

## Fluid-Structure Interaction via an Adaptive Finite Element Immersed Boundary Method

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One of the main difficulties that arises when dealing with visco-elasticity and fluid structure interaction problems is the fact that fluids and elastic materials have a different “natural” framework.

The usual way of characterizing a fluid motion is the Eulerian framework, where the system is described using the velocity and pressure fields. On the other hand, when dealing with elasticity, it is customary to express the stress as a function of the displacements of the material particles from their reference, or Lagrangian, position, which is not directly available in the Eulerian formulation.

The Immersed Boundary (IB) method, introduced by Peskin to study blood flow around the heart valves (see [5] for a comprehensive introduction to the IB method) gives one way to link the two frameworks together and deploy the strengths of both formulations at the same time.

Consider a region of space occupied by a continuum of material  $\mathcal{B}$  (be it solid or fluid), evolving in time as  $\mathcal{B}_t$ . We define the following mappings

$$(0.1) \quad \begin{aligned} \mathbf{X} : \mathcal{B} \subset \mathbb{R}^d \times [0, T] &\mapsto \mathcal{B}_t \subset \mathbb{R}^d \\ \mathbf{p} : \mathcal{B}_t \subset \mathbb{R}^d \times [0, T] &\mapsto \mathcal{B} \subset \mathbb{R}^d, \end{aligned}$$

representing the trajectory of a material point  $\mathbf{s}$  (identified with its position in the reference undeformed configuration  $\mathcal{B}$ ) and its inverse mapping that associates with each point  $\mathbf{x} \in \mathcal{B}_t \subset \mathbb{R}^d$  the material point that happens to be there.

We define the velocity field  $\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(\mathbf{s}, t)|_{\mathbf{s}=\mathbf{p}(\mathbf{x}, t)}$  and its total time derivative by  $\dot{\mathbf{u}}(\mathbf{x}, t) = \frac{\partial^2 \mathbf{X}}{\partial t^2}(\mathbf{s}, t)|_{\mathbf{s}=\mathbf{p}(\mathbf{x}, t)} = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t)$  and we will adopt the following notation: capital letters refer to functions whose domain is the Lagrangian one (e.g.  $\mathbf{X}(\mathbf{s}, t)$ ), while lower case letters refer to functions whose domain is the Eulerian or spatial one (e.g.  $\mathbf{u}(\mathbf{x}, t)$  or  $p(\mathbf{x}, t)$ ). Bold letters are vector fields (e.g.  $\mathbf{u}$ ) and boldface or greek letters are tensor fields respectively in the Lagrangian coordinate system (e.g.  $\mathbb{P}$ ) or in the Eulerian one (e.g.  $\boldsymbol{\sigma}$ ).

Equations for incompressible continuum mechanics derive from the conservation of mass and of linear and angular momentum. Via a localization technique, it is possible to show that this is

equivalent to the existence of a symmetric tensor  $\boldsymbol{\sigma}$ , usually referred to as Cauchy stress tensor, such that

$$(0.2) \quad \dot{\mathbf{u}} = \nabla \cdot \boldsymbol{\sigma} \quad \text{in } \mathcal{B}_t,$$

$$(0.3) \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \mathcal{B}_t.$$

Navier-Stokes equations for incompressible fluids derive from modeling the Cauchy stress tensor as

$$(0.4) \quad \boldsymbol{\sigma}_f = -p\mathbb{I} + 2\nu D(\mathbf{u}) = -p\mathbb{I} + \nu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

where  $\nu$  is the so called kinematic viscosity and the pressure  $p \in L_0^2(\Omega)$  is the Lagrange multiplier associated with the incompressibility constraint  $\nabla \cdot \mathbf{u} = 0$ .

When dealing with elasticity it is customary to express the Cauchy stress tensor in the reference (Lagrangian) variables. To do so consider a smooth portion  $\mathcal{P}$  of  $\mathcal{B}$  evolving as  $\mathcal{P}_t$ . We define the  $d \times d$  tensor field  $\mathbb{P}$ , referred to as the first Piola-Kirchhoff stress tensor, as

$$(0.5) \quad \int_{\partial \mathcal{P}_t} \boldsymbol{\sigma} \mathbf{n} da = \int_{\partial \mathcal{P}} \mathbb{P} \mathbf{N} dA,$$

where  $\mathbf{N}$  is the outer normal to the region  $\mathcal{P}$  in the Lagrangian coordinates while  $da$  and  $dA$  are the area differentials in dimension  $d - 1$ .

