GENERALIZING DOUBLE GRAPHS

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ABSTRACT. In this paper we study the graphs which are direct product of a simple graph $G$ with the graphs obtained by the complete graph $K_k$ adding a loop to each vertex; thus these graphs turn out to be a generalization of the double graphs.

1. Introduction

Let $G$ be a finite simple graph, i.e. a graph without loops and multiple edges. In [1] it is introduced and studied the graph, said double of $G$ and denoted $D(G)$, obtained by taking two copies of $G$ and joining every vertex $v$ in one component to every vertex $w'$ in the other component corresponding to a vertex $w$ adjacent to $v$ in the first component. The above construction can be generalized in the following way.

As usual $V(G)$ and $E(G)$ denote the set of vertices and edges of $G$, respectively, and $\text{adj}$ denote the adjacency relation of $G$. For all definitions not given here see [2, 3, 4, 5, 6].

The direct product $G \times H$ of two graphs $G$ and $H$ is the graph with $V(G \times H) = V(G) \times V(H)$ and with adjacency defined by $(v_1, w_1) \text{ adj } (v_2, w_2)$ if and only if $v_1 \text{ adj } v_2$ in $G$ and $w_1 \text{ adj } w_2$ in $H$.\[5\]

The total graph $T_n$ on $n$ vertices is the graph associated to the total relation (where every vertex is adjacent to every vertex). It can be obtained by the complete graph $K_n$ adding a loop to every vertex. In [5] it is denoted by $K_n^*$. We define the $k$-fold of $G$ as the graph $D[k](G) = G \times T_k$; clearly for $k = 2$ we obtain the double graphs.

Since the direct product of a simple graph with any graph is always a simple graph, it follows that the $k$-fold of a simple graph is still a simple graph.

In $D[k](G)$ we have $(v, a) \text{ adj } (w, b)$ if and only if $v \text{ adj } w$ in $G$. Then, if $V(T_k) = \{0, 1, \ldots, k-1\}$, we have that $G_i = \{(v, i) : v \in V(G)\}$, $0 \leq i \leq k-1$, are $k$ subgraphs of $D[k](G)$ isomorphic to $G$ such that $G_0 \cap G_1 \cap \cdots \cap G_{k-1} = \emptyset$ and $G_0 \cup G_1 \cup \cdots \cup G_{k-1}$ is a spanning subgraph of $D[k](G)$. Moreover we have an edge between $(v, i)$ and $(w, j)$ and similarly we have an edge between $(v, j)$ and $(w, i)$, where $0 \leq i, j \leq k-1$, whenever $v \text{ adj } w$ in $G$. We will call $\{G_0, G_1, \ldots, G_{k-1}\}$ the canonical decomposition of $D[k](G)$.

The lexicographic product (or composition) of two graphs $G$ and $H$ is the graph $G \circ H$ with $V(G) \times V(H)$ as vertex set and with adjacency defined by $(v_1, w_1) \text{ adj } (v_2, w_2)$
if and only if \( v_1 = v_2 \) and \( w_1 \) adj \( w_2 \) in \( H \) or \( v_1 \) adj \( v_2 \) in \( G \). The graph \( G \circ H \) can be obtained from \( G \) replacing each vertex \( v \) of \( G \) by a copy \( H_v \) of \( H \) and joining every vertex of \( H_v \) with every vertex of \( H_w \) whenever \( v \) and \( w \) are adjacent in \( G \) [5, p. 185].

**Lemma 1.** For any graph \( G \) we have \( G \times T_n \simeq G \circ N_n \), where \( N_n \) is the graph on \( n \) vertices without edges.

**Proof.** For simplicity consider \( T_n \) and \( N_n \) on the same vertex set. Then the function \( f : G \times T_n \to G \circ N_n \), defined by \( f(v, k) = (v, k) \) for every \( (v, k) \in V(G \times T_n) \), is a graph isomorphism. Indeed, since \( N_n \) has no edges, we have that \( (v, k) \) adj \( (w, k) \) in \( G \circ N_n \) if and only if \( v \) adj \( w \) in \( G \). \( \Box \)

### 2. Some basic properties of \( k \)-fold graphs

In this section we will review some elementary properties of the \( k \)-fold graphs. We will write \( D^2[G] \) for the double of the double of \( G \). More generally \( D^k(G) \) is obtained by multiplying \( G \) by \( T_2 \) \( k \) times, i.e., \( D^k(G) = G \times T_{2k} \) and \( D^k(G) = D^k(G) \).

