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Methods of Mathematical Physics Applied to Polymer Composites

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Abstract.

The aim of this lecture note is twofold:

1. To provide a detailed analytical derivation of the Kerner's classical homogenization formulae because the original paper by E.H. Kerner is particularly concise and difficult to understand. Moreover, in the past, the geometrical assumption the model uses to justify the mathematical treatment of the homogenizing process was subject to great criticism. For this reason, the original paper was critically re-examined, the homogenizing procedure newly stated as weight average of matrix/particle and matrix/matrix stress distribution and the equations of the model were re−derived accordingly. This critical effort confirmed that Kerner's original formulae are correct.
2. To set−up a novel experimental method to predict the elastic properties of $MgCl_2$ particles and their derived Ziegler−Natta catalysts which are spherical in shape and have an average diameter of few tenths of a micron. The novel method is derived by inverting Kerner's classical model which describes a particulate filled material consisting of micro-spheres randomly dispersed in a matrix and perfectly bound to the matrix.

(S.M. − Ferrara, 2007−2008)

1. PREDICTION OF COMPOSITE ELASTIC PROPERTIES.

1.1. *Introduction.*

Several theoretical and semi–empirical models have been used to predict the elastic properties of materials filled with particulate matter. Most of these equations are derived from the classical Kerner model[1] in which spheres are either randomly or periodically dispersed in a matrix and perfectly bound to that matrix. The two elastic constants – shear modulus and bulk modulus – of a homogeneous, macroscopically isotropic composite were found in terms of the modules and concentrations of its components. The original paper by E.H. Kerner is particularly concise and not easily understandable. Moreover, the geometrical assumptions the model uses to justify the mathematical treatment of the homogenizing process gave rise to a great deal of criticism.[2] For this reason, the original paper by Kerner has been critically re–examined, the homogenizing procedure newly stated as the weight average of matrix/particle and matrix/matrix stress distribution and the model equations were then rederived accordingly. This critical effort confirmed that Kerner's original formulae are correct.

The micro–mechanic basis for Kerner's model requires knowledge of the matrix/particle interaction in terms of stress distribution. To this end, Kerner's model makes use of the calculation, obtained by J.N. Goodier,[3] of the stress distribution of an isolated spherical inclusion embedded in an infinite matrix.

From a practical point of view, it is necessary to specify the meaning of the term "isolated particle". From the Saint Venant principle, we already know that the disturbing effect of any small spherical inclusion is confined to the neighborhood of the inclusion itself. In fact, from the Goodier solutions themselves, it appears that, at a distance of one particle radius from the elastic inclusion, the stress distribution, which would be uniform without inclusion, is not modified by more than a few percentage units. This distance could be the repeating unit at which particles behave as though they were isolated from each other.

The elastic properties of the spherical inclusions can be identified by analyzing the experimental elastic properties of several composites obtained at increasing inclusion concentrations through linear regression of Kerner's equation. According to the experimental setup, the elastic property could be the bulk modulus or shear modulus of the spherical particle.

Once the elastic properties of both the composite and the matrix are known, calculation of the elastic modules of the spherical inclusions is an inverse mathematical problem. Generally, inverse problems are more complex than the associated direct problem and this is mainly because of solution numerical instability and insufficient accuracy of the input data which can lead to unacceptable error propagation with the inverse solution.

1.2. *Mathematical model by E.H. Kerner*

1.2.1. *Critical analysis of the foundation hypotheses of the Kerner model*

It is worth mentioning that the original paper by E.H. Kerner[1] is particularly concise and, consequently, not easily understandable. The geometric model – described by Kerner as a support to justify his mathematical treatment of the homogenizing procedure – may induce misunderstanding or confusion.[2]

In this paper, on the contrary, we have developed a homogenization model that differs from the one presented by Kerner and, we hope, is more convincing than the original one. However, the final formulae are exactly the same as derived by Kerner. This attempt to establish more sound foundations for these famous formulae arises from the fact that, when worked out for a binary system, the formulae provide exactly the lower Hashin–Shtriman variational limit[4] when the matrix is softer than the inclusion, while the formulae provide the upper limit in the opposite case. Consequently, Kerner's formulae do not violate the variational limits which are valid for arbitrary inclusion shapes.

The analytical derivation of the formulae will use exclusively a J.N. Goodier classical, linear elastic result[3] which gives the stress field around a spherical inclusion embedded in an infinite matrix, undergoing uniform, uniaxial traction at infinity. The linear elastic problem is analytically solved by Goodier in terms of spherical harmonics, the special functions of the Higher Analysis.

We shall take into account such analytical solution, in particular the stress and deformation along the uniaxial tension application axis OZ. Our aim is to establish a reasonable link between the macroscopic behavior of a given spherical composite material and the microscopic matrix/particle interaction described by Goodier's analysis.

Within the framework of micromechanics, let us first evaluate the average values of stress $\langle\sigma^i_{zz}\rangle$ and deformation $\langle\varepsilon^i_{zz}\rangle$ along the principal axis OZ, referred to the whole sphere of radius a occupied by the particle or the matrix, respectively. After calculation, (see Appendix sect. 2.4), we obtain:

$$\begin{aligned}\langle\sigma^i_{zz}\rangle &= H_i + 4\left(F_i\frac{7}{5}\right)G_i a^2\\ \langle\varepsilon^i_{zz}\rangle &= (3\lambda_i + 2\mu)\,H_i + 8\mu_i\left(F_i + \frac{7}{5}G_i a^2\right)\end{aligned} \tag{1}$$

where F_i G_i H_i are given constants obtained as solutions of Goodier's problem. Let us now highlight and separate the dilatational and deviatoric components:

$$\begin{aligned}\langle\varepsilon^i_{zz}\rangle &= \frac{1}{3}\langle\varepsilon^i\rangle + \langle e^i\rangle\\ \langle\sigma^i_{zz}\rangle &= \frac{1}{3}\langle\sigma^i\rangle + \langle s^i\rangle\end{aligned} \tag{2}$$

After using the definition of *volume bulk modulus*[5]

$$\kappa = \frac{1}{3}\,(3\lambda + 2\mu) \tag{3}$$

the dilatational component becomes:

$$\begin{aligned}\langle\varepsilon^i\rangle &= 3H_i\\ \langle\sigma^i\rangle &= (3\lambda_i + 2\mu_i)\,3H_i \equiv 9\kappa H_i = 3\kappa\langle\varepsilon^i\rangle\end{aligned} \tag{4}$$

while the deviatoric component remains:

$$\langle e^i \rangle = 4\left(F_i + \frac{7}{5}G_i a^2\right)$$
$$\langle s^i \rangle = 8\mu_i \left(F_i \frac{7}{5} + G_i a^2\right) = 2\mu_i \langle e^i \rangle \tag{5}$$

1.2.2. *Calculation of the volume bulk modulus of a composite once the elastic properties of the components are known (direct mathematical problem)*

We shall first take into account the dilatational component, but analogous treatment will follow for the deviatoric component.