If we wanted to treat incompressibility and viscosity in their natural Eulerian framework and elasticity in the Lagrangian one, we could write a variational formulation for incompressible visco-elastic materials as

Given  $\mathbf{u}_0 \in H_0^1(\mathcal{B}_0)^d$  and  $\mathbf{X}_0 : \mathcal{B} \rightarrow \mathcal{B}_0$ , for all  $t \in [0, t]$ , find  $\mathbf{u}(t) \in H_0^1(\mathcal{B}_t)^d$  and  $\mathbf{X} : \mathcal{B} \times [0, T] \rightarrow \Omega$ , such that

$$(0.6) \quad (\dot{\mathbf{u}}(t), \mathbf{v}) + (\boldsymbol{\sigma}_f(t), \nabla \mathbf{v}) + ((\mathbb{P}_s(t), \nabla_s(\mathbf{v} \circ \mathbf{X}))) = 0 \quad \forall \mathbf{v} \in H_0^1(\mathcal{B}_t)^d$$

$$(0.7) \quad (\nabla \mathbf{u}(t), q) = 0 \quad \forall q \in L_0^2(\mathcal{B}_t)$$

$$(0.8) \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \forall \mathbf{x} \in \mathcal{B}_t$$

$$(0.9) \quad \frac{\partial \mathbf{X}}{\partial t}(\mathbf{s}, t) = \mathbf{u}(\mathbf{X}(\mathbf{s}, t), t) \quad \forall \mathbf{s} \in \mathcal{B}$$

$$(0.10) \quad \mathbf{X}(\mathbf{s}, 0) = \mathbf{X}_0(\mathbf{s}) \quad \forall \mathbf{s} \in \mathcal{B},$$

where  $(\cdot, \cdot)$  is the inner product in the Eulerian space, while  $((\cdot, \cdot))$  is the inner product in the Lagrangian one.

Here  $\boldsymbol{\sigma}_f$  contains the viscosity and pressure terms as in (0.4), i.e. only the terms easily described by the Eulerian formulation, while  $\mathbb{P}_s$  contains only the terms depending on the material response to deformation.

The extension to fluid structure interaction is immediate. Consider a region  $\Omega \subset \mathbb{R}^d$  containing  $\mathcal{B}_t$ , filled up with a fluid described by the Cauchy stress tensor (0.4) and suppose the region  $\mathcal{B}_t$  to be filled with a visco-elastic material which, in addition to the stress tensor (0.4), responds to elastic deformations through a given formulation of the first Piola-Kirchhoff stress tensor  $\mathbb{P}_s$ . This formulation makes sense also in the cases where the Lagrangian domain is a closed curve of co-dimension one (from which the method gets its name).

Let  $\mathcal{T}_h$  be a subdivision of  $\Omega$  into triangles or rectangles if  $n = 2$ , tetrahedrons or parallelepipeds if  $n = 3$ . We then consider two finite dimensional subspaces  $V_h \subseteq H_0^1(\Omega)^d$  and

$Q_h \subseteq L_0^2(\Omega)$ . It is well known that the pair of spaces  $V_h$  and  $Q_h$  need to satisfy the inf-sup condition in order to have existence, uniqueness and stability of the discrete solution of the Navier-Stokes problem (0.6)-(0.7) (see [4]). We chose the inf-sup stable ‘‘Q2-P1’’ finite element pair for the fluid discretization.

Next consider a subdivision  $\mathbf{S}_h$  of  $\mathcal{B}$  into segments, triangles or tetrahedrons (respectively  $m = 1, 2, 3$ ). We will denote by  $\mathbf{s}_i, i = 1, \dots, M$  the vertices of the triangulation  $\mathbf{S}_h$  and by  $T_i, i = 1, \dots, Me$  the elements of  $\mathbf{S}_h$ . Let  $\mathbf{S}_h$  be the finite element space of piecewise linear  $d$ -vectors defined on  $\mathcal{B}$  as follows

$$(0.11) \quad \mathbf{S}_h = \{ \mathbf{Y} \in C^0(\mathcal{B}; \Omega) : \mathbf{Y}|_{T_i} \in \mathcal{P}^1(T_i)^d, i = 1, \dots, Me \},$$

where  $\mathcal{P}^1(T_i)$  stands for the space of affine polynomials on the element  $T_i$ . For an element  $\mathbf{Y} \in \mathbf{S}_h$  we shall use also the following notation  $\mathbf{Y}_i = \mathbf{Y}(\mathbf{s}_i)$  for  $i = 0, \dots, M$ .

In the immersed boundary method the Lagrangian component of the stress can be interpreted as a source term for the underlying Navier-Stokes equations. In particular in the finite element framework that we described (see also [2] and [3]) the computation of the Lagrangian part of the stress (which depends on  $\mathbb{F} = \nabla_s \mathbf{X}$ , therefore is piecewise constant in each element) is given by, for all  $\mathbf{X} \in \mathbf{S}_h$  and for all  $\mathbf{v} \in V_h$ ,

$$(0.12) \quad ((\mathbb{P}, \nabla_s(\mathbf{v} \circ \mathbf{X}))) = - \langle \mathbf{f}(t), \mathbf{v} \rangle = \sum_{e \in \mathcal{E}} \int_e [[\mathbb{P}]] \cdot \mathbf{v}(\mathbf{X}) dA,$$

where  $\mathcal{E}$  is the set of the edges of the triangulation  $\mathbf{S}_h$  and  $[[\cdot]]$  denotes the jump across the interface.

