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THE FAMILY OF OPERATORS ASSOCIATED WITH A CAPITALIZATION LAW

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ABSTRACT. In the present paper we conduct a deep study on some basic concepts of financial mathematics. The financial mathematics involves several mathematical tools (even though at elementary level). But these tools are not organized in a theoretical framework. This circumstance affected badly on the development of financial mathematics and its applications. The aim of this paper is to present the building of a theoretical structure for the basic part of financial mathematics, all with elementary mathematics. In particular, the goal consists of describing the "evolution" of an amount of money as a time dependent-process, using the one parameter families of operators. The generalization to the case of the evolution of a portfolio of n goods is very natural.

In section 1, we define the capitalizations and the family of the capitalization factors of a capitalization.

In section 2, we give a definition of capitalization law, more general than the usual one. Furthermore, we introduce the main concept of the paper: the family of operators of a capitalization law.

In section 3, we connect the financial laws to the capitalizations.

In section 4, we introduce a subclass of capitalization laws (the self-actualizing capitalization laws). Finally, we characterize them and the classic separable financial laws in the new framework.

We have to note that almost all the proofs of the paper are straightforward computations. Nevertheless, we resolved to give them, we did it not so many for desire of completness, but, above all, to build a bridge beetwen the language and thoughts of economists and mathematicians.

1. Capitalizations on compact intervals

We bigin formalizing the basic object of financial mathematics, i.e., the financial evolution of a capital (or a debt).

Definition 1.1 (of capitalization on a compact interval). *Let $[t_0, T]$ be an interval of the real line. We define capitalization (resp. debt-evolution) on $[t_0, T]$ a function $K : [t_0, T] \to \mathbb{R}$ such that*

1) $K(t_0) > 0$ (resp. $K(t_0) < 0$),

2) K is not-decreasing (resp. not-increasing).

The real number t_0 is called the initial time of K and T the end-time of K. The difference $T - t_0$ is called the time-length of K, the interval $[t_0, T]$ is called the space of times of K, and the interval $[0, T - t_0]$ the space of time-lengths of K.

Moreover, $K(t_0)$ is called the initial capital and $K(t)$ the capital at t, for every $t \in \mathbb{R}$. □

The first concept that we want to examine is the capitalization factor.

Proposition 1.1 (on the capitalization factor). *Let K be a capitalization. Then there is a unique function $f_{t_0} : [0, T - t_0] \to \mathbb{R}$ such that $K(t) = K(t_0) f_{t_0}(t - t_0)$, for every $t \in [t_0, T]$.*

Proof. Existence. Since, $K(t_0) \neq 0$, we can put

$$f_{t_0}(h) = \frac{K(t_0 + h)}{K(t_0)}.$$

Obviously, it follows $K(t) = K(t_0) f_{t_0}(t - t_0)$.

Uniqueness. It's obvious. (For completeness. If $K(t) = K(t_0) g(t - t_0)$, for every t, one has

$$K(t_0) g(t - t_0) = K(t_0) f_{t_0}(t - t_0),$$

for every t, and thus $g = f_{t_0}$). ■

Definition 1.2 (of capitalization factor at the initial time). *The unique function of above proposition is called the capitalization factor of K at t_0.* □

Economic motivations. Usually, the function of the preceding definition is called, simply, the capitalization factor of the capitalization K. But, this name does not underline the dependence of the capitalization factor on the time t_0. This dependence on t_0 is strong as we shall see. On the contrary, in this paper, we desire to find concepts not depending on a particular instant of time. For this reason we give the following new definition.

Definition 1.3 (the capitalization factor at a time for a capitalization). *Let $t_1 \in [t_0, T]$, the unique function $f_{t_1} : [t_0 - t_1, T - t_1] \to \mathbb{R}$, such that, for every $t \in [t_0, T]$, $K(t) = K(t_1) f_{t_1}(t - t_1)$ is called the capitalization factor of K at t_1.* □

Note that, since K is not-decreasing, $K(t_1)$ is not zero for every $t_1 \in [t_0, T]$, so that the existence and uniqueness of f_{t_1} can be proved as in the case of f_{t_0}.