The following homogenizing criterion will be considered as a real axiomatic definition: *The linear invariants of a composite material's deformation $\langle \varepsilon^0 \rangle$ and stress $\langle \sigma^0 \rangle$ are obtained as weight averages of the deformation invariants (series model) and of the stress invariants (parallel model) of the component materials (i)*), the weight being the volume fraction ϕ_i of the i–th component:

$$\langle \varepsilon^0 \rangle \equiv \sum_{i=1}^{N} \phi_i \langle \varepsilon^i \rangle$$
$$\langle \sigma^0 \rangle \equiv \sum_{i=1}^{N} \phi_i \langle \sigma^i \rangle \tag{6}$$

For the case of a binary system – matrix–(m)/particle–(p)–one obtains, fig.1:

$$\langle \varepsilon^0 \rangle \equiv (1-\phi)\langle \varepsilon^m \rangle + \phi \langle \varepsilon^p \rangle$$
$$\langle \sigma^0 \rangle \equiv (1-\phi)\langle \sigma^m \rangle + \phi \langle \sigma^p \rangle \tag{7}$$

It is worth noting that the averages referred to the matrix are also evaluated over a volume that is exactly equal to that occupied by the inclusions. This homogenizing criterion takes into account the contribution of the homogeneous matrix/matrix stress field, with weight $(1-\phi)$, and the heterogeneous matrix/particle stress field, with weight ϕ . After substitution of the averages, both members of relationship (7) become explicitly:

$$H_0 = (1-\phi) \cdot H_m + \phi \cdot H_p$$
$$\kappa_0 H_0 = (1-\phi) \cdot \kappa_m H_m + \phi \cdot \kappa_p H_p \tag{8}$$

From these relationships, member–to–member division gives the bulk modulus of the composite:

$$\kappa_0 = \frac{(1-\phi)\cdot \kappa_m + \phi \cdot \kappa_p \dfrac{H_p}{H_m}}{(1-\phi) + \phi \dfrac{H_p}{H_m}} \tag{9}$$

The values assumed by the constants H_m H_p are obtained as solution of Goodier's problem[3]

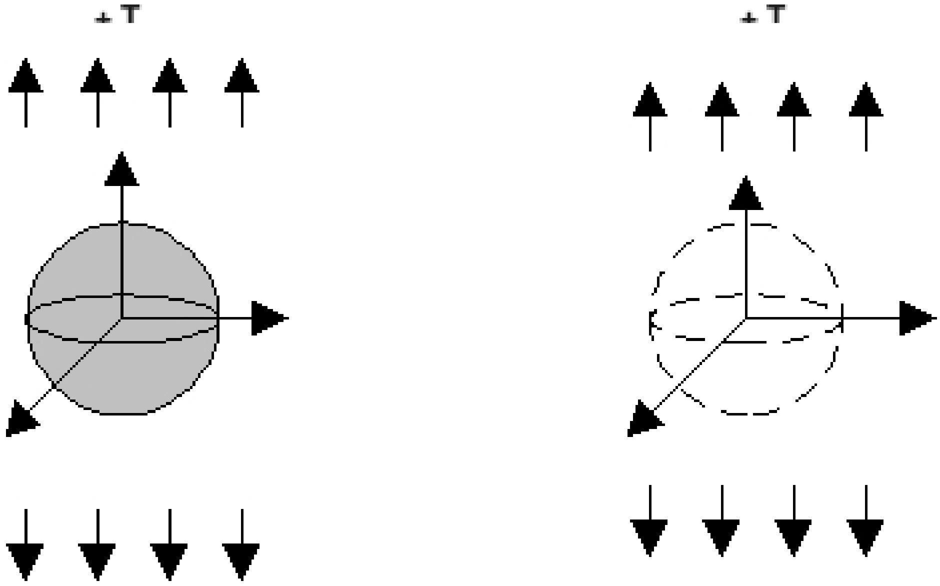

Figure 1. *Geometry of stress distribution* Matrix–(m)/Particle–(p) interaction and Matrix–(m)/Matrix–(m) interaction

for both matrix/matrix and matrix/particle interaction (υ being the Poisson ratio):

$$\begin{aligned} H_m &= \frac{2}{3} \cdot \frac{1-2\upsilon_m}{1+\upsilon_m} \cdot \frac{T}{4\upsilon_m} \\ H_p &= \frac{1-\upsilon_m}{1+\upsilon_m} \cdot \frac{2\,(1-2\upsilon_p)}{2\,(1-2\upsilon_p) + \dfrac{\mu_p}{\mu_m}\,(1+\upsilon_p)} \cdot \frac{T}{4\mu_m} \end{aligned} \tag{10}$$

Making use of the definition of volume bulk modulus κ, the ratio between the two constants can be written as:

$$\frac{H_p}{H_m} = \frac{3\kappa_m + 4\mu_m}{3\kappa_p + 4\mu_m} \tag{11}$$

Substituting this ratio in equation (9) gives rise to the celebrated Kerner formula for the volume bulk modulus κ_0 of a binary matrix–(m)/particle–(p) system:

$$\kappa_0 = \frac{\dfrac{(1-\phi)\cdot\kappa_m}{3\kappa_m + 4\mu_m} + \dfrac{\phi\cdot\kappa_p}{3\kappa + 4\mu_m}}{\dfrac{(1-\phi)}{3\kappa_m + 4\mu_m} + \dfrac{\phi}{3\kappa_m + 4\mu_m}} \tag{12}$$

This formula solves the direct mathematical problem of predicting the volume bulk modulus of a binary composite material– where a given volumetric fraction of inclusions is randomly dispersed within the matrix – once the elastic properties of both matrix and spherical inclusions are known. Another form of the same Kerner formula, but which is more convenient for numerical

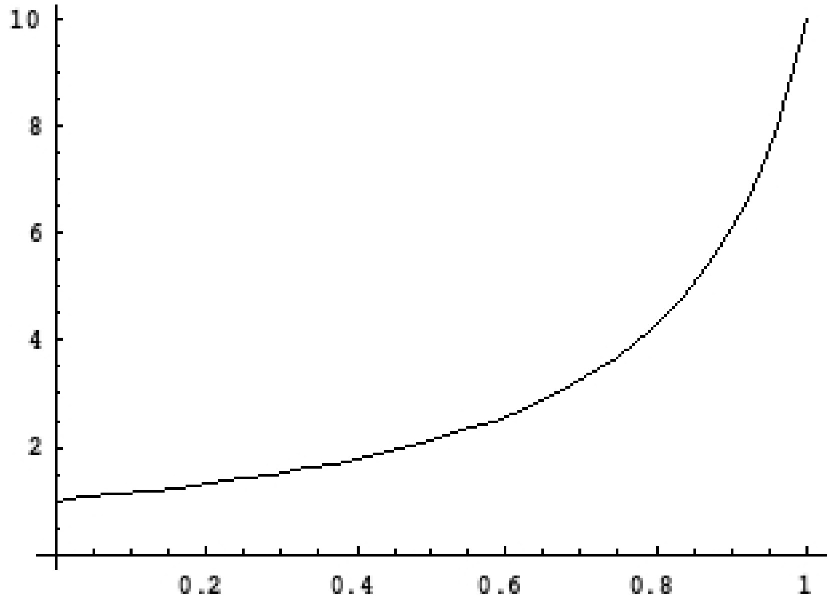

Figure 2. Composite bulk modulus as a function of the volume fraction of inclusions

