates for the components and their derivatives which are used in the mesh construction. The solution y has the following decomposition:

$$(0.2) \quad y = v + w_L + w_R,$$

where

$$\begin{aligned} Lv &= f, \quad v(0), v(1) \quad \text{chosen} \\ Lw_L &= 0, \quad w_L(0) = y(0) - v(0), \quad w_L(1) = 0 \\ Lw_R &= 0, \quad w_R(0) = 0, \quad w_R(1) = y(1) - v(1). \end{aligned}$$

Here v is the regular component of the solution and

$$\|v^{(k)}\| \leq M, \quad k = 0, 1, 2, \quad \|v^{(3)}\| \leq M\varepsilon^{-1}.$$

The singular components w_L and w_R satisfy bounds of the following lemma. **Lemma 1**
[1] *When the solution of (0.1) is decomposed as in (0.2) we have*

$$|w_L(x)| \leq Me^{-\theta_1 x}, \quad |w_R(x)| \leq Me^{-\theta_2(1-x)}$$

where

$$\theta_1 = \frac{\mu\alpha + \sqrt{\mu^2\alpha^2 + 4\varepsilon\beta}}{2\varepsilon}, \quad \theta_2 = \frac{-\mu A + \sqrt{\mu^2 A^2 + 4\varepsilon\beta}}{2\varepsilon}$$

and $A = \max_{0 \leq x \leq 1} |a(x)|$. θ_1 and θ_2 are respectively, the positive roots of the equations

$$\varepsilon\theta_1^2 - \mu\alpha\theta_1 - \beta = 0 \quad \text{and} \quad \varepsilon\theta_2^2 + \mu A\theta_2 - \beta = 0.$$

We approximate the solution y of the problem (0.1) with quadratic spline on a piecewise uniform Shishkin mesh $\Delta_n : x_0 = 0, x_1, x_2, \dots, x_n = 1$, where

$$x_i = \begin{cases} \frac{4\sigma_1 i}{n} & i \leq \frac{n}{4} \\ \sigma_1 + \frac{2}{n}(i - \frac{n}{4})(1 - \sigma_1 - \sigma_2) & \frac{n}{4} \leq i \leq \frac{3n}{4} \\ 1 - \sigma_2 + (i - \frac{3n}{4})\frac{4\sigma_2}{n} & \frac{3n}{4} \leq i \leq n, \end{cases}$$

$$\sigma_1 = \min\left\{\frac{1}{4}, \frac{2}{\theta_1} \ln n\right\}, \quad \sigma_2 = \min\left\{\frac{1}{4}, \frac{2}{\theta_2} \ln n\right\}.$$

Using quadratic spline $u(x) \in C^1[0, 1]$ (like in [3]) we have

$$(0.3) \quad \begin{aligned} u(x) &= u_i + (x - x_i)u'_i + \frac{(x - x_i)^2}{2}u''_i, \quad x \in [x_i, x_{i+1}] \\ u_{i+1} &= u_i + h_{i+1}u'_i + \frac{h_{i+1}^2}{2}u''_i, \quad h_i = x_i - x_{i-1} \\ u'_{i+1} &= u'_i + h_{i+1}u''_i. \end{aligned}$$

We chose collocation points in a nonstandard way:

$$\xi_i = \alpha_{1i}x_{i-1} + (1 - \alpha_{1i})x_i, \quad \text{on } [x_{i-1}, x_i],$$

$$\eta_i = \alpha_{2i}x_i + (1 - \alpha_{2i})x_{i+1}, \quad \text{on } [x_i, x_{i+1}]$$

where $0 \leq \alpha_{1i}, \alpha_{2i} \leq 1$. The scheme with collocation points

$$\xi_i = \alpha_i x_{i-1} + (1 - \alpha_i)x_i$$

on all intervals $[x_{i-1}, x_i]$ is derived in [4]. In order to relax the conditions from [4] and improve stability we use different collocation points on intervals $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$. For the standard collocation points we have $\alpha_{1i} = \alpha_{2i} = \frac{1}{2}$. But in the case of equation (0.1),