Economic interpretation. At this point, we have associated with a capitalization K a family of functions $f = (f_t)_{t \in [t_0, T]}$. This family gives us a gain in symmetry: it allow us to go back in past. In fact, even if $t_1, t_2 \in [t_0, T]$ and $t_1 \leq t_2$, we have

$$K(t_1) = K(t_2) f_{t_2}(t_1 - t_2).$$

Moreover, note that, until now, the space of time-displacements was defined only as the set $\mathrm{dom} f_{t_0} = [0, T - t_0]$, and the time-displacements were, consequently, only non negative. Instead, now we propose a new space of time-displacements for a capitalization, the following set

$$t \in [t_0, T] \cup \mathrm{dom} f_t = [t_0 - T, T - t_0] = \overline{B}(0, T - t_0).$$

In this way, the space of time-displacements is symmetric around 0.

Resuming, we can give the following new definition.

Definition 1.4 (the family of capitalization factors of a capitalization). *Let $[t_0, T]$ be an interval of the real line, let $K : [t_0, T] \to \mathbb{R}$ be a capitalization on $[t_0, T]$. $f = (f_t)_{t \in [t_0, T]}$, defined by*

$$f_{t_1} : [t_0 - t_1, T - t_1] \to \mathbb{R} : h \mapsto \frac{K(t_1 + h)}{K(t_1)},$$

for every $t_1 \in [t_0, T]$, is called the family of capitalization factors of K. Moreover the set

$$t \in [t_0, T] \cup \mathrm{dom} f_t = [t_0 - T, T - t_0] = \overline{B}(0, T - t_0),$$

is called the space of time-displacements of K. □

Now, naturally, arises the following question.

What is the relation between capitalization factors at different times ?

The following proposition and its corollary answer completely to the problem, giving us a transformation law.

Proposition 1.2. *Let K be a capitalization and $f = (f_t)_{t \in [t_0, T]}$ its family of capitalization factors. Then we have*

$$f_{t_2} = \frac{K(t_1)}{K(t_2)} \left(f_{t_1} \circ \tau_{t_2 - t_1} \right),$$

for every $t_1, t_2 \in [t_0, T]$, where $\tau_{t_2 - t_1}$ is the $(t_2 - t_1)$-translation on $\mathbb{R}$.

Proof. Let $t, t_1, t_2 \in [t_0, T]$, we have $K(t) = K(t_1) f_{t_1}(t - t_1)$, and $K(t) = K(t_2) f_{t_2}(t - t_2)$, hence

$$K(t_1) f_{t_1}(t - t_1) = K(t_2) f_{t_2}(t - t_2),$$

now, setting $h = t - t_2$ we have $h \in [t_0 - t_2, T - t_2]$, and we deduce

$$f_{t_2}(h) = \frac{K(t_1)}{K(t_2)} f_{t_1}(h + (t_2 - t_1)) = \frac{K(t_1)}{K(t_2)} f_{t_1}(\tau_{t_2 - t_1}(h)),$$

where we used the v-translation on $\mathbb{R}$, $\tau_v : \mathbb{R} \to \mathbb{R} : x \mapsto x + v$, so

$$f_{t_2}(h) = \frac{K(t_1)}{K(t_2)} \left(f_{t_1} \circ \tau_{t_2 - t_1} \right)(h).$$

Note that, if $h \in [t_0 - t_2, T - t_2]$, one has $h + t_2 - t_1 \in [t_0 - t_1, T - t_1]$. ■

Corollary 1.1. *Let K be a capitalization and $f = (f_t)_{t \in [t_0, T]}$ its family of capitalization factors. Then we have*

$$f_{t_2} = \frac{1}{f_{t_1}(t_2 - t_1)} \left(f_{t_1} \circ \tau_{t_2 - t_1} \right).$$

Proof. Let $t, t_1, t_2 \in [t_0, T]$, we have

$$f_{t_2}(t - t_2) = \frac{K(t_1)}{K(t_2)} f_{t_1}(t - t_1),$$

this implies

$$f_{t_2}(t_2 - t_2) = \frac{K(t_1)}{K(t_2)} f_{t_1}(t_2 - t_1),$$

but $f_{t_2}(t_2 - t_2) = f_{t_2}(0) = 1$, thus

$$\frac{1}{f_{t_1}(t_2 - t_1)} = \frac{K(t_1)}{K(t_2)},$$

substituting in

$$f_{t_2} = \frac{K(t_1)}{K(t_2)} (f_{t_1} \circ \tau_{t_2 - t_1}),$$

we obtain the goal. ■

2. The family of operators associated with a capitalization law

Another important concept of the classic financial mathematics is that of capitalization law. We give below a definition more general that the usual ones.