evaluations, is the following:

$$\frac{\kappa_0}{\kappa_m} = 1 + \frac{\left(\frac{\kappa_p}{\kappa_m}\right) \cdot \phi}{1 + \left(\frac{\kappa_p}{\kappa_m} - 1\right) \cdot (1 - \phi) \cdot a} \tag{13}$$

where:

$$\alpha = \frac{1}{3} \cdot \frac{1 + \upsilon_m}{1 - \upsilon_m} \tag{14}$$

The behavior of the ratio $\frac{\kappa_0}{\kappa_m}$, as a function of the volume fraction of inclusions ϕ, is plotted in fig. 2 for a given value of the reinforcing ratio $\frac{\kappa_p}{\kappa_m} = 10$ and of the Poisson ratio $\upsilon_m = \frac{1}{3}$.

1.2.3. *Calculation of the shear modulus of a composite once the elastic properties of the components are known (direct mathematical problem)*

Let us now take into account the deviatoric component of the stress distribution and impose the following homogenizing criterion to be considered as a real axiomatic definition: –*The linear invariants of deformation $\langle e^0 \rangle$ and stress $\langle s^0 \rangle$ of the composite material are obtained as weight average of the deformation invariants (series model) and of the stress invariants (parallel model) of the component materials (i)), the weight being the volume fraction ϕ of the i–th component (in symbols):*

$$\begin{aligned} \langle e^0 \rangle &\equiv \sum_{i=1}^{N} \phi_i \langle e^i \rangle \\ \langle s^0 \rangle &\equiv \sum_{i=1}^{N} \phi_i \langle e^i \rangle \end{aligned} \tag{15}$$

For the case of a binary system, matrix–(m)/particle–(p) one obtains:

$$\begin{aligned} \langle e^0\rangle &\equiv (1-\phi)\,\langle e^m\rangle + \phi\langle e^p\rangle \\ \langle s^0\rangle &\equiv (1-\phi)\,\langle s^m\rangle + \phi\langle s^p\rangle \end{aligned} \tag{16}$$

It is worth noting that the averages referred to the matrix are also evaluated over a volume that is exactly equal to that occupied by the inclusions. This homogenizing criterion takes into account the contribution of the homogeneous matrix/matrix stress field, with weight $(1-\phi)$, and the heterogeneous matrix/particle stress field, with weight ϕ. After the averages have been substituted, both members of the relationship (16) become:[3]

$$\begin{aligned} \left(F_0+\frac{7}{5}G_0a^2\right) &= (1-\phi)\cdot\left(F_m+\frac{7}{5}G_ma^2\right)+\phi\cdot\left(F_p+\frac{7}{5}G_pa^2\right) \\ \mu_0\left(F_0+\frac{7}{5}G_0a^2\right) &= (1-\phi)\,\mu_m\cdot\left(F_m+\frac{7}{5}G_ma^2\right)+\phi\mu_p\cdot\left(F_p+\frac{7}{5}G_pa^2\right) \end{aligned} \tag{17}$$

From these relationships, member–to–member division gives the composite shear modulus:

$$\mu_0=\frac{(1-\phi)\cdot\mu_m+\phi\cdot\mu_p\dfrac{F_p+\frac{7}{5}G_pa^2}{F_m+\frac{7}{5}G_ma^2}}{(1-\phi)+\phi\cdot\dfrac{F_p+\frac{7}{5}G_pa^2}{F_m+\frac{7}{5}G_ma^2}} \tag{18}$$

The values assumed by the constants F_m F_m F_p G_p are obtained as solution of Goodier's problem[3] for both matrix/matrix and matrix/particle interaction.
In the homogeneous case, e.g. matrix/matrix interaction, we obtain:

$$\begin{aligned} F_m &= \frac{1}{3}\cdot\frac{T}{4\mu_m} \\ G_m &= 0 \end{aligned} \tag{19}$$

While, in the heterogeneous case, e.g. matrix/particle interaction (υ is the Poisson ratio) we obtain:

$$\begin{aligned} F_p &= \frac{T}{8\mu_m}\cdot\frac{10\,(1-\upsilon_m)\,\mu_m}{(7-5\upsilon_m)\,\mu_m+(8-10\upsilon_m)\,\mu_p} \\ G_p &= 0 \end{aligned} \tag{20}$$

The fortuitous circumstance that the G constants are both equal to zero, $G_m = G_p = 0$,implies that prediction of the composite shear modulus is independent of the size of the spherical inclusions and depends only on the inclusion volume fraction as was the case for the previously derived bulk modulus.

Therefore:

$$\mu_0 = \frac{(1-\phi)\cdot\mu_m + \phi\cdot\mu_p\dfrac{F_p}{F_m}}{(1-\phi)+\phi\cdot\dfrac{F_p}{F_m}} \tag{21}$$

Substitution of the ratio between F_p and F_m in equation (21) gives rise to the Kerner's celebrated formula for the shear modulus μ_0 of the binary matrix–(m)/particle–(p) system:

$$\mu_0 = \mu_m \cdot \frac{\dfrac{(1-\phi)}{15\,(1-\upsilon_m)} + \dfrac{\phi\cdot\mu_p}{(7-5\upsilon_m)\,\mu_m + (8-10\upsilon_m)\,\mu_p}}{\dfrac{(1-\phi)}{15\,(1-\upsilon_m)} + \dfrac{\phi\cdot\mu_m}{(7-5\upsilon_m)\,\mu_m + (8-10\upsilon_m)\,\mu_p}} \tag{22}$$

This formula solves the direct mathematical problem of predicting the shear modulus of a binary composite material – where a given volumetric fraction of inclusions is randomly dispersed in the matrix – once the elastic properties of both matrix and spherical inclusions are known. Another form of the same Kerner formula, but which is more convenient for numerical evaluations, is the following:

$$\frac{\mu_0}{\mu_m} = 1 + \frac{\left(\dfrac{\mu_p}{\mu_m}-1\right)\cdot\phi}{1+\left(\dfrac{\mu_p}{\mu_m}-1\right)\cdot(1-\phi)\cdot\beta} \tag{23}$$

where:

$$\beta = \frac{2}{15}\cdot\frac{4-5\cdot\upsilon_m}{1-\upsilon_m} \tag{24}$$

The behavior of the ratio $\dfrac{\mu_0}{\mu_m}$, as a function of the volume fraction of inclusions ϕ, is plotted