In particular \( D[k](G) \simeq D^m(G) \) when \( k = 2^m \). In the following proposition the converse statement is proved.

**Proposition 2.** Let \( k \) and \( m \) positive integers. Then \( D[k](G) \simeq D^m(G) \) if and only if \( k = 2^m \).

**Proof.** We prove the "only if" part. Let \( n \) be the number of vertices of \( G \). The assumption that \( D[k](G) \simeq D^m(G) \) implies \( |V(D[k](G))| = |V(D^m(G))| \). Thus, because \( |V(D[k](G))| = kn \) and \( |V(D^m(G))| = 2^m n \), the result follows. \( \Box \)

Recall the following theorem.

**Theorem 3** ([5, p. 190]). If \( G \circ H \simeq G' \circ H' \) and \( |V(H)| = |V(H')| \), then \( H \simeq H' \) and \( G \simeq G' \).

An immediate consequence of the theorem is the following

**Theorem 4.** Two graphs \( G_1 \) and \( G_2 \) are isomorphic if and only if \( D[k](G_1) \) and \( D[k](G_2) \) are isomorphic.

**Proof.** By Lemma 1 \( D[k](G_1) = G_1 \circ N_k \) and \( D[k](G_2) = G_2 \circ N_k \); then the claim holds. \( \Box \)

**Proposition 5.** The \( k \)-fold graph \( D[k](G) \) of a graph \( G \) on \( n \) vertices contains at least \((2^n - 2)(k - 1)k + k\) subgraphs isomorphic to \( G \) itself.

**Proof.** Let \( \{G_0, G_1, \ldots, G_{k-1}\} \) be the canonical decomposition of \( D[k](G) \). Let \( S_0 \) be any subset of \( V(G_0) \) and \( C_0 \) the complementary set of \( S_0 \) with respect to \( V(G_0) \); moreover let \( S_i \), where \( 1 \leq i \leq k-1 \), be subsets of \( V(G_i) \) and \( C_i \) their complementary sets with respect to \( V(G_i) \). Then the subsets \( S_i \cup C_j \), where \( 1 \leq i, j \leq k - 1 \), \( i \neq j \), are isomorphic to \( G \). The number of subsets \( S_i \) is \( 2^n - 2 \), because we exclude the cases of \( S_0 = \emptyset \) and \( S_0 = G_0 \); finally we have to add the \( k \) subsets \( G_0, G_1, \ldots, G_{k-1} \). \( \Box \)

**Proposition 6.** For any graph \( G \), \( G \) is bipartite if and only if \( D[k](G) \) is bipartite.
Proof. Let \( \{G_0, G_1, \ldots, G_{k-1}\} \) be the canonical decomposition of \( D^{[k]}(G) \). If \( G \) is bipartite then also \( G_j \), where \( 0 \leq j \leq k-1 \), are bipartite. Let \( \{V, W\} \) be the partite sets of \( G \) and \( \{V_j, W_j\} \), be the corresponding partite sets of \( G_j \). Every edge of \( D^{[k]}(G) \) has one extreme in \( \bigcup_{j=0}^{k-1} V_j \), and the other in \( \bigcup_{j=0}^{k-1} W_j \) and hence also \( D^{[k]}(G) \) is bipartite.

Conversely, if \( D^{[k]}(G) \) is bipartite then it does not contain odd cycles. Hence also the subgraph \( G_0 \simeq G \) does not contain odd cycles and then it is bipartite.