Definition 2.1 (of capitalization law). *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a map, F is said to be capitalization law if*

1) $F(0, t_0, t) = 0$ for every $t_0, t \in \mathbb{R}$;

2) $F(C, t_0, t_0) = C$ for every $C, t_0 \in \mathbb{R}$;

3) $F(C, t_0, \cdot)$ is not decreasing for every $t_0 \in \mathbb{R}$ and for $C \in \mathbb{R}_0^+$, and it is not increasing for every $t_0 \in \mathbb{R}$ and for $C \in \mathbb{R}^-$.

4) $F(C, \cdot, t)$ is not increasing for every $t \in \mathbb{R}$ and $C \in \mathbb{R}_0^+$, and it is not decreasing for every $t \in \mathbb{R}$ and for $C \in \mathbb{R}^-$.

5) $F(\cdot, t_0, t)$ is increasing.

Now we can give the main definition of the paper.

Definition 2.2 (the family of operators associated with a capitalization law). *Let $h \in \mathbb{R}$ and let F be a capitalization law. We define h-financial time translation induced by F the operator $T_h : \mathbb{R}^2 \to \mathbb{R}^2$ defined as follows, for every $(t, C) \in \mathbb{R}^2$,*

$$T_h(t, C) = (t + h, F(C, t, t + h)).$$

Moreover, we define the family of operators associated with F the following family $T = (T_h)_{h \in \mathbb{R}}$.

As usual, a family $T = (T_h)_{h \in \mathbb{R}}$ is, by definition, a surjective map defined on the real line. The h-component of T is the correspondent of h with respect to T, and it is denoted by $T(h)$ or T_h indifferently.

Now we examine the properties of the above family of operators.

Remark 2.1. By the *axiom 1*, of the definition of capitalization law, we deduce

$$T_h(t, 0) = (t + h, F(0, t, t + h)) = (t + h, 0).$$

Then the restriction of T_h to the set

$$\mathrm{span}(1, 0) = \{(t, 0)\}_{t \in \mathbb{R}}$$

is simply the geometric translation by the vector $(h, 0)$.

By *axiom 2,* we deduce the following proposition of basic importance for our goal.

Proposition 2.1. *Let F be a capitalization law and let T the family of operators associated with F. Then, we have*

$$T\left(0\right)=\mathbb{I}_{\mathbb{R}^2}.$$

Proof. In fact, for every $(t, C) \in \mathbb{R}^2$, we have

$$T_0\left(t,C\right)=\left(t+0,F\left(C,t,t\right)\right)=\left(t,C\right)$$

by the second property of capitalization law, so $T_0 = \mathbb{I}_{\mathbb{R}^2}$. ■

Remark 2.2. By the *axiom 3*, we deduce that, fixed C and t_0, the mapping

$$K_{(t_0,C)}:[t_0,+\infty[\to\mathbb{R}:t\mapsto F(C,t_0,t),$$

is a capitalization (when $C > 0$) or a debt-evolution (when $C < 0$). Moreover, we have

$$T_h\left(t_0,C\right)=\left(t_0+h,K_{(t_0,C)}\left(t_0+h\right)\right).$$

In other words, T_h sends forward the financial-event (t_0, C) to the point of the graph of $K_{(t_0,C)}$ whose time-distance is h.

Remark 2.3. By the *axiom 5,* of the above definition, we deduce that the section

$$F\left(\cdot,t_0,t\right):\mathbb{R}\to\mathbb{R}$$

is an injective function. So that T_h is always injective (in fact, if (t, C) is different from (t', C') is $t \neq t'$ vel $C \neq C'$, if the first case happens $T_h\,(t, C)$ differs from $T_h\,(t', C')$ for the first component, if the second case happens $T_h\,(t, C)$ differs from $T_h\,(t', C')$ for the second component, by injectivity of $F\,(\cdot, t_0, t)$).