the corresponding discrete analogue does not satisfy the discrete minimum principle, i.e. the corresponding matrix is not inverse monotone. We want to obtain inverse monotone matrix in order to increase stability of the system and apply the barrier function method for the proof of the convergence. Thus, from (0.3) and

$$\begin{aligned}\varepsilon u''(\xi_i) + \mu a(\xi_i)u'(\xi_i) - b(\xi_i)u(\xi_i) &= f(\xi_i), & \xi_i \in [x_{i-1}, x_i] \\ \varepsilon u''(\eta_i) + \mu a(\eta_i)u'(\eta_i) - b(\eta_i)u(\eta_i) &= f(\eta_i), & \eta_i \in [x_i, x_{i+1}]\end{aligned}$$

where $u''(\xi_i) = u''_{i-1}$, $u''(\eta_i) = u''_i$, we obtain the scheme

$$(0.4) \quad \begin{aligned}L_n u_i &:= R_i^- u_{i-1} + R_i^c u_i + R_i^+ u_{i+1} = q^- f(\xi_i) + q^+ f(\eta_i), \\ u_0 &= p_0, \quad u_n = p_1,\end{aligned}$$

where

$$\begin{aligned}R_i^- &= \frac{S_i}{2D_i}, & R_i^+ &= \frac{h_i Q_i}{2h_{i+1} P_i}, \\ R_i^c &= -1 + \frac{h_i h_{i+1} b(\eta_i)}{2P_i} + \frac{Q_i h_i}{2h_{i+1} P_i} - \frac{\Omega_i}{2D_i}, \\ q^- &= -\frac{h_i^2}{2D_i}, & q^+ &= -\frac{h_i h_{i+1}}{2P_i}, \\ S_i &= -2\varepsilon + 2\mu a(\xi_i)h_i \alpha_{1i} + b(\xi_i)h_i^2 \alpha_{1i}^2, \\ D_i &= -2\varepsilon + 2\mu a(\xi_i)h_i(2\alpha_{1i} - 1) - b(\xi_i)h_i^2(1 - \alpha_{1i})^2, \\ Q_i &= -2\varepsilon - 2\mu a(\eta_i)(1 - \alpha_{2i})h_{i+1} + b(\eta_i)(1 - \alpha_{2i})^2 h_{i+1}^2, \\ P_i &= -2\varepsilon + \mu a(\eta_i)(2\alpha_{2i} - 1)h_{i+1} - b(\eta_i)\alpha_{2i}(1 - \alpha_{2i})h_{i+1}^2, \\ \Omega_i &= 2\varepsilon + 2\mu a(\xi_i)h_i(1 - \alpha_{1i}) - b(\xi_i)h_i^2(1 - \alpha_{1i})^2.\end{aligned}$$

We want to determine α_{1i} and α_{2i} such that the corresponding matrix has L-form, i.e. $R_i^- \geq 0$, $R_i^+ \geq 0$, $R_i^c < 0$. The first condition $R_i^- \geq 0$ is satisfied if α_{1i} is determined from the conditions $\alpha_{1i} \leq 1/2$, and $S_i \leq 0$. Further, $R_i^+ \geq 0$ if α_{2i} is determined such that $Q_i \leq 0$, and $P_i \leq 0$. With the α_{1i} and α_{2i} determined above, we have $R_i^c < 0$.

Theorem 1 (*Discrete Minimum Principle*) *Let the parameters α_{1i} be determined so that $\alpha_{1i} \leq \frac{1}{2}$ and $S_i \leq 0$, α_{2i} so that $Q_i \leq 0$ and $P_i \leq 0$. If W is any mesh function with properties $L_n W \leq 0$, $W_0 \geq 0$, $W_n \geq 0$ then $W \geq 0$.*

Remark. We put $\alpha_{1i} = \alpha_{2i} = \frac{1}{2}$ whenever the mesh step supplies $R_i^- \geq 0$ and $R_i^+ \geq 0$. Otherwise, the collocation points are moved in a way to obtain $R_i^- \geq 0$ and $R_i^+ \geq 0$.

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