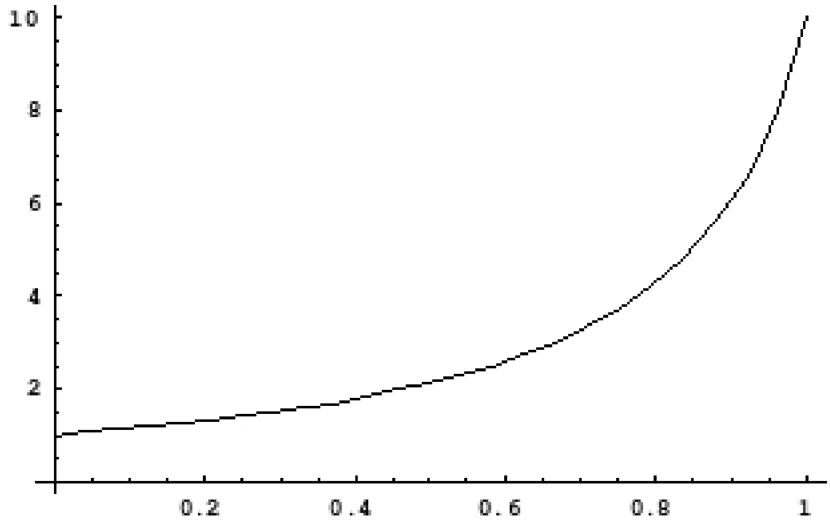

Figure 3. Shear modulus as a function of the volume fraction of inclusions.

in fig.3 for a given value of the reinforcing ratio $\dfrac{\mu_p}{\mu_m} = 10$ and of the Poisson ratio $\upsilon_m = \dfrac{1}{3}$.

Remark Kerner's formula for the shear modulus μ_0 assumes a particularly simple form in the limit case of a composite with a highly diluted concentration of rigid spheres in a compressible liquid matrix having elastic constants:

$$\begin{aligned} \upsilon_m &= \frac{1}{2} \\ \kappa_m &\leq +\infty \end{aligned} \tag{25}$$

In such a case we obtain the asymptotic formula:

$$\begin{cases} \phi \rightarrow 0 \\ \dfrac{\mu_p}{\mu_0} \rightarrow +\infty \\ \mu_0 \cong \mu_m \left(1 + \dfrac{\phi}{\beta}\right) = \mu_m \left(1 + \dfrac{5}{2} \cdot \phi\right) \end{cases} \tag{26}$$

This is Einstein's classical equation for the viscosity of a liquid filled with a diluted suspension of rigid spheres. This limiting result further supports the consistency of the Kerner's formulae.

1.2.4. *Prediction of the elastic moduli of the spherical inclusions once the elastic properties of both the composite and the matrix are known (inverse mathematical problem)*

Generally, inverse problems are more complex than the associated direct problem and this is mainly because of solution numerical instability and insufficient accuracy of the input data which can lead to unacceptable error propagation with the inverse solution.

Kerner's celebrated equation for the bulk modulus and the shear modulus of a composite can be set into an equation of the general form:

$$\frac{M_0}{M_m} = \frac{(1 + A \cdot B \cdot \phi)}{1 - B \cdot \phi} \tag{27}$$

In this equation M_0 and M_m are the elastic modules of the composite material and of the matrix, respectively, while ϕ is the volume fraction of the filler phase.
The constant B takes into account the relative modules of the filler and matrix phases, and it is defined as:

$$B = \frac{M_p/M_m - 1}{M_p/M_m - A} \tag{28}$$

while A is a constant that takes into account such factors as filler geometry and matrix Poisson ratio. For the bulk modulus of the composite, it assumes the following value:

$$\begin{cases} A &= \dfrac{1-\alpha}{\alpha} \\ \alpha &= \dfrac{1}{3} \cdot \dfrac{1+\upsilon_m}{1-\upsilon_m} \end{cases} \tag{29}$$

For the shear modulus of the composite, it assumes the following value:

$$\begin{cases} A &= \dfrac{1-\beta}{\beta} \\ \alpha &= \dfrac{2}{15}\cdot\dfrac{4-5\upsilon_m}{1-\upsilon_m} \end{cases} \tag{30}$$

The elastic properties of the spherical inclusions can be identified by analyzing the experimental elastic properties of several composites obtained at increasing inclusion concentrations through linear regression of Kerner's equation (30) after rearrangement in the form:

$$\left[\frac{1}{\phi}\left(\frac{M_0}{M_m}\right)\right]_i = A\cdot B + B\cdot\left[\frac{M_0}{M_m}\right]_i \tag{31}$$

In this equation, the terms in square brackets are known from the experiments; consequently equation (31) represents a straight line of the form:

$$[y]_i\, a + b\cdot[x]_i \tag{32}$$

The data best fit to this linear equation provides $a = A\cdot B$ as the intercept and $b = B$ as the slope, $i = 1, 2, \ldots n$ being the number of available experiments.

The final identification formula for the elastic property of the spherical particle is the following equation:

$$M_p = M_m\frac{1 + A\cdot B}{1-B} \tag{33}$$

According to the experimental setup, the elastic property could be the bulk modulus or the shear modulus of the spherical particle.

Moreover, from the best fit value of A, it is also possible to estimate the Poisson ratio of the matrix:

$$\upsilon = \frac{2-A}{4+A} \tag{34}$$

From the theory of elasticity we know that the Poisson ratio can assume values in the range:

$$-q \le \upsilon \le \frac{1}{2} \tag{35}$$

The physical meaning of the above limitation can easily be deduced by defining the Poisson ratio in terms of the two other elastic constants,(κ, μ) , e.g. the bulk modulus and the shear modulus:

$$\upsilon = \frac{1}{2}\cdot\frac{3\cdot\kappa - 2\cdot\mu}{3\cdot\kappa+\mu} \tag{36}$$

The upper limit $\upsilon = \dfrac{1}{2}$ implies:

1. either $\mu = 0$, which means ideal non–viscous fluid,
2. or $\kappa \to \infty$, which means ideal incompressible fluid.

The lower limit $\upsilon = -1$ implies:

3 either $\kappa = 0$, which means ideal foam–like fluid.
4 or $\mu \to \infty$, which means ideal fiber reinforced material.

1.2.5. *Discussion*

Some peculiar features of the Kerner formulae are worth mentioning.

i) The micro–mechanic basis of the Kerner model requires knowledge of the matrix/particle interaction in terms of stress distribution. To this end, Kerner's model makes use of the calculation, obtained by J.N. Goodier,[3] of the stress distribution of an isolated spherical inclusion embedded in an infinite matrix.
From a practical point of view, it is necessary to specify the meaning of the term "isolated particle". From the Saint Venant principle, we already know that the disturbing effect of any small spherical inclusion is confined to the neighborhood of the inclusion itself. In fact, from the Goodier solutions themselves, it appears that, at a distance of one particle radius from the elastic inclusion, the stress distribution, which would be uniform without inclusion, is not modified by more than a few percentage units. This distance could be the repeating unit at which particles behave as though they were isolated from each other. As a consequence, the predictions drawn from the Kerner model can be considered valid for a volume fraction of the inclusions up to $\phi_{\max}$, estimated in compliance with the inter–particle distance as above:

$$\phi_{\max} = \frac{\frac{4}{3}\pi \cdot r^3}{\frac{4}{3}\pi \cdot \left(\frac{3}{2}r\right)^3} \cong 30\ldots\% \tag{37}$$

ii) It is a fortuitous circumstance that the size of the spherical inclusions does not appear explicitly in the formulae used to predict the composite bulk and shear modulus. Consequently Kerner's formulae are not affected by either the particle size or the particle size distribution of the inclusions.

