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Recovery based error estimation for plate problems

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A posteriori error estimation is an important tool in finite element software development, since it allows to verify and validate the finite element simulations, as well as to improve results and control the error, when combined with adaptivity. An efficient and practical way to derive a posteriori error estimators is offered by recovery procedures. The error, generally in stress based norms, is estimated by comparing the original finite element solution with the recovered one. The major steps forward in using recovery procedures were made with the Superconvergent Patch Recovery (SPR) and the Recovery by Equilibrium in Patches (REP), both successfully applied to plate problems, see for example [1]. Recently, a new superconvergent procedure called Recovery by Compatibility in Patches (RCP) has been proposed and shown to provide an excellent basis for error estimation in 2D problems [2,3].
The present paper aims at presenting an extension of the RCP-based error estimation to Reissner-Mindlin plates finite element analysis. The basic idea of the procedure is to recover stress resultants by enforcing compatibility over patches of elements. Displacements computed by the finite element analysis are prescribed on the boundary of the patch, and improved stress resultants are computed by minimizing the complementary energy of such a sub-model. The resulting procedure is simple, efficient, numerically stable and does not need any knowledge of superconvergent points. Its performance is evaluated in a numerical test, using both displacement elements and the new 9βQ4 hybrid stress elements recently proposed in [4].

PLATE EQUATIONS AND FINITE ELEMENT ANALYSIS

Consider a plate referred to a Cartesian reference frame (O, x, y, z) with the origin O on the mid-surface Ω and the z-axis in the thickness direction, $-h/2 \leq z \leq h/2$ where h is the thickness. Let $\partial\Omega$ be the boundary of Ω. The Reissner-Mindlin theory is employed. The compatibility equations are

$$\chi = \mathbf{D}_b \theta, \quad \gamma = \mathbf{D}_s w + \hat{\mathbf{I}}\theta, \tag{0.1}$$

where w is the transverse displacement, $\theta^{\mathrm{T}} = \left[\begin{array}{cc} \theta_x & \theta_y \end{array} \right]$, $\chi^{\mathrm{T}} = \left[\begin{array}{ccc} \chi_x & \chi_y & \chi_{xy} \end{array} \right]$, $\gamma^{\mathrm{T}} = \left[\begin{array}{cc} \gamma_x & \gamma_y \end{array} \right]$ are the vectors that collect the rotations, the curvatures and the shear strains, and

$$\mathbf{D}_b = \left[\begin{array}{cc} 0 & -\partial/\partial x \\ \partial/\partial y & 0 \\ \partial/\partial x & -\partial/\partial y \end{array} \right], \quad \mathbf{D}_s = \left[\begin{array}{c} \partial/\partial x \\ \partial/\partial y \end{array} \right], \quad \hat{\mathbf{I}} = \left[\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right]. \tag{0.2}$$

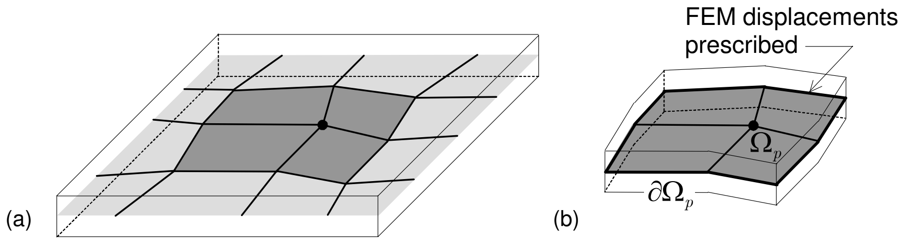

Figure 0.1: Example of patch: (•) assembly node defining the patch.

The equilibrium equations can be obtained *via* the principle of virtual work in the form

$$\mathbf{D}_b^*\mathbf{M} + \hat{\mathbf{I}}^{\mathrm{T}}\mathbf{S} = \mathbf{m}, \quad \mathbf{D}_s^*\mathbf{S} = q, \tag{0.3}$$

where vectors $\mathbf{M}^{\mathrm{T}} = \begin{bmatrix} M_x & M_y & M_{xy} \end{bmatrix}$ and $\mathbf{S}^{\mathrm{T}} = \begin{bmatrix} S_x & S_y \end{bmatrix}$ collect the moment and shear resultants, $\mathbf{D}_b^*$ and $\mathbf{D}_s^*$ are differential operators adjoint to $\mathbf{D}_b$ and $\mathbf{D}_s$, respectively, and q and $\mathbf{m}$ are the prescribed generalized loads. For a isotropic linearly elastic material, the constitutive equations can be written in the form

$$\mathbf{M} = \mathbf{C}_b\chi\,, \mathbf{S} = \mathbf{C}_s\gamma, \tag{0.4}$$

$$\mathbf{C}_b = \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \mathbf{C}_s = \frac{5Eh}{12(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

being E the Young's modulus and ν the Poisson's ratio.

The 9β Q4 *element.* Recently, a new quadrilateral 4-node element for the analysis of shear deformable plates has been presented in [4] and labelled 9βQ4. The formulation is of hybrid stress type, involving equilibrating stress resultants within each element and compatible displacements along the interelement boundary. The assumed stress approximation, expressed in terms of skew coordinates, has the minimum number of stress modes (9 modes) and is coordinate invariant. Displacements are modelled using a linked interpolation ruled by the standard nodal degrees of freedom (3 dofs per node). The hybrid stress approach, where displacement acts on element boundary only, together with the linked interpolation allow to derive an element which is locking-free and passes all the patch tests. Moreover, the resulting element has been tested to be stable, accurate and relatively insensitive to geometry distortions.