A vertex cut of a graph \( G \) is a subset \( S \) of \( V(G) \) such that \( G \setminus S \) is disconnected. The connectivity \( \kappa(G) \) of \( G \) is the smallest size of a vertex cut of \( G \). A point of articulation (resp. bridge) is a vertex (resp. edge) whose removal augment the number of connected components. A block is a connected graph without articulation points. In the following proposition we present some properties of the \( k \)-fold graphs, whose proof is perfectly similar to the proof in the case of the double graphs [1].

Proposition 7. For any graph \( G \neq K_1 \) the following properties hold.

1. \( G \) is connected if and only if \( D^{[k]}(G) \) is connected.
2. If \( G \) is connected, then every pair of vertices of \( D^{[k]}(G) \) belongs to a cycle.
3. Every edge of \( D^{[k]}(G) \) belongs to a 4-cycle.
4. In a \( k \)-fold graph there are neither bridges nor articulation points.
5. If \( G \) is connected, then \( D^{[k]}(G) \) is a block.
6. The connectivity of \( D^{[k]}(G) \) is \( \kappa(D^{[k]}(G)) = 2^k \kappa(G) \).

A graph \( G \) is Hamiltonian if it has a spanning cycle.

Proposition 8. If a graph \( G \) is Hamiltonian, then also \( D^{[k]}(G) \) is Hamiltonian.

Proof. Let \( \{G_0, G_1, \ldots, G_{k-1}\} \) be the canonical decomposition of \( D^{[k]}(G) \). Let \( \gamma \) be a spanning cycle of \( G \), \( vw \) be an edge of \( \gamma \) and \( \gamma' \) be the path obtained by \( \gamma \) removing the edge \( vw \). Let \( \gamma'_i \) be the corresponding path in \( G_i \), for \( i = 0, 1, \ldots, k-1 \). Then \( \gamma_0 \cup \{(w, 0), (v, 1)\} \cup \gamma'_1 \cup \{(w, 1), (v, 2)\} \cup \cdots \cup \gamma'_{k-1} \{(w, k-1), (v, 0)\} \) is a spanning cycle of \( D^{[k]}(G) \).

Proposition 9. For any graph \( G_1 \) and \( G_2 \) the following properties hold:

1. \( D^{[k]}(G_1 \times G_2) = G_1 \times D^{[k]}(G_2) = D^{[k]}(G_1) \times G_2 \)
2. \( D^{[k]}(G_1 \circ G_2) = G_1 \circ D^{[k]}(G_2) \).

Proof. The first identity comes from the definition of \( k \)-fold graphs and \( (G_1 \times G_2) \times T_k = G_1 \times (G_2 \times T_k) \), while the second one comes from \( (G_1 \circ G_2) \circ N_k = G_1 \circ (G_2 \circ N_k) \).

Let \( J_k \) be the matrix of all ones of order \( k \). From the definition it follows immediately that

Proposition 10. Let \( A \) be the adjacency matrix of \( G \). Then the adjacency matrix of \( D^{[k]}(G) \) is

\[
D^{[k]}[A] = \begin{bmatrix} A & A & \ldots & A \\ A & A & \ldots & A \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \ldots & A \end{bmatrix} = A \otimes J_k.
\]
The rank $r(G)$ of a graph $G$ is the rank of its adjacency matrix. Then from the above proposition it follows that

**Proposition 11.** For any graph $G$, $r(D^{[k]}(G)) = r(G)$.

In the sequel we will use the property that two graphs are isomorphic if and only if their adjacency matrices are similar by means of a permutation matrix.

Let $G_1$ and $G_2$ be two graphs. The sum $G_1 + G_2$ of $G_1$ and $G_2$ is the disjoint union of the two graphs. The complete sum $G_1 \oplus G_2$ of $G_1$ and $G_2$ is the graph obtained by $G_1 + G_2$ joining every vertex of $G_1$ to every vertex of $G_2$. A graph is decomposable if it can be expressed as sums and complete sums of isolated vertices [6, p.183].

**Proposition 12.** For any graph $G_1$ and $G_2$ the following properties hold:

1. $D^{[k]}(G_1 + G_2) = D^{[k]}(G_1) + D^{[k]}(G_2)$
2. $D^{[k]}(G_1 \oplus G_2) = D^{[k]}(G_1) \oplus D^{[k]}(G_2)$
3. The $k$-fold of a decomposable graph is decomposable.