2. Appendix A: The linear elastic problem by J.N. Goodier

Our aim is to solve the indefinite linear elastic equilibrium equations for a heterogeneous isotropic body, made up of an infinite matrix in which a unique, perfectly adhering spherical inclusion is embedded. The composite body is subjected to uniform, uniaxial stress at infinity. The elastic constants of the matrix, $(\kappa_m; \mu_m)$, and of the spherical particle, $(\kappa_p; \mu_p)$, are known. The analytical solution to this problem of elasticity is obtained in three subsequent steps:

2.1 Analytical solution, using spherical coordinates, of the three dimensional equilibrium equations for a homogeneous elastic body undergoing uniform uniaxial traction T at infinity.
2.2 Specialization of this general solution to a region external to a spherical surface, e.g. the matrix, and to a region internal to a spherical surface, e.g. the particle.
2.3 Solution of the *boundary value problem* obtained by imposing the continuity of displacements and radial tractions at the interface between matrix and particle in order to maintain the heterogeneous body in equilibrium.

The boundary value problem generates a linear algebraic system of 6 equations in 6 unknowns which completely[3] determines the elastic problem of stress distribution within the heterogeneous matrix/inclusion body.

2.1. *Particular three-dimensional solutions of the indefinite equilibrium equations*

The three–dimensional Cartesian components of the displacement $\overline{u} = (u, v, w)$ of a linear elastic body in equilibrium, free of any body forces, satisfy the three equations:[6]

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \text{div} \cdot \overline{u} + (1 - 2v)\, \nabla^2 \cdot \overline{u} = \overline{0} \tag{1}$$

where

$$(1 - 2v) = \frac{\lambda}{\lambda + \mu} \tag{2}$$

is the relationship between the *Poisson* ratio v and the *Lamè* constants (λ, μ),

$$\text{div} \cdot \overline{u} \equiv \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{3}$$

is the definition of divergence of vector $\overline{u}$ and

$$\nabla^2 \cdot \overline{u} \equiv \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}\right) \cdot (u, v, w) \tag{4}$$

Some special solutions to the above equations can be obtained as will be described below. Given a vector $\overline{u}$ that satisfies equation (1), displacement divergence is a harmonic function

$$\nabla^2\, (\text{div} \cdot \overline{u}) = 0 \tag{5}$$

and, at the same time, the displacement is a bi–harmonic function

$$\nabla^2 \left(\nabla^2 \overline{u}\right) = 0 \tag{6}$$

Particular solutions to equation (1) can be obtained starting from equations (5) and (6), e.g. taking into account vectors $\overline{u}$ such that div $\cdot\overline{u}$ and $\nabla^2\overline{u}$ are harmonic functions. This special type of solution was first obtained by Lord Kelvin.

The three–dimensional solutions of the Laplace equation, in polar coordinates, are called spherical harmonic functions. When written in cartesian coordinates (x, y, z), such functions are homogeneous functions of positive or negative integer degree n. A generic spherical harmonic function V_n, written in polar coordinates (r, ϑ, ψ) reads as

$$V_n = r^n S_n(\vartheta, \psi) \tag{7}$$

where S_n is a function of the coordinates (ϑ, ψ) only, but is independent of the radius $\sqrt{x^2 + y^2 + z^2}$. The factor S_n is called the surface spherical harmonic. At this point we are now ready to build two classes of particular solutions, e.g. ϕ *-solutions* and ω–solutions.

2.1.1. *Particular ϕ–solutions*

It is easy to check that a displacement $\overline{u}$ of the form:

$$\overline{u} = \left(\frac{\partial \phi_n}{\partial x}, \frac{\partial \phi_n}{\partial y}, \frac{\partial \phi_n}{\partial z}\right) \tag{8}$$

satisfies equation (1) if ϕ_n is an integer harmonic function of degree n. In fact, it ensues that:

$$\begin{cases} \text{div} \cdot\overline{u} = \text{div} \cdot \text{grad}\, \phi_n = \nabla^2 = 0 \\ \nabla^2\overline{u} = \nabla^2 \,\text{grad}\, \phi_n = \text{grad} \cdot\nabla^2\phi_n = 0 \end{cases} \tag{9}$$

2.1.2. *Particular ω–solutions*

A second type of solution to equation (1) is supplied by displacements of the form:

$$\overline{u} = r^2 \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \omega_n + (x, y, z) \cdot \omega_n \cdot \alpha_n \tag{10}$$

where ω_n is an integer harmonic function of degree n.

In this case equation (1) is satisfied by virtue of the properties:

$$\begin{cases} \nabla^2(x\omega_n) = 2 \cdot \dfrac{\partial \omega_n}{\partial x}, \dots \\ \nabla^2(r^m\omega_n) = m \cdot (m + 2n + 1) \cdot r^{m-2}\omega_n \end{cases} \tag{11}$$

and the *Euler* relationship, which holds for every homogeneous function:

$$x \cdot \frac{\partial \omega_n}{\partial x} + y \cdot \frac{\partial \omega_n}{\partial y} + z \cdot \frac{\partial \omega_n}{\partial z} = n \cdot \omega_n \tag{12}$$

provided that α_n is given by the value:

$$\alpha_n = -2 \cdot \frac{3n + 1 - 2 \cdot (2n + 1) \cdot v}{n + 5 - 4v} \tag{13}$$

Therefore, it follows that:

$$\begin{cases} \operatorname{div} \cdot \overline{u} = [2n + (3 + n) \cdot \alpha_n]\, \omega_n \\ \nabla^2 \overline{u} = 2 \cdot [2n + 1 + \alpha_n] \cdot \operatorname{grad} \omega_n \end{cases} \tag{14}$$

To obtain solutions symmetrical around an axis –e.g. the OZ axis of application of the uniaxial tension T – we may use *zonal spherical harmonics*, ϕ_n, ω_n, that is, functions that are independent of the ψ coordinate. In this way, the displacements will be limited to meridian planes having a component u_r along the radius r and a component u_ϑ at right angle to the radius, in the direction of increasing ϑ.