Let  $\Delta t$  denote the time step and let us indicate by the superscript  $n$  an unknown function at time  $t_n = n\Delta t$ , so that the number of time steps needed to reach the final time  $T$  is  $N$ . The time-space discrete problem that we solve is given by the following steps.

Given  $\mathbf{u}_0 \in V_h$  and  $\mathbf{X}_0 \in \mathbf{S}_h$ , set  $\mathbf{u}^0 = \mathbf{u}_0$  and  $\mathbf{X}^0 = \mathbf{X}_0$ , then for  $n = 0, 1, \dots, N - 1$

**Step 1.** compute the source term

$$\langle \mathbf{f}^{n+1}, \mathbf{v} \rangle = - \sum_{e \in \mathcal{E}} \int_e [[\mathbb{P}(\mathbf{X}_h^n)]] \cdot \mathbf{v}(\mathbf{X}_h^n) dA \quad \forall \mathbf{v} \in V_h;$$

**Step 2.** find  $(\mathbf{u}^{n+1}, p^{n+1}) \in V_h \times Q_h$ , such that

$$\begin{aligned} \left( \frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}, \mathbf{v} \right) + (\mathbf{u}^{n+1} \cdot \nabla \mathbf{u}^{n+1}, \mathbf{v}) + \nu (\nabla \mathbf{u}^{n+1}, \nabla \mathbf{v}) \\ - (\nabla \mathbf{v}, p^{n+1}) = \langle \mathbf{f}^{n+1}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_h \\ (\nabla \mathbf{u}^{n+1}, q) = 0 \quad \forall q \in Q_h \end{aligned}$$

**Step 3.** find  $\mathbf{X}^{n+1} \in \mathbf{S}_h$ , such that

$$\frac{\mathbf{X}_i^{n+1} - \mathbf{X}_i^n}{\Delta t} = \mathbf{u}^{n+1}(\mathbf{X}_i^n) \quad \forall i = 1, \dots, M.$$

The main characteristic of this method is the possibility to decouple spatially the lagrangian domain from the Eulerian one, making it possible to build a fast black box Navier-Stokes solver on a fixed lattice grid.

One disadvantage of this approach is the loss of accuracy around the boundary of the moving Lagrangian domain. To address this issue we propose a non-matching refinement strategy, where the cartesian grid elements of the Eulerian mesh are trivially refined in sub-elements with the same aspect-ratio according to whether or not they contain also some Lagrangian elements. Figure 0.1 is an example of such a procedure in a two dimensional domain with anisotropic linear visco-elasticity coupled with Navier-Stokes equations.

All computations were performed using the `deal.II` Finite Element Library (see [1] for a technical reference).

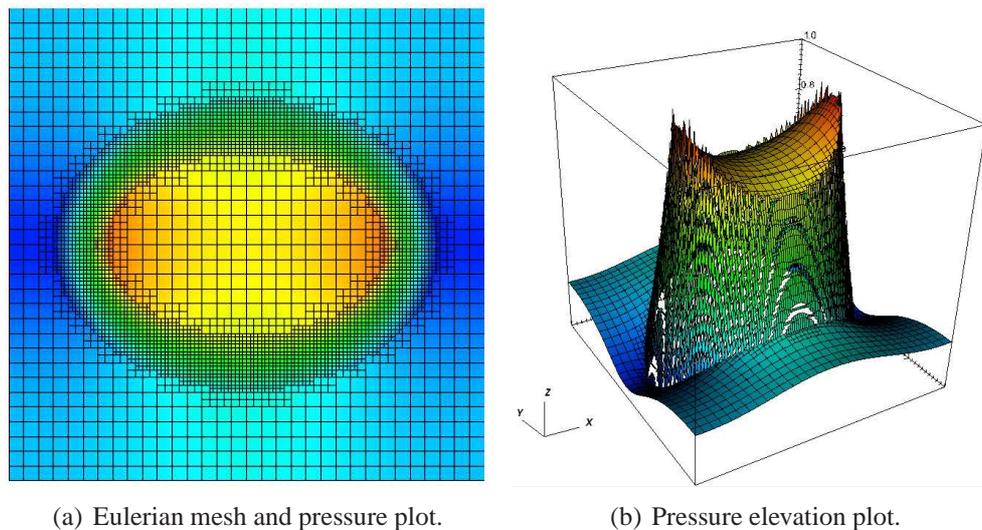


Figure 0.1: Deformed elastic shell immersed in a two-dimensional domain: pressure plots and Eulerian computational domain.

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