**Proof.** The first two properties can be proved simultaneously as follows. Let $A_1$ and $A_2$ be the adjacency matrices of $G_1$ and $G_2$, respectively. Then $\begin{bmatrix} A_1 & X \\ X & A_2 \end{bmatrix}$ is the adjacency matrix of $G_1 + G_2$ when $X = O$ and of $G_1 \oplus G_2$ when $X$ is the matrix $J$ all of whose entries are 1’s. Then the adjacency matrix of the $k$-fold graph is

$\begin{bmatrix} A_1 & X \\ X & A_2 \end{bmatrix} \otimes J_k.$

Interchanging first the columns in even positions with those in odd positions and similarly for the rows, we obtain the matrix

$\begin{bmatrix} A_1 \otimes J_k & X \otimes J_k \\ X \otimes J_k & A_2 \otimes J_k \end{bmatrix}$

which is the adjacency matrix of $D^{[k]}(G_1) + D^{[k]}(G_2)$ when $X = O$ and of $D[G_1] \oplus D[G_2]$ when $X = J$. These properties are also implied by the right-distributive laws of the lexicographic product [5, pp. 185-186]. Finally the third property follows from the fact that $D^{[k]}$ preserves sums and complete sums and $D^{[k]}(K_1) = N_k = K_1 + K_1 + \cdots + K_1$.

**Examples**

1. If $N_n$ is the graph on $n$ vertices without edges, then $D^{[k]}(N_n) = N_{kn}$, while $D^{[k]}(N_n) = N_{2^k \cdot n}$.

2. Let $K_{m,n}$ be a complete bipartite graph. Then $D^{[k]}(K_{m,n}) = K_{km, kn}$. Similarly, if $K_{m_1, \ldots, m_n}$ is a complete $n$-partite graph we have $D^{[k]}(K_{m_1, \ldots, m_n}) = K_{km_1, \ldots, km_n}$. In particular, if $K_{m(n)}$ is the complete $m$-partite graph $K_n, \ldots, n$, then $D^{[k]}(K_{m(n)}) = K_{m(kn)}$. Since $K_n = K_n(1)$ it follows that the $k$-fold of the complete graph $K_n$ is the graph $H^{[k]}_{kn} = K_{n(k)}$, which turns out to be a generalization of the hyperoctahedral graph.
(3) For $n \geq 2$, let $K_{n}^-$ be the graph obtained by the complete graph $K_n$ deleting any edge. Then $K_{n}^- = N_2 \boxplus K_{n-2}$ and $D^{[k]}(K_{n}^-) = D^{[k]}(N_2) \boxplus D^{[k]}(K_{n-2}) = N_{2k} \boxplus H_{n-2}^{[k]}$, that is $D^{[k]}(K_{n}^-) = K_{2k,k,...,k}$.

A graph $G$ is circulant when its adjacency matrix $A$ is circulant, i.e. when every row distinct from the first one, is obtained from the preceding one by shifting every element one position to the right. Let $C(a_1, \ldots, a_n)$ be the circulant graph where $(a_1, \ldots, a_n)$ is the first row of the adjacency matrix (for a suitable ordering of the vertices).

**Proposition 13.** A graph $G$ is circulant if and only if $D^{[k]}(G)$ is circulant. Specifically

$$D^{[k]}(C(a_1, \ldots, a_n)) = C(a_1, \ldots, a_n, a_1, \ldots, a_n, \ldots, a_1).$$

Let $R[G] = G \times K_2$ be the canonical double covering of $G$ [7]. In a way similar to the case of double graphs it is possible to prove the following proposition.

**Proposition 14.** $D^{[k]}$ and $R$ commutes, that is $D^{[k]}(R[G]) = R[D^{[k]}(G)]$ for every graph $G$.

### 3. Spectral properties of k-fold graphs

The eigenvalues, the characteristic polynomial and the spectrum of a graph are the eigenvalues, the characteristic polynomial and the spectrum of its adjacency matrix [3, p. 12].