The particular solutions given by Lord Kelvin are written in Cartesian coordinates; however, operating in polar coordinates is quite convenient as this makes use of formulae for vector transformation from cartesian to curvilinear coordinates.[7]

$$\begin{cases} u_r = u_x \sin\vartheta \cos\psi + u_y \sin\vartheta \cos\psi + u_z \cos\vartheta \\ u_\vartheta = u_x \sin\vartheta \cos\psi + u_y \cos\vartheta \sin\psi + u_z \sin\vartheta \\ u_\psi = -u_x \sin\psi + u_y \cos\psi \end{cases} \tag{15}$$

Therefore, for the *ϕ–solutions*, we obtain the following displacements:

$$\begin{cases} u_r = \dfrac{\partial \phi_n}{\partial r} \\ u_\vartheta = \dfrac{1}{r} \dfrac{\partial \phi_n}{\partial \vartheta} \\ u_\psi = 0 \end{cases} \tag{16}$$

While for the *ω–solutions* we obtain the following displacements:

$$\begin{cases} u_r = r^2 \dfrac{\partial \omega_n}{\partial r} + \alpha_n r \omega_n \\ u_\vartheta = r \dfrac{\partial \omega_n}{\partial \vartheta} \\ u_\psi = 0 \end{cases} \tag{17}$$

As functions of the displacements, the non zero components of the strains, in polar coordinates, assume the form:

$$\begin{cases} \epsilon_{rr} = \dfrac{\partial u_r}{\partial r} \\ \epsilon_{\vartheta\vartheta} = \dfrac{1}{r}\dfrac{\partial u_\vartheta}{\partial \vartheta} + \dfrac{u_r}{r} \\ \epsilon_{\psi\psi} = \operatorname{div} \overline{u} - (\epsilon_{rr}\epsilon_{\vartheta\vartheta}) \\ \epsilon_{r\vartheta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \vartheta} + r\dfrac{\partial}{\partial r}\left(\dfrac{u_\vartheta}{r}\right) \end{cases} \tag{18}$$

while, the stress–strain relationship, e.g. *Hooke's law*, accordingly becomes:

$$\begin{cases} (\sigma_{rr}, \sigma_{\vartheta\vartheta}, \sigma_{\psi\psi}) = 2\mu \cdot \left[\dfrac{v}{1-2v} \operatorname{div} \cdot \overline{u} + (\epsilon_{rr}, \epsilon_{\vartheta\vartheta}, \epsilon_{\psi\psi})\right] \\ \sigma_{r\vartheta} = \mu \cdot \epsilon_{r\vartheta} \end{cases} \tag{19}$$

2.2. *Indefinite solution for the spherical particle (p) and for the matrix (m)*

Having obtained, in the previous section, some particular solutions for the three–dimensional, indefinite equilibrium equation, it is relatively easy to specialize such solutions to a region inside a spherical surface, e.g. the particle, and to a region outside a spherical surface, e.g. the matrix.

The solution for the particle is obtained as linear combination of three particular solutions:

$$\begin{cases} \phi_2 = \dfrac{1}{2} r^2 \left(3\cos^2\vartheta - 1\right) \cdot 2F_0 \\ \omega_2 = \dfrac{1}{2} r^2 \left(3\cos^2\vartheta - 1\right) \cdot \dfrac{2}{3}(7-4v) \cdot G_0 \\ \omega_0 = -\dfrac{1}{2} \cdot \dfrac{5-4v}{1-2v} \cdot H_0 \end{cases} \tag{20}$$

The constants F_0, G_0, H_0 will be determined by the boundary conditions (see sect. 2.3). Such a choice for the three particular solutions satisfies the requirement of regularity at the origin and, in all three solutions, it vanishes at $r = 0$.

The solution for the matrix is obtained as a linear combination of three particular solutions, congruent with the previous three:

$$\begin{cases} \phi_{-1} = \dfrac{1}{r} A_2 \\ \phi_{-3} = \dfrac{1}{2r^3}\left(3\cos^2\vartheta - 1\right) \cdot 4B_2 \\ \omega_{-3} = -\dfrac{1}{2r^3}\left(3\cos^2\vartheta - 1\right) \cdot \dfrac{4}{3} C_2 \end{cases} \tag{21}$$

This choice satisfies the requirement of regularity at infinity. As before, the three constants A_2, B_2, C_2 are determined by the boundary conditions (see sect. 2.3).

To complete this analysis, the *far field* conditions must be specified. The composite material undergoes uniaxial tension which is uniform at infinity. In cartesian coordinates, the stress tensor possesses only one non zero component, in the same OZ direction, and this amounts to:

$$\sigma_{zz}^{\infty} = +T \tag{22}$$

Due to the spherical geometry of the inclusion, the applied stress tensor must be represented in spherical coordinates. Using the appropriate tensor transformation formulae,[9] in spherical coordinates, at infinity, the non–zero stress tensor components become:

$$\begin{cases} \sigma_{rr}^{\infty} = +\frac{\pi}{2}\left(1+\cos 2\vartheta\right) \\ \sigma_{\vartheta\vartheta}^{\infty} = +\frac{\pi}{2}\left(1-\cos 2\vartheta\right) \\ \sigma_{\psi\psi}^{\infty} = 0 \\ \sigma_{r\vartheta}^{\infty} = -\frac{\pi}{2}\sin 2\vartheta \end{cases} \tag{23}$$

Likewise, at infinity, the strains assume the values:

$$\begin{cases} u_r^{\infty} = +\frac{\pi \cdot r}{4\mu} \cdot \left(\frac{1-v}{1+v} + \cos 2\vartheta\right) \\ u_{\vartheta}^{\infty} = -\frac{\pi \cdot r}{4\mu} \cdot \sin 2\vartheta \end{cases} \tag{24}$$

Using the *Saint Venant* principle, the stress distribution within the matrix is obtained by superpositioning the same components drawn from the stress distribution at infinity in the indefinite solution. In other words, the inclusion induces local disturbance in the stress distribution, disturbance that is confined to the vicinity of the inclusion itself. At a distance from the inclusion, the stress distribution in the matrix will return to uniform.

Now we are ready to build the indefinite solutions – e.g. displacements, stresses and strains for the matrix and the spherical particle – through a linear combination of the ϕ–solutions and ω–solutions. After calculations we obtain:

2.2.1. *The indefinite solution for the matrix (m)*

Displacements:

$$\begin{cases} u_r^{m} = -\frac{A}{r^2} - \frac{3B}{r^4} - \left(\frac{9B}{r^4} - \frac{5-4v_m}{1-2v_m} \cdot \frac{C}{r^2}\right) \cdot \cos 2\vartheta + \frac{T \cdot r}{4\mu_m}\left(\frac{1-v_m}{1+v_m} + \cos 2\vartheta\right) \\ u_{\vartheta}^{m} = -\left(\frac{6B}{r^4} + \frac{2C}{r^2}\right) \cdot \sin 2\vartheta - \frac{T \cdot r}{4\mu_m}\left(\sin 2\vartheta\right) \end{cases} \tag{25}$$