Concluding, if we restrict the second set of T_h to the set $T_h(\ \mathbb{R}^2)$ the resulting map is injective and surjective, for every $h \in \mathbb{R}$, we denote its inverse by T_h^- (or obviously by $T(h)^-$) then T induces a family of bijections defined on $\mathbb{R}^2$, and so T can be viewed as a family of changes of coordinates on $\mathbb{R}^2$.

From an economic point of view, this circumstance is of fundamental importance we can consider the family $(T_h^-)_{h\in\mathbb{R}}$. We call this family the actualization family of the financial law F. By definition $T_h^- : T_h(\mathbb{R}^2) \to \mathbb{R}^2$ and $T_h^-(T_h(t, C)) = (t, C)$

Example 2.1. It's easy to see that, if $r \in \mathbb{R}^+$ (from an economic point of view r is a rate of interest), we have that the scalar field

$$F:\mathbb{R}^3\to\mathbb{R}:(C,t_0,t)\mapsto C\left(1+r\right)^{t-t_0}$$

is a capitalization law.

The family of operators associated with F is the map defined by

$$T_h\left(t,C\right)=\left(t+h,C\left(1+r\right)^h\right).$$

In this particular case, we can associate to F the following curve of smooth diffeomorphisms

$$\gamma:\mathbb{R}\to\text{Diff}^{\,\infty}\left(\mathbb{R}^2,\ \mathbb{R}^2\right):h\mapsto T_h.$$

3. Law at a time for a capitalization

Definition 3.1 (law at a time). *Let K be a capitalization on the period $[t_0, T]$, i.e., a not-decreasing function $K : [t_0, T] \to \mathbb{R}$ such that $K(t_0) > 0$. A scalar field $F : \mathbb{R}^3 \to \mathbb{R}$ is said to be a law for K at $t_1 \in [t_0, T]$ if, for every $t \in [t_0, T]$, we have*

$$K(t) = F(K(t_1), t_1, t). \ \square$$

It is not true, in general, that, if K is a capitalization, a law for F at the initial time is a law for F at every time, as shows the following example.

Example 3.1. Let $r, C_0 > 0$, and let

$$K : [t_0, T] \to \mathbb{R} : t \mapsto C_0 (1 + r(t - t_0))$$

(with $t_0, T \in \mathbb{R}$).

The scalar field

$$F : \mathbb{R}^3 \to \mathbb{R} : F(C, \alpha, \beta) = C(1 + r(\beta - \alpha))$$

is a law for K at t_0. In fact, for every $t \in [t_0, T]$, we have

$$K(t) = C_0 (1 + r(t - t_0)) = K(t_0)(1 + r(t - t_0)) = F(K(t_0), t_0, t),$$

thus F is a law at t_0 for K.

We shall see that F is a law for K only at t_0. In fact, let $t_1, t \in [t_0, T]$, we have

$$\begin{aligned} K(t) &= C_0 (1 + r(t - t_0)) = \\ &= C_0 (1 + r(t - t_1 + t_1 - t_0)) = \\ &= C_0 (1 + r(t_1 - t_0)) + C_0 r (t - t_1). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} F(K(t_1), t_1, t) &= K(t_1)(1 + r(t - t_1)) = \\ &= C_0 (1 + r(t_1 - t_0))(1 + r(t - t_1)) = \\ &= C_0 (1 + r(t_1 - t_0)) + C_0 (1 + r(t_1 - t_0)) r (t - t_1). \end{aligned}$$

Hence, we have

$$F(K(t_1), t_1, t) = K(t)$$

if and only if

$$C_0 (1 + r(t_1 - t_0)) r (t - t_1) = C_0 r (t - t_1).$$

That is

$$(1 + r(t_1 - t_0) - 1)(t - t_1) = 0.$$

Concluding, we have

$$K(t) = F(K(t_1), t_1, t)$$

if and only if

$$r(t_1 - t_0) = 0,$$

i.e., $t_1 = t_0$.