**Proposition 15.** The characteristic polynomial of the $k$-fold of a graph $G$ on $n$ vertices is

$$\varphi(D^{[k]}(G); \lambda) = (k\lambda^{k-1})^n \varphi(G; \lambda/k).$$

In particular the spectrum of $D^{[k]}(G)$ is $\{0, \ldots, 0, k\lambda_1, \ldots, k\lambda_n\}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $G$ and 0 is taken $(k-1)n$ times.

**Proof.** By Proposition 10 it follows that

$$\varphi(D^{[k]}(G); \lambda) = \begin{vmatrix}
\lambda I - A & -A & \ldots & -A \\
-A & \lambda I - A & \ldots & -A \\
& \ldots & \lambda I - A
\end{vmatrix} = \begin{vmatrix}
\lambda I - kA & -A & \ldots & -A \\
-A & \lambda I - A & \ldots & -A \\
& \ldots & \lambda I - A
\end{vmatrix}$$

$$= \begin{vmatrix}
\lambda I - kA & -A & \ldots & -A \\
0 & \lambda I & \ldots & 0 \\
& \ldots & \lambda I
\end{vmatrix}.$$

□

An integral graph is a graph all of whose eigenvalues are integers [3, p. 266].

**Proposition 16.** A graph $G$ is integral if and only if $D^{[k]}(G)$ is an integral graph.

**Proof.** Since the characteristic polynomial of a graph is monic with integer coefficients its rational roots are necessarily integers. Then the claim immediately follows from Proposition 15. □
Two graphs are *cospectral* when they are non-isomorphic and have the same spectrum [2], [3]. From Proposition 15 and Theorem 4 we have the following property.

**Proposition 17.** Two graphs $G_1$ and $G_2$ are cospectral if and only if $D[k][G_1]$ and $D[k][G_2]$ are cospectral.

Therefore given two cospectral graphs $G_1$ and $G_2$, it is always possible to construct an infinite sequence of cospectral graphs. Indeed $D[k](G_1)$ and $D[k](G_2)$ are cospectral for every $k \in \mathbb{N}$.

The relation between the spectrum of a graph $G$ and its $k$-fold graph has a consequence for the strongly regular graphs. First recall that a graph $G$ is $d$-regular if every vertex has degree $d$; then a graph $G$ is $d$-regular if and only if $D[k][G]$ is $kd$-regular.

A simple graph $G$ is *strongly regular* with parameters $(n, d, \lambda, \mu)$ when it has $n$ vertices, is $d$-regular, every adjacent pair of vertices has $\lambda$ common neighbors and every nonadjacent pair has $\mu$ common neighbors.[8]

Connected strongly regular graphs, distinct from the complete graph, are characterized [3, p. 103] as the connected regular graphs with exactly three distinct eigenvalues.

Strongly regular graphs with one zero eigenvalue are characterized as follows [3, p. 163]: a regular graph $G$ has eigenvalues $k$, $0$, $\lambda_3$ if and only if the complement of $G$ is the sum of $1 - k/\lambda_3$ complete graphs of order $-\lambda_3$. Equivalently, a regular graph has three distinct eigenvalues of which one is zero if and only if it is a multipartite graph $K_{m(n)}$.

We are able now to characterize the strongly regular $k$-fold graphs in the following proposition proved in a perfectly similar way as in the case of the double graphs.

**Proposition 18.** For any graph $G$ the following characterizations hold.

1. $D[k](G)$ is a connected strongly regular graph if and only if $G$ is a complete multipartite graph $K_{m(n)}$.
2. $D[k](G)$ is a disconnected strongly regular graph if and only if $G$ is a completely disconnected graph $N_{kn}$.

Moreover, since complete bipartite graphs are characterized by their spectrum, we have that

**Proposition 19.** Strongly regular $k$-fold graphs are characterized by their spectrum.

4. Complexity and Laplacian spectrum

Let $t(G)$ be the complexity of the graph $G$, i.e. the number of its spanning trees. It is well known [9] that

$$t(G) = \frac{1}{n^2} \det(L + J)$$

where $n$ is the number of vertices of $G$, $L$ is the Laplacian matrix of $G$ and $J$, as before, is the $n \times n$ matrix all of whose entries are equal to 1.