Strains:

$$\begin{cases} \epsilon^m_{rr} = +\dfrac{2A}{r^3} + \dfrac{12B}{r^5} + \left(\dfrac{36B}{r^5} - 2\cdot\dfrac{5-4v_m}{1-2v_m}\cdot\dfrac{C}{r^3}\right)\cdot\cos 2\vartheta \\ \qquad + \dfrac{T}{4\mu_m}\left(\dfrac{1-v_m}{1-v_m} + \cos 2\vartheta\right) \\ \epsilon^m_{\vartheta\vartheta} = -\dfrac{A}{r^3} - \dfrac{3B}{r^5} - \left(\dfrac{21B}{r^5} - \dfrac{1+4v_m}{1-2v_m}\cdot\dfrac{C}{r^3}\right)\cdot\cos 2\vartheta + \dfrac{T}{4\mu_m}\left(\dfrac{1-v_m}{1-v_m} + \cos 2\vartheta\right) \\ \epsilon^m_{\psi\psi} = -\dfrac{A}{r^3} - \dfrac{9}{r^5} - \dfrac{2C}{r^3} - \left(\dfrac{15B}{r^5} - \dfrac{3}{1-2v_m}\cdot\dfrac{C}{r^3}\right)\cdot\cos 2\vartheta + \dfrac{T}{4\mu_m}\left(\dfrac{1-v_m}{1-v_m} - 1\right) \\ \epsilon^m_{r\vartheta} = +\left(\dfrac{48B}{r^5} - 4\cdot\dfrac{1+v_m}{1-2v_m}\cdot\dfrac{C}{r^3}\right)\cdot\sin 2\vartheta - \dfrac{T}{4\mu_m}(\sin 2\vartheta) \end{cases} \tag{26}$$

Stress distribution:

$$\begin{cases} \sigma^m_{rr} = 2\mu_m\left[-\dfrac{A}{r^3} - \dfrac{3B}{r^5} - \dfrac{2v_m}{1-2v_m}\cdot\dfrac{C}{r^3} - \left(\dfrac{21B}{r^5} - \dfrac{C}{r^3}\right)\cdot\cos 2\vartheta + \dfrac{T}{4\mu_m}(1 - cos2\vartheta)\right] \\ \sigma^m_{\psi\psi} = 2\mu_m\left[-\dfrac{9B}{r^5} - 2\cdot\dfrac{1-v_m}{1-2v_m}\cdot\dfrac{C}{r^3} - \left(\dfrac{15B}{r^5} - 3\cdot\dfrac{C}{r^3}\right)\cdot\cos 2\vartheta\right] \\ \sigma^m_{r\vartheta} = 2\mu_m\left[\left(+\dfrac{24B}{r^5} - 2\cdot\dfrac{1+v_m}{1-2v_m}\cdot\dfrac{C}{r^3}\right)\cdot\sin 2\vartheta - \dfrac{T}{4\mu_m}\sin 2\vartheta\right] \end{cases} \tag{27}$$

2.2.2. *The indefinite solution for the spherical particle (p)*

Displacements:

$$\begin{cases} u^p_r = F\cdot r + 3F\cdot r\cdot\cos 2\vartheta + 2v_pG\cdot r^3\cos 2\vartheta + H\cdot r \\ u^p_\vartheta = -3F\cdot r\cdot\sin 2\vartheta - (7-4v_p)\cdot G\cdot r^3\sin 2\vartheta \end{cases} \tag{28}$$

Strains:

$$\begin{cases} \epsilon^p_{rr} = +F + 3F\cos 2\vartheta + 6v_pG\cdot r^2 + 18v_pG\cdot r^2\cos 2\vartheta + H \\ \epsilon^p_{\vartheta\vartheta} = +F - 3F\cos 2\vartheta + 2v_pG\cdot r^2 - 14\cdot(1-v_p)\cdot G\cdot r^2\cos 2\vartheta + H \\ \epsilon^p_{\psi\psi} = -2F - (7-6v_p)\cdot G\cdot r^2 - (7-10v_p)\cdot G\cdot r^2\cos 2\vartheta + H \\ \epsilon^p_{r\vartheta} = -2F\sin 2\vartheta - 2\cdot(7+2v_p)\cdot G\cdot r^2\sin 2\vartheta \end{cases} \tag{29}$$

Stress distibution:

$$\begin{cases} \sigma^p_{rr} = 2\mu_p\left[F + 3F\cos 2\vartheta - v_pG\cdot r^2 - 3v_pG\cdot r^2\cos 2\vartheta + \dfrac{1+v_p}{1-v_p}\cdot H\right] \\ \sigma^p_{\vartheta\vartheta} = 2\mu_p\left[F - 3F\cos 2\vartheta - 5v_pG\cdot r^2 - 7\cdot(2+v_p)\cdot G\cdot r^2\cos 2\vartheta + \dfrac{1+v_p}{1-2v_p}\cdot H\right] \\ \sigma^p_{\psi\psi} = 2\mu_p\left[-2F - (7+v_p)\cdot G\cdot r^2 - (7+11v_p)\cdot G\cdot r^2\cos 2\vartheta + \dfrac{1+v_p}{1-2v_p}\cdot H\right] \\ \sigma^p_{r\vartheta} = 2\mu_p\left[-3F\sin 2\vartheta - (7+2v_p)\cdot G\cdot r^2\sin 2\vartheta\right] \end{cases} \tag{30}$$

2.3. *Solution of the boundary value problem*

In physical terms, for the matrix(m)/particle(p) composite body to remain in equilibrium, there must be a balance between the displacements and radial tractions of the two adherent bodies at the surface of the common boundary, $(r = a)$, between matrix and inclusion. In mathematical terms, the continuity condition at the interface generates a so–called *boundary value problem*. In symbols:

$$\begin{cases} u_r^p(a) = u_r^m(a) \\ u_{r\vartheta}^p(a) = u_{r\vartheta}^m(a) \\ \sigma_{rr}^p(a) = \sigma_{rr}^m(a) \\ \sigma_{r\vartheta}^p(a)\sigma_{r\vartheta}^m(a) \end{cases} \tag{31}$$

Due to the fact that u_r and σ_{rr} are functions of the form $(\alpha + \beta \cdot \cos 2\vartheta)$ and the equations must hold for every value of the angle ϑ, the four conditions of continuity at the common interface, $(r = a)$, actually generate 6 equations. Consequently we have a linear system of 6 equations in 6 unknowns A, B, C, F, G, H. Explicitly the conditions (31) become:

$$\begin{aligned}
&+F \quad + 2v_pG \cdot a^2 \quad +H = -\frac{A}{a^3} - \frac{3B}{a^5} \quad + \frac{T}{4\mu_m} \cdot \frac{1-v_m}{1-v_m} \\
&3F \quad - (7-4v_p) \cdot G \cdot a^2 \quad = \quad -\frac{9B}{a^5} \quad + \frac{5-4v_m}{1-2v_m} \cdot \frac{C}{a^3} + \frac{T}{4\mu_m} \\
&-3F \quad + 6v_pG \cdot a^2 \quad = \quad = -\frac{6B}{a^5} \quad -\frac{2C}{a^3} + \frac{T}{4\mu_m} \\
&+\eta F \quad - \eta v_pG \cdot a^2 \quad + \eta\frac{1+v_p}{1-2v_p} \cdot H = \frac{2A}{a^3} + \frac{12B}{a^5} \quad - \frac{2v_m}{1-2v_m} \cdot \frac{C}{a^3} \quad + \frac{T}{4\mu_m} \\
&+\eta 3F \quad - \eta 3v_pG \cdot a^2 \quad = \quad \frac{36B}{a^5} - 2\frac{5-v_m}{1-2v_m} \cdot \frac{C}{a^5} \quad + \frac{T}{4\mu_m} \\
&+\eta 3F \quad - \eta(7+2v_p) \cdot G \cdot a^2 \quad = \quad \frac{24B}{a^5} - 2\frac{1+v_m}{1-2v_m} \cdot \frac{C}{a^3} \quad - \frac{T}{4\mu_m}
\end{aligned}$$

where $\eta = \mu_p/\mu_m$.

Construction of an occurrence matrix for the unknowns in the equations will prove useful in establishing a strategy for analytical solution of the system. In this case, the occurrence matrix for the 6 unknowns A, B, C, F, G, H appearing in the six equations, numbered in order $(1) \div (6)$, is the (6×6)–square matrix: The first operation required is to partition the initial system into

	F	G	H	A	B	C
1	x	x	x	x	x	
2	x	x			x	x
3	x	x			x	x
4	x	x	x	x	x	x
5	x	x			x	x
6	x	x			x	x

sub–systems of lower order. This result can be obtained by proper exchange of rows or columns in the occurrence matrix. In our case we obtain: The permutated occurrence matrix now clearly shows that there are two sub–systems to be solved in sequence: the first made up of equations

	F	G	H	A	B	C
2	x	x	x	x		
3	x	x	x	x		
5	x	x	x	x		
6	x	x	x	x		
1	x	x	x		x	x
4	x	x	x	x	x	x

(2), (3), (5), (6) referring only to unknowns (F, G, B, C) and the second made up of equations (1), (4) for the real determination of unknowns (H, A).

After patient calculations the real value of the six unknowns was obtained, as follows:

$$\begin{cases} \dfrac{A}{a^3} = -\dfrac{T}{8\mu_m} \cdot \dfrac{3 \cdot (1-\eta) + 10 \cdot (1-v_m)}{(7-5v_m) + \eta \cdot (8-10v_m)} \\ \qquad + \dfrac{T}{4\mu_m} \cdot \dfrac{(1-v_m)}{(1+v_m)} \cdot \dfrac{\eta \cdot (1-v_p)}{2 \cdot (1-2v_p) + \eta \cdot (1+v_p)} \\ \dfrac{B}{a^5} = +\dfrac{T}{8\mu_m} \cdot \dfrac{(1-\eta)}{(7-5v_m) + \eta \cdot (8-10v_m)} \\ \dfrac{C}{a^3} = +\dfrac{T}{8\mu_m} \cdot \dfrac{5 \cdot (1-2v_m) \cdot (1-v_m)}{(7-5v_m) + \eta \cdot (8-10v_m)} \\ F = +\dfrac{T}{8\mu_m} \cdot \dfrac{10 \cdot (1-v_m)}{(7-5v_m) + \eta \cdot (8-10v_m)} \\ G \cdot a^2 = 0 \\ H = +\dfrac{T}{4\mu_m} \cdot \dfrac{(1-v_m)}{(1+v_m)} \cdot \dfrac{2 \cdot (1-2v_p)}{2 \cdot (1-2v_p) + \eta \cdot (1+v_p)} \end{cases} \tag{32}$$

This noteworthy analytical result was first obtained by J.N. Goodier[3] . It allows the complete analytical calculation of the stress distribution in the matrix/inclusion composite and was the basis for introduction of the stress intensity factor concepts of fracture mechanics.

3. Appendix B: Average value of stress and deformation in a sphere

The analytical solution of *Goodier's problem* (see section 2.1 − 2.3) provides the complete analytical map of the stress distribution within the matrix/particle composite. The next step, required by *Kerner's model*, is to calculate the average stress $\langle \sigma^i_{zz} \rangle$ and deformation $\langle \epsilon^i_{zz} \rangle$ along OZ, the axis of application of the uniaxial tension T which is uniform at infinity. *Kerner's model* requires that the average value be extended to all the spheres of radius $r = a$ occupied by both the particle and the matrix. Given that the solution to *Goodier's problem* is written in spherical coordinates, it is necessary to return to Cartesian coordinates through use of the appropriate tensor transformation formulae.[9] For deformation we have:

$$\epsilon_{zz} = \epsilon_{rr} \cos^2 \vartheta + \epsilon_{\vartheta\vartheta} \sin^2 \vartheta - \epsilon_{r\vartheta} \frac{1}{2} \sin 2\vartheta \tag{1}$$

After substitution of deformations in the spherical coordinates we obtain:

$$\epsilon_{zz} = H + 4F + 2 \cdot Gr^2 \left[2v + 10v\cos^2\vartheta + (7 - 8v)\sin 2\vartheta\right] \tag{2}$$

When extended to the whole sphere of radius $r = a$, the average value, $\langle\epsilon^i_{zz}\rangle$, is:

$$\epsilon_{zz} = \frac{\iiint \epsilon_{zz} \cdot 2r^2 \sin\vartheta \cdot dr d\vartheta d\psi}{\frac{4}{3}\pi \cdot a^3} = H + 4F + \frac{28}{5}Ga^2 \tag{3}$$

Likewise, for the stress $\langle\sigma^i_{zz}\rangle$ along the principal direction OZ we obtain:[9]

$$\sigma_{zz} = \lambda \cdot (\epsilon_{rr} + \epsilon_{\vartheta\vartheta} + \epsilon_{\psi\psi}) + 2\mu \cdot \epsilon_{zz} \tag{4}$$

When extended to the whole sphere of radius $r = a$, the average value $\langle\sigma^i_{zz}\rangle$ is:

$$\sigma_{zz} = \frac{\iiint \sigma_{zz} \cdot 2r^2 \sin\vartheta \cdot dr d\vartheta d\psi}{\frac{4}{3}\pi \cdot a^3} = (3\lambda + 2\mu) \cdot H + 8\mu \cdot \left(F + \frac{7}{5}Ga^2\right) \tag{5}$$

where F, G, H are given constants as solutions of the *Goodier problem.*

Notation

Latin letters

a, Radius of the spherical inclusion
F_i, G_i, H_i, Constants of given value as solutions of *Goodier's problem*
i, Pedix = (m)–matrix; (p)–particle, (0)−composite
T, Uniaxial tension uniform at infinity

Greek letters

$\langle \epsilon^i_{zz} \rangle$, Average deformation along the OZ axis, within a sphere of radius $r = a$
$\langle \epsilon^i \rangle$, Dilatation component of deformation
$\langle e^i \rangle$, Deviatoric component of deformation
$\langle e^0 \rangle$, Linear invariant of deformation
$\langle \sigma^i_{zz} \rangle$, Average stress along the OZ axis, within a sphere of radius $r = a$
$\langle \sigma^i \rangle$, Dilatation component of stress
$\langle s^i \rangle$, Deviatoric component of stress
$\langle \sigma^0 \rangle$, Linear invariant of stress
ϕ_i, Volume fraction of inclusions
λ_i, *Lamè* constant
κ_i, Volume bulk modulus
μ_i, Shear modulus
v_i, *Poisson* ratio.

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