Resuming, the law

$$F : \mathbb{R}^3 \to \mathbb{R} : F(C, \alpha, \beta) = C(1 + r(\beta - \alpha))$$

is a law for K in t_1 if and only if $t_1 = t_0$. $\triangle$

But we can give a definitive result about the financial laws that are laws at every time for a capitalization. We need the following definition.

Definition 3.2 (of separable law with respect to a capitalization). *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a financial law and K be a capitalization on $[t_0, T]$. F is called K-separable if, for every $t_1, t_2 \in \mathbb{R}$ one has*

$$F\left(F\left(K(t_0), t_0, t_1\right), t_1, t_2\right) = F\left(K(t_0), t_0, t_2\right).$$

Theorem 3.1. *Let K be a capitalization on $[t_0, T]$. Then, a scalar field $F : \mathbb{R}^3 \to \mathbb{R}$ is a law for K at every $t \in [t_0, T]$ if and only if F is a law at t_0 and it is K-separable.*

Proof. $\Rightarrow$) Let the scalar field $F : \mathbb{R}^3 \to \mathbb{R}$ be a law for K at every $t \in [t_0, T]$. Then, for every $t_1, t_2 \in [t_0, T]$, since F is a law at t_1 , we have

$$K\left(t_2\right) = F\left(K\left(t_1\right), t_1, t_2\right).$$

On the other hand, F is a law at t_0, hence

$$F\left(K(t_0), t_0, t_2\right) = F\left(F\left(K(t_0), t_0, t_1\right), t_1, t_2\right).$$

$\Leftarrow$) Let F be K-separable. Then, for every $t_1, t_2 \in \mathbb{R}$, one has

$$F\left(F\left(K(t_0), t_0, t_1\right), t_1, t_2\right) = F\left(K(t_0), t_0, t_2\right).$$

Let $t_1 \in [t_0, T]$, we have to prove that, for every t, we have $K\left(t\right) = F\left(K\left(t_1\right), t_1, t\right)$. It's obvious, because we have

$$F(K(t_1), t_1, t) = F\left(F\left(K(t_0), t_0, t_1\right), t_1, t\right) = F\left(K(t_0), t_0, t\right) = K(t). \blacksquare$$

4. Separable financial law and one parameter group

Definition 4.1 (of self-actualizing capitalization law). *Let F be a capitalization law. We say that F is self-actualizing if, for every $C \in \mathbb{R}$ an for every t_1 and t_2, we have*

$$F\left(F\left(C, t_1, t_2\right), t_2, t_1\right) = C.$$

As we shall see in the following theorem, the family of operators associated with a self-actualizing capitalization law is an *invertible one parameter family,* in the sense that, for every real h, $T(h) = T_h$ is invertible and moreover is

$$T\left(h\right) \circ T\left(-h\right) = \mathbb{I}_{\mathbb{R}^2}.$$

Theorem 4.1 (characterization of self-actualizing laws). *Let F be a capitalization law. Then F is self-actualizing if and only if the family of operators associated with F is an invertible one parameter family.*

Proof. $\Rightarrow$) Let F be self-actualizing, then

$$F\left(F\left(C, t, t+h\right), t+h, t\right) = C,$$

for every $C \in \mathbb{R}, t \in \mathbb{R}, h \in \mathbb{R}$. We have to prove that for every $h \in \mathbb{R}$,

$$T\left(h\right) \circ T\left(-h\right) = \mathbb{I}_{\mathbb{R}^2}.$$

Let us examine the above composition

$$\begin{aligned} T(h)(T(-h)(t,C)) &= T(h)(t-h, F(C,t,t-h)) = \\ &= ((t-h)+h, F(F(C,t,t-h).t-h,(t-h)+h)) = \\ &= (t,C). \end{aligned}$$

Concluding, we have that $T(h)$ is invertible, for every real h, and moreover

$$(T(h))^{-} = T(-h).$$

$\Leftarrow$) Now we assume that $T(h)$ is invertible, for every real h, and moreover

$$(T(h))^{-} = T(-h),$$

for every $h \in \mathbb{R}$. We have to prove that

$$F(F(C,t,t+h), t+h, t) = C.$$

We have $T(h)(T(-h)(t,C)) = (t,C)$, so that

$$\begin{aligned} (t,C) &= T(h)(T(-h)(t,C)) = \\ &= T(h)(t-h, F(C,t,t-h)) = \\ &= ((t-h)+h, F(F(C,t,t-h), t-h, t)), \end{aligned}$$

hence

$$F(F(C,t,t-h), t-h, t) = C$$

for every $h \in \mathbb{R}$, as desired.