**Theorem 20.** The complexity of the $k$-fold of a graph $G$ on $n$ vertices with degrees $d_1, d_2, \ldots, d_n$ is
\[ t(D^{[k]}(G)) = k^{kn-2} d_1^{k-1} d_2^{k-1} \cdots d_n^{k-1} t(G) . \]

**Proof.** Let \( v_1, \ldots, v_n \) be the vertices of \( G \) and \( d_1, \ldots, d_n \) their degrees. As known the Laplacian matrix \( L \) of \( G \) is equal to \( D - A \) where \( D \) is the diagonal matrix \( \text{diag}(d_1, \ldots, d_n) \) and \( A \) is the adjacency matrix of \( G \). Then the Laplacian matrix of \( D^{[k]}(G) \) is

\[ D^{[k]}(L) = D^{[k]}(D) - D^{[k]}(A) = \begin{bmatrix} kD & O & \cdots & 0 \\ O & kD & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & kD \end{bmatrix} - \begin{bmatrix} A & A & \cdots & A \\ A & A & \cdots & A \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \cdots & A \end{bmatrix}, \]

then

\[ D^{[k]}(L) = \begin{bmatrix} kD - A & -A & \cdots & -A \\ -A & kD - A & \cdots & -A \\ \vdots & \vdots & \ddots & \vdots \\ -A & -A & \cdots & kD - A \end{bmatrix}. \]

Hence it follows that

\[ t(D^{[k]}(G)) = \frac{1}{(kn)^2} \det(D^{[k]}(L) + J) = \frac{1}{(kn)^2} \det \begin{bmatrix} kD - A + J & -A + J & \cdots & -A + J \\ -A + J & kD - A + J & \cdots & -A + J \\ \vdots & \vdots & \ddots & \vdots \\ -A + J & -A + J & \cdots & kD - A + J \end{bmatrix}. \]

Summing to the first the remaining columns, we have

\[ t(D^{[k]}(G)) = \frac{1}{(kn)^2} \begin{bmatrix} kD - kA + kJ & -A + J & \cdots & -A + J \\ kD - kA + kJ & kD - A + J & \cdots & -A + J \\ \vdots & \vdots & \ddots & \vdots \\ kD - kA + kJ & -A + J & \cdots & kD - A + J \end{bmatrix} \]

\[ = \frac{1}{(kn)^2} \begin{bmatrix} kD - kA + kJ & -A + J & \cdots & -A + J' \\ 0 & kD & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & kD \end{bmatrix}. \]

Then

\[ t(D^{[k]}(G)) = \frac{1}{(kn)^2} \left| kD - kA + kJ \right| |kD|^{k-1} = k^{nk-2} t(G), (d_1)^{k-1}, (d_2)^{k-1} \cdots (d_n)^{k-1} \]

and the theorem follows. \( \square \)

As an immediate consequence we have the following

**Theorem 21.** The complexity of the double of a \( d \)-regular graph \( G \) on \( n \) vertices is

\[ t(D^{[k]}(G)) = k^{nk-2} t(G), d^n(k-1). \]
Finally, from (5), it can be proved the following

**Proposition 22.** Let $G$ be a graph on $n$ vertices with degrees $d_1, d_2, \ldots, d_n$ and let $\{\lambda_1, \ldots, \lambda_n\}$ be its Laplacian spectrum. Then the Laplacian spectrum of $D[k](G)$ is $\{kd_1, \cdots, kd_n, k\lambda_1, \ldots, k\lambda_n\}$. In particular, $G$ has an integral Laplacian spectrum if and only if the same hold for $D[k](G)$.