RCP RECOVERY FOR PLATE PROBLEMS AND ERROR ESTIMATION

The RCP recovery for plate problems recovers generalized stresses locally, over patches of elements. Fig. 0.1 shows an example of patch. The key idea to recover stresses over the patch is analogous to the RCP for 2D problems [2]: each patch Ω_p is considered as a separate system on which generalized stresses are improved by enhancing equilibrium and relaxing compatibility. To this purpose, the rotations θ^h and transverse displacement w^h resulting from the finite element analysis are prescribed along the patch boundary $\partial\Omega_p$, Fig. 0.1(b), and the local generalized stresses $\mathbf{M}_p^*$, $\mathbf{S}_p^*$ are determined by minimizing the complementary energy associated to this separate system:

$$\Pi = \frac{1}{2}\int_{\Omega_p} (\mathbf{M}_p^{*\,\mathrm{T}}\mathbf{C}_b^{-1}\mathbf{M}_p^* + \mathbf{S}_p^{*\,\mathrm{T}}\mathbf{C}_s^{-1}\mathbf{S}_p^*)\,\mathrm{d}\Omega - \int_{\partial\Omega_p} (\theta^{h\,\mathrm{T}}\mathbf{N}_b^{\mathrm{T}}\mathbf{M}_p^* + w^{h\,\mathrm{T}}\mathbf{N}_s^{\mathrm{T}}\mathbf{S}_p^*)\,\mathrm{d}(\partial\Omega),$$

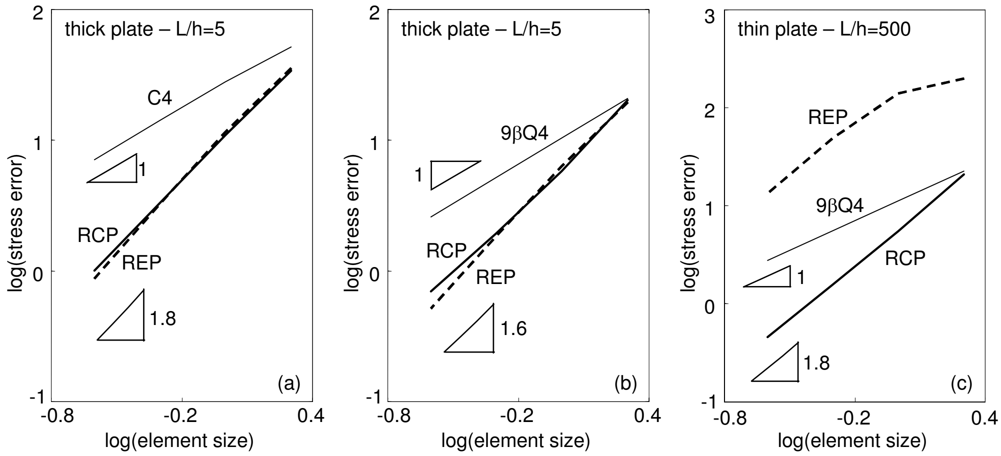

Figure 0.2: Relative percentage error: (a) and (b) thick plate; (c) thin plate.

with $\mathbf{N}_b$ and $\mathbf{N}_s$ matrices of the components of the unit outward normal to $\partial\Omega_p$, among a set of generalized stresses which satisfy interior equilibrium, Eq.(0.3), within the patch. The local generalized stresses are decomposed as

$$\left[\begin{array}{c}\mathbf{M}_p^*\\ \mathbf{S}_p^*\end{array}\right]=\left[\begin{array}{c}\mathbf{M}_{pp}^*\\ \mathbf{S}_{pp}^*\end{array}\right]+\left[\begin{array}{c}\mathbf{M}_{ph}^*\\ \mathbf{S}_{ph}^*\end{array}\right], \tag{0.5}$$

where the first term is a particular solution of Eq.(0.3) within the patch and the second term is the homogeneous solution of the same equation, approximated over the patch as:

$$\left[\begin{array}{c}\mathbf{M}_{ph}^*\\ \mathbf{S}_{ph}^*\end{array}\right]=\mathbf{P}\,\beta, \tag{0.6}$$

being $\mathbf{P}$ a matrix of self-equilibrated stress modes and β a vector of unknown parameters. Here, the 11β-approximation developed in [4] has been used for $\mathbf{P}$. Minimizing functional Π leads to a system of linear algebraic equations whose solution permits to determine parameters β and, hence, the value of the recovered generalized stresses in the assembly node of the patch. The improved stress distribution $\mathbf{M}^*$, $\mathbf{S}^*$ is then obtained by interpolating these nodal values with the same shape functions used for displacements. Once the improved solution has been obtained, the error in the finite element solution can be estimated as:

$$\|e^*\|^2=\frac{1}{2}\int_{\Omega}[\left(\mathbf{M}^h-\mathbf{M}^*\right)^{\mathrm{T}}\mathbf{C}_b^{-1}\left(\mathbf{M}^h-\mathbf{M}^*\right)+\left(\mathbf{S}^h-\mathbf{S}^*\right)^{\mathrm{T}}\mathbf{C}_s^{-1}\left(\mathbf{S}^h-\mathbf{S}^*\right)]\mathrm{d}\Omega$$

where $\mathbf{M}^h$, $\mathbf{S}^h$ are the finite element generalized stresses.
A simply supported (SS2) square plate under uniform transverse load, analyzed using conventional 4-node elements (C4) and 9βQ4 elements, is considered. The convergence of the global error in energy norm, reported in Fig. 0.2, demonstrates an excellent performance of the present procedure which appears to be competitive with the REP procedure.

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