An alternative proof can be the following one:

$$\begin{aligned} (t,C) &= T(-h)(T(h)(C,t,)) = \\ &= T(-h)(t+h, F(C,t,t+h)) = \\ &= (t+h-h, F(F(C,t,t+h), t+h, (t+h)-h)) = \\ &= (t, F(F(C,t,t+h), t+h, t)). \ \blacksquare \end{aligned}$$

Consider a capitalization law F. We recall that we defined actualization family associated with F, the below family

$$A(h) = T(h)^{-}.$$

At this point the following corollary is obvious.

Corollary 4.1. *Let F be a capitalization law. Then, F is self-actualizing if and only if*

$$A(h) = T(-h).$$

Proof. $\Rightarrow$) Obviously, we have

$$A(h) = (T(h))^{-} = T(-h).$$

$\Leftarrow$) If $A(h) = T(-h)$ then,

$$T(-h) = A(h) = (T(h))^{-}. \ \blacksquare$$

We recall that, if $(G,*)$ is a group, a one parameter group on $(G,*)$ is a family T in G indexed by $\mathbb{R}$ such that

1) $T(0) = \mathbb{I}_*$
2) $T(h+k) = T(h) * T(k)$, $\forall h, k \in \mathbb{R}$.

Definition 4.2 (of separable law). *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a law. F is called separable if, for every $C, t_0, t_1, t_2 \in \mathbb{R}$ one has*

$$F(F(C, t_0, t_1), t_1, t_2) = F(C, t_0, t_2).$$

Theorem 4.2. *If F is a separable financial law then F is self-actualizing.*

Proof. Let $C, t_0, t_1, t_2 \in \mathbb{R}$, one has

$$F(F(C, t_0, t_1), t_1, t_0) = F(C, t_0, t_0) = C. \blacksquare$$

Theorem 4.3 (characterization of separable laws). *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a capitalization law and let T be its family of operators. Then F is separable if and only if T is a one parameter group.*

Proof. $\Rightarrow$) Let F be separable and let T be the family of operators associated with F, we already know that

$$T(0)(t, C) = (t + 0, F(C, t, t + 0)) = (t, C),$$

then $T(0) = \mathbb{I}_{\mathbb{R}^2}$.

Now,

$$T(h+k)(t, C) = (t + h + k, F(C, t, t + h + k));$$

and, on the other hand, we have

$$\begin{aligned} T(h) \circ T(k)(t, C) &= T(h)(T(k)(t, C)) = \\ &= T(h)(t + k, F(C, t, t + k)) = \\ &= (t + h + k, F(F(C, t, t + k), t + k, t + k + h) = \\ &= (t + h + k, F(C, t, t + h + k)). \end{aligned}$$

$\Leftarrow$) Let T be the family of operators associated with F, and assume T be a one parameter group. For every $C, t_0, t_1, t_2 \in \mathbb{R}$ one has

$$\begin{aligned} (t_2, F(F(C, t_0, t_1), t_1, t_2)) &= T_{t_2 - t_1}(t_1, F(C, t_0, t_1)) = \\ &= T_{t_2 - t_0}(T_{t_0 - t_1}(t_1, F(C, t_0, t_1)) = \\ &= T_{t_2 - t_0}(t_0, F(F(C, t_0, t_1), t_1, t_0)) = \\ &= T_{t_2 - t_0}(t_0, F(C, t_0, t_0)) = \\ &= T_{t_2 - t_0}(t_0, C) = \\ &= (t_2, F(C, t_0, t_2)). \end{aligned}$$

This concludes the proof. $\blacksquare$

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