5. Independent sets

An independent set of vertices of a graph $G$ is a set of vertices in which no pair of vertices is adjacent. Let $\mathcal{I}_h[G]$ be the set of all independent subsets of size $h$ of $G$ and let $i_h(G)$ be its size. The independence polynomial of $G$ is defined as

$$I(G; x) = \sum_{h \geq 0} \sum_{S \in \mathcal{I}_h[G]} x^{|S|} = \sum_{h \geq 0} i_h(G) x^h.$$  

**Proposition 23.** For any graph $G$ we have $\mathcal{I}_h[D[k](G)] \simeq \mathcal{I}_h[G] \times k^h$, where $k = \{0, 1, \cdots, k-1\}$. In particular $i_h(D[k](G)) = k^h i_h(G)$ and $I(D[k](G); x) = I(G, kx)$.

**Proof.** Let the vertices of $G$ be linearly ordered in some way. Let $S = \{(v_1, w_1), \ldots, (v_h, w_h)\}$ be an independent set of $D[k](G) = G \times T_k$. Since $T_k$ is a total graph, it follows that $\pi_1(S) = \{v_1, \ldots, v_h\}$ is an arbitrary independent subset of $G$ and $\pi_2(S)$ is equivalent to an arbitrary sequence $(w_1, \ldots, w_h)$ of length $h$ (where the order is established by the order of $\pi_1(S)$ induced by the order of $V(G)$). The claim follows.

The (vertex) independence number $\alpha(G)$ of a graph $G$ is the maximum size of the independent sets of vertices of $G$. Equivalently, $\alpha(G)$ is the degree of the polynomial $I(G, x)$. Then Proposition 23 implies the following

**Proposition 24.** For any graph $G$ we have that $\alpha(D[k](G)) = k \alpha(G)$.

6. Morphisms

A morphism $f : G \to H$ between two graphs $G$ and $H$ is a function from the vertices of $G$ to the vertices of $H$ which preserves adjacency (i.e. $v$ adj $w$ implies $f(v)$ adj $f(w)$, for every $v, w \in V(G)$) [10, 11]. An isomorphism between two graphs is a bijective morphism whose inverse function is also a morphism.

Let $\text{Hom}(G, H)$ be the set of all morphisms between $G$ and $H$ and let $k^{V[G]}$ be the set of all functions from $V(G)$ to $k = \{0, 1, \cdots, k-1\}$.

**Lemma 25.** For every graph $G$ and $H$, $\text{Hom}(G, D[k](H)) = \text{Hom}(G, H) \times k^{V[G]}$.

**Proof.** From the universal property of the direct product (in the categorical sense [12]) we have $\text{Hom}(G, G_1 \times G_2) = \text{Hom}(G, G_1) \times \text{Hom}(G, G_2)$. Since $D[k](G) = G \times T_k$ and $\text{Hom}(G, T_k) = k^{V[G]}$, the lemma follows.

We now extend $D[k]$ to morphisms in the following way: for any graph morphism $f : G \to H$ let $D[k][f] : D[k](G) \to D[k](H)$ be the morphism defined by $D[k][f](v, k) = (f(v), k)$ for every $(v, k) \in D[G]$. In this way $D[k]$ is an endofunctor of the category of finite simple graphs and graph morphisms.
A morphism $r : G \to H$ between two graphs $G$ and $H$ is a retraction if there exists a morphism $s : H \to G$ such that $r \circ s = 1_H$. If there exists a retraction $r : G \to H$ then $H$ is a retract of $G$. Since $\mathcal{D}^k$ is a functor it preserves retractions and retracts.

**Proposition 26.** Every graph $G$ is a retract of $\mathcal{D}^k(G)$. More generally every retract of $G$ is also a retract of $\mathcal{D}^k(G)$.

**Proof.** Consider the morphisms $r : \mathcal{D}^k(G) \to G$ and $s : G \to \mathcal{D}^k(G)$ defined by $r(v,k) = v$ for every $(v,k) \in V(\mathcal{D}^k(G))$ and $s(v) = (v,0)$ for every $v \in V(G)$. Then $r$, which is the projection of $G \times T_k$ on $G$, is a retraction. The second part of the proposition follows from the fact that $\mathcal{D}^k$ is a functor and the composition of retractions is a retraction. 

**References**


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