

TEMPERATURE COMPONENT METHOD FOR HEAT CONDUCTION PROBLEMS

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ABSTRACT. The work includes a solving proposal for initial-boundary value 3D heat conduction problems. The proposal is based on an extension of the body model region to the whole space where the space integral as a particular solution to the initial-boundary value problem is derived. Temperature component is separated from the space integral. The component admissibility conditions are formulated. For numerical purposes the approximated integral with a discrete set of fictitious components is proposed. The fictitious component intensities are determined on an approximate way from the boundary condition. An approximate solution of the heat conduction problem is obtained by extension in time and contraction in space of the approximated integral.

1. Introduction

The method of fictitious sources for heat conduction has been developed by Stefaniak [1]. The solution to direct and inverse problems may be found in [2-5]. Comparison with the results obtained with FEM and BEM is presented in [6]. A generalization of the method on the problems with an inhomogeneous initial condition appeared in [7]. The paper [8] refers to the method of temperature source applied to the problems with inhomogeneous initial conditions. A method of temperature component presented here contains some generalizations of the above ideas made to the three-dimensional problems.

The *temperature component method* (TCM) enables solving linear problems of heat conduction for homogeneous and isotropic bodies. In particular, the temperature field and heat fluxes may be determined this way. The boundary conditions divide the boundary surface into separate parts at which the 3rd kind thermal boundary condition, imposing a connection between heat flux and temperature field, the 2nd kind condition with given heat flow, and the 1st kind, with imposed temperature distribution are given. Nevertheless, in the present paper the last two cases are considered as well as some realizations of the 3-rd kind condition. The TCM required the space value problem to be formulated. Fictitious extension of the body region to the whole space leads to the space value problem with known solution. The field obtained this way turn out a particular solution of the primary initial-boundary value problem. Verification of boundary condition with the help of the particular solution gives an integral equation for the load determined beyond the body region. Therefore, a proposal was made that consisted in replacing multiple integrals of this load for another function referred to as temperature component. Expressing of the temperature component in terms of fundamental solution for the parabolic operator [9] in order

to satisfy some component admissibility conditions leads to an integral equation. The determination of the exact temperature component is often inefficient for the various shapes of the boundary. Therefore, the temperature component is determined in an approximate form named fictitious components, enabling to satisfy the boundary condition with imposed accuracy, and, at the same time, exactly observing the governing equation and the initial condition.

Intensity values of the fictitious components located beyond the body region are determined in fictitious instants which takes on a discrete set named fictitious time. For this purpose a bounded time interval named current time was introduced, in which the boundary condition is verified. The testing instants divide the current time into particular sections and create the set called testing time. The verifying algorithm of the boundary condition, that is of recursive character with regard to the testing time, is based on the properties of the fundamental solution for the parabolic operator.

The TCM is reckoned among the modified Trefftz methods searching an approximate solution of a boundary-value problem in the form of linear combination of singular fundamental solutions for the partial-differential operator, while the coefficients are determined from the boundary conditions in the collocation nodes. The Trefftz's idea in the presented work was extended and generalized in several directions. First of all, the method is extended to initial-boundary value problems. In effect particular solution of the problem is obtained as a new mathematical quality, as compared to the functions satisfying the governing equation appropriate for the Trefftz method. This enables accurate consideration of the initial condition in the approximate solution. Secondly, the functions resulting from an extension of the domain of the thermal field on the whole space (as opposed to the functions satisfying the governing equation of the Trefftz method) allow formulating the solution satisfying the governing equation and initial condition strictly and the boundary condition in the collocation nodes and the testing instants only. Thirdly, the TCM introduces the component admissibility condition that ensure satisfaction of the governing equation and the initial condition. This enables finding the solution for an arbitrary load in a natural way. In result a universal method is obtained, enabling solving the 3D heat conduction problems for imposed initial conditions and loads of the area of any types of boundary conditions and sufficiently smooth boundary surfaces.

2. Governing equation

The theory of heat conduction relates to a homogeneous and isotropic model of a body provided that the heat sources are subject to time-changes. Let \mathbb{R} is for the set of real number, and Θ the absolute temperature. The open set D of points in Euclidean space Z occupied by a places inside a body model is referred to as the *region*. One shall call the body as *thermal free*, if heat sources are absent inside and on the boundary of the region D occupy in Euclidean space Z by the body. Let the body will be thermal free at the moment ${}_0t \in \mathbb{R}$ and has the constant reference temperature ${}_0T$ of surroundings. In such a case the thermal state shall meet the relationship

$$(1) \quad \Theta(\vec{x}, {}_0t) = {}_0T \quad \vec{x} \in D$$

For a new scale of temperature $T = \Theta - {}_0T$ and an initial instant $0 \in \mathbb{R}$ the process is described by the temperature field: $T(\vec{x}, t)$, $\vec{x} \in D$, $t > 0$, where \vec{x} determines the

place with regard to the reference system, and t is the current instant. The interval $(0, \infty)$ is referred to as the *time*, while $(0, t]$ is the *current time*. A homogeneous and isotropic material is described by Fourier relation

$$(2) \quad q_i(\vec{x}, t) = -\lambda T_{,i}(\vec{x}, t), \quad \vec{x} \in D, \quad t > 0, \quad i = 1, \bar{3}$$

where q_i is the heat flux vector and λ the thermal conductivity. Physical law for the process is of the form of linear equation of heat conduction ([11], p. 13)

$$(3) \quad q_{j,j}(\vec{x}, t) + \epsilon c \frac{\partial T}{\partial t}(\vec{x}, t) - W(\vec{x}, t) = 0, \quad \vec{x} \in D, \quad t > 0$$

where W is the heat generated per unit volume and time, ϵc the specific heat at constant strain. Let us replace the q_j vector in (3) relationship for the Fourier relation (2)

$$(4) \quad T_{,jj}(\vec{x}, t) = \frac{1}{\kappa} \frac{\partial T}{\partial t}(\vec{x}, t) - \frac{1}{\lambda} W(\vec{x}, t) = 0, \quad \vec{x} \in D, \quad t > 0$$

The thermal diffusivity $\kappa = \frac{\lambda}{\epsilon c}$ determines the rate of temperature changes. *Governing equation*(4) may describe the phenomena by means of the temperature, provided that boundary loads and initial state are taken into account. This will lead to the initial-boundary value problem, referred to as a model of the thermal process. The set $\{T(\vec{x}, t), \vec{x} \in D\}$ is called the *global*, while the value $T(\vec{x}, t)$ the *local thermal state*.

Thermal flow through a face determined by a normal versor ν_i is defined by

$$(5) \quad \nu q(\vec{x}, t) = q_j(\vec{x}, t) \nu_j(\vec{x}, t)$$

It allows to follow heat fluxes in some arbitrary given directions.

3. Heat conduction problem

The formula (4) is not sufficient for description of some thermal phenomena, if the initial state and the surroundings interactions on the boundary are not taken into account. This results from the definitions of the region and the time as some open sets. Let us subtract the region from its closure, thus obtaining the difference $\partial D = \bar{D} \setminus D$ as a model, called the boundary, of an object where the body get into touch with surround. Similar operation is performed with the time that is subtracted from its closure providing the difference $\{0\} = [0, \infty) \setminus (0, \infty)$ as a model of touch of the time with its surround, making an initial instant, when the infinite end of time is not a real number. Description of the process should refer to the model of the body contact with an environment at the space-time boundary called a limiting condition.

Let ν_i denotes the outward normal to the boundary ∂D . For the process undergoing in the time $(0, \infty)$ in the D domain the limiting condition is considered as an initial condition for $t = 0$ and boundary condition at ∂D . The limiting problem includes the governing equation

$$(6) \quad T_{,jj}(\vec{x}, t) - \frac{1}{\kappa} \frac{\partial T}{\partial t}(\vec{x}, t) = -\frac{1}{\lambda} W(\vec{x}, t), \quad \vec{x} \in D, \quad t > 0$$

initial condition

$$(7) \quad T(\vec{x}, 0) = g(x), \quad \vec{x} \in D$$

and boundary condition

$$(8) \quad {}_0\lambda T_{,j}(\vec{r}, t) \nu_j(\vec{r}, t) + \alpha T(\vec{r}, t) = V(\vec{r}, t) + \alpha_b T(\vec{r}, t), \quad \vec{r} \in \partial D, \quad t > 0$$

The symbol g stands here for the initial state, V the heat generated per unit surface and time, α the surface conductance ([13], p. 19), ${}_bT$ the surrounding medium temperature. Formulae (7)-(9) are referred to as the heat conduction problem. Boundary condition exhaust the boundary surface. Thermal 3rd type condition is defined at all parts of the boundary. The 2nd type condition is obtained for $\alpha \rightarrow 0$, while the 1st one for $\alpha \rightarrow \infty$. Hence, the α parameter defines actual type of the condition assigned to a particular parts of the boundary surface. In the (7)-(9) system of equations the local thermal state appears at the left sides of the equal signs. Let us called this state an unknown. The unknown is a subject of mathematical operations defined as an operator. On the other hand, the right-hand sides of the equations will be referred to as thermal load.

4. Space problem

Let us make a model assumption that the body region fictitiously occupies the whole space Z . Let us define, for the model process undergoing in the time $(0, \infty)$ and in the space Z , a continuous and bounded in infinity fictitious load, making a model of interactions on the body

$$(9) \quad \begin{aligned} \tilde{g} : Z &\rightarrow \mathbb{R}, \quad \lim_{z \rightarrow \infty} |\tilde{g}(\vec{z})| < \infty, \\ \tilde{W} : Z \times (0, \infty) &\rightarrow \mathbb{R}, \quad \lim_{z \rightarrow \infty} |\tilde{W}(\vec{z}, t)| < \infty, \\ &\vec{z} \in Z, \quad t > 0 \end{aligned}$$

We assume that new loads imply similar process as the one occurring in the considered body region

$$(10) \quad \tilde{g}(\vec{z}) = g(\vec{z}), \quad \tilde{W}(\vec{z}, t) = W(\vec{z}, t), \quad \vec{z} \in D, \quad t > 0$$

For the heat conduction problem in fictitious region such temperature field is required

$$(11) \quad \tilde{T} : Z \times (0, \infty) \rightarrow \mathbb{R}$$

that satisfy the following space problem:

$$(12) \quad \begin{aligned} \tilde{T}_{,jj}(\vec{z}, t) - \frac{1}{\kappa} \frac{\partial \tilde{T}}{\partial t}(\vec{z}, t) &= -\frac{1}{0\lambda} \tilde{W}(\vec{z}, t), \quad \vec{z} \in Z, \quad t > 0 \\ \tilde{T}(\vec{z}, 0) &= \tilde{g}(\vec{z}), \quad \vec{z} \in Z \\ \lim_{z \rightarrow \infty} |\tilde{T}(\vec{z}, t)| &< \infty, \quad \vec{z} \in Z, \quad t > 0 \end{aligned}$$

Let us consider an auxiliary problem

$$(13) \quad \begin{aligned} {}_1T_{,jj}(\vec{z}, t) - \frac{1}{\kappa} \frac{\partial {}_1T}{\partial t}(\vec{z}, t) &= 0, \quad \vec{z} \in Z, \quad t > 0 \\ {}_1T(\vec{z}, 0) &= \tilde{g}(\vec{z}), \quad \vec{z} \in Z \\ \lim_{z \rightarrow \infty} |{}_1T(\vec{z}, t)| &< \infty, \quad \vec{z} \in Z, \quad t > 0 \end{aligned}$$

Let F means a fundamental solution to the parabolic operator

$$(14) \quad F(\vec{z}, t) = \begin{cases} 0, & t < 0, \vec{z} \in Z \\ 0, & t = 0, \vec{z} \neq \vec{0} \\ (2\sqrt{\pi\kappa t})^{-3} \exp\left(-\frac{|\vec{z}|^2}{4\kappa t}\right), & t > 0, \vec{z} \in Z \end{cases}$$

The F function satisfies the relationship

$$(15) \quad F_{,jj}(\vec{z}, t) - \frac{1}{\kappa} \frac{\partial F}{\partial t}(\vec{z}, t) = 0, \quad \vec{z} \in D, \quad t > 0$$

For a bounded and continuous function f the fundamental solution enables determining the limit ([17], p. 347)

$$(16) \quad \int_Z f(\vec{y}) F(\vec{z} - \vec{y}, t) d\vec{y} \rightarrow f(\vec{z}), \quad t \rightarrow 0$$

Properties of the F function allow expressing the solution to the problem (14) by the formula ([17], p. 361)

$$(17) \quad \begin{aligned} {}_1T(\vec{z}, t) &= \tilde{g}(Z) * F(\vec{z}, t), \\ \tilde{f}(Z) * F(\vec{z}, t) &\equiv \int_Z f(\vec{y}) F(\vec{z} - \vec{y}, t) d\vec{y}, \quad \vec{z} \in Z, t > 0 \end{aligned}$$

The denotation $(Z)*$ will be called a space convolution multiplication on the set Z , and its result is a space convolution. The improper boundary condition is met thanks to the properties of the fundamental solution to the parabolic operator and from condition (10) that \tilde{g} is the load limited in infinity. Solution to the next auxiliary limiting problem

$$(18) \quad \begin{aligned} {}_2T_{,jj}(\vec{z}, t) - \frac{1}{\kappa} \frac{\partial {}_2T}{\partial t}(\vec{z}, t) &= -\frac{1}{{}_0\lambda} \tilde{W}(\vec{z}, t), \quad \vec{z} \in D, \quad t > 0 \\ {}_2T(\vec{z}, 0) &= 0, \quad \vec{z} \in Z \\ \lim_{z \rightarrow \infty} |{}_2T(\vec{z}, t)| &< \infty, \quad \vec{z} \in Z, t > 0 \end{aligned}$$

has a form ([17], p. 367)

$$(19) \quad {}_2T(\vec{z}, t) = \frac{\kappa}{{}_0\lambda} \tilde{W}(Z \times (0, t)) * F(\vec{z}, t), \quad \vec{z} \in Z, t > 0$$

$$\tilde{f}(Z \times (0, t)) * F(\vec{z}, t) \equiv \int_0^t \int_Z f(\vec{y}, s) F(\vec{z} - \vec{y}, t - s) d\vec{y} ds$$

Symbol $(Z \times (0, t))*$ is named a space-time convolution multiplication on the set $Z \times (0, t)$, and its result a space-time convolution. For an arbitrary integrable function f the following formula is valid

$$(20) \quad \frac{\partial}{\partial t} [f(Z \times (0, t)) * F(\vec{z}, t)] - \kappa [f(Z \times (0, t)) * F(\vec{z}, t)] = f(\vec{z}, t)$$

Similarly to the solution (18) the improper boundary condition is met thanks to the properties of the F function and from condition (10) that the load is limited in infinity. Sum of the solutions (18) and (20) is the exact solution to the space problem

$$(21) \quad \tilde{T}(\vec{z}, t) = \frac{\kappa}{0\lambda} \tilde{W}(Z \times (0, t)) * F(\vec{z}, t) + \tilde{g}(Z) * F(\vec{z}, t), \quad \vec{z} \in Z$$

The functions convolutionally multiplied by F are assumed in continuous and bounded manner, hence, the formula (22) also presents a continuous and bounded function in the Cartesian product $Z \times (0, t)$, where ([17], pp. 361, 363). The limit transition $t \rightarrow 0$ transforms the (22) formula into the Cauchy condition

$$(22) \quad \lim_{t \rightarrow 0} \tilde{T}(\vec{z}, t) = \tilde{g}(\vec{z}), \quad \vec{z} \in Z$$

This results from the property of (17) and zero-measure of current time in the space-time convolution. An interesting property of the F function arises in consequences of the definition (16)

$$(23) \quad f(Z) * \left[\kappa F_{,jj}(\vec{z}, t) - \frac{\partial}{\partial t} F(\vec{z}, t) \right] = 0, \quad \vec{z} \in Z, t > 0$$

The function (22) should describe thermal phenomena in the D region in a finite time.

5. Space integral

The formula (22) provides

$$(24) \quad \begin{aligned} \tilde{T}_{,jj}(\vec{x}, t) - \frac{1}{\kappa} \frac{\partial}{\partial t} \tilde{T}(\vec{x}, t) &= \tilde{g}(Z) * \left[\kappa F_{,jj}(\vec{x}, t) - \frac{1}{\kappa} \frac{\partial}{\partial t} F(\vec{x}, t) \right] + \\ &+ \frac{\kappa}{0\lambda} \tilde{W}(Z \times (0, t)) * F_{,jj}(\vec{x}, t) - \frac{1}{0\lambda} \frac{\partial}{\partial t} [\tilde{W}(Z \times (0, t)) * F(\vec{x}, t)] \end{aligned}$$

$$\vec{z} \in Z, t > 0$$

The equation (7) will be met provided the (24), (21), (11) formulae are used. Approaching the equation (23) gives the condition (8) from the (11) formula. The formula (22) is referred to as a space integral. It is substituted to the boundary condition (9)

$$(25) \quad \begin{aligned} &[\kappa \tilde{W}(Z \times (0, t)) * F(\vec{z}, t) + \tilde{g}(Z) * F(\vec{z}, t)] \nu_j(\vec{r}, t) + \\ &\alpha \left[\frac{\kappa}{0\lambda} \tilde{W}(Z \times (0, t)) * F(\vec{z}, t) + \tilde{g}(Z) * F(\vec{z}, t) \right] = V(\vec{r}, t) + \alpha_b T(\vec{r}, t), \end{aligned}$$

$$\vec{r} \in \partial D, t > 0$$

The expression (26) makes an integral equation with two unknown functions.

6. Temperature fictitious component

Let us make use of the law of additivity of the integral as the set function

$$(26) \quad \tilde{T}(\vec{z}, t) = \frac{\kappa}{0\lambda} W(D \times (0, t)) * F(\vec{z}, t) + g(D) * F(\vec{z}, t) + {}_fW(\vec{z}, t)$$

The (27) formula is called component integral. Its last term is

$$(27) \quad {}_fW(\vec{z}, t) \equiv \frac{\kappa}{0\lambda} \tilde{W}((Z \setminus D) \times (0, t)) * F(\vec{z}, t) + \tilde{g}(Z \setminus D) * F(\vec{z}, t)$$

The function ${}_fW$ will be called the temperature fictitious source. The conditions required for the function (28) to satisfy the formulae (7)-(8)

$$(28) \quad {}_fW_{,jj}(\vec{z}, t) - \frac{1}{\kappa} \frac{\partial}{\partial t} {}_fW(\vec{z}, t) = 0, \quad {}_fW(\vec{z}, 0) = 0, \quad \vec{z} \in Z, t > 0$$

are called the source admissibility conditions. The (9) formula enables determining of the temperature fictitious source

$$(29) \quad 0\lambda {}_fW_{,j}(\vec{r}, t) \nu_j(\vec{r}, t) + \alpha {}_fW(\vec{r}, t) = f(\vec{r}, t), \quad \vec{r} \in \partial D, \quad t > 0$$

The following denotation is used here

$$(30) \quad \begin{aligned} & f(\vec{r}, t) \equiv V(\vec{r}, t) + \alpha_b T(\vec{r}, t) \\ & -\kappa W(Z \times (0, t)) * F(\vec{r}, t) + g(Z) * F(\vec{r}, t) \nu_j(\vec{r}, t) \\ & - \frac{\kappa}{0\lambda} [W(Z \times (0, t)) * F(\vec{r}, t) + g(Z) * F(\vec{r}, t)] \end{aligned}$$

Nevertheless, the temperature fictitious source determined this way usually do not satisfy the admissibility conditions (29). The function (27) satisfies the equations (7)-(8). Should the temperature fictitious source (28) meet the equation (30), a contracted (to the region D) source integral may be obtained

$$(31) \quad \bar{T}(\vec{x}, t) \equiv \frac{\kappa}{0\lambda} W(D \times (0, t)) * F(\vec{x}, t) + g(D) * F(\vec{x}, t) + {}_fW(\vec{x}, t)$$

This could be an accurate solution to the heat conduction problem in the body region.

7. Approximate sources

Let $\bar{D} = D \cup \partial D$ is a body region, inclusive of its boundary. The time $(0, \infty)$ will be contracted to the interval $(0, \bar{t}]$ called the finite time. Replacement ${}_fW$ for the fundamental solution (15) leads to satisfying the source admissibility conditions. Therefore, an approximate temperature fictitious source is defined as

$$(32) \quad {}^a_fW(\vec{z}, t) \equiv \sum_{l=1}^p \sum_{m=1}^n w_l^m F(\vec{z} - {}_m\vec{y}, t - {}^l s), \quad \vec{z} \in \bar{D}, t \in (0, \bar{t}], {}^m y \in Z \setminus \bar{D}$$

The function (33) is referred to as approximate sources, ${}_m\vec{y}$ a fictitious place of the m -th source, ${}^l s$ the l -th fictitious instant, w_l^m the capacity in the ${}_m\vec{y}$ place at the ${}^l s$ instant, the set $\{{}_m\vec{y} : m = 1, \bar{n}\}$ fictitious location, and $\{{}^l s : m = 1, \bar{n}\}$ a fictitious time. The fictitious source (28) occurring in the formula (27) shall be replaced for its approximate form (33)

$$(33) \quad \hat{T}(\vec{z}, t) \equiv \frac{\kappa}{0\lambda} W(D \times (0, t)) * F(\vec{z}, t) + g(D) * F(\vec{z}, t) + {}^a_fW(\vec{z}, t)$$

This is an approximate integral. In the finite time $(0, \tilde{t}]$ the boundary surface may be so modeled as to make it independent on t . The outward normal appears in the approximate boundary condition does not depend on time. A sufficient condition for the temperature fictitious source is the approximate integral satisfying the (9) formula

$$(34) \quad {}_0\lambda_f^\alpha W_{,j}(\vec{r}, t) \nu_j(\vec{r}, t) + \alpha_f^\alpha W(\vec{r}, t) = f(\vec{r}, t), \quad \vec{r} \in \partial D, t \in (0, \tilde{t}]$$

This is an equation for the capacities of approximate sources (33). Now the definition (33) is substituted into the equation (35) that provides

$$(35) \quad \sum_{l=1}^p \sum_{m=1}^n w_l^m G_m^l(\vec{r}, t) = f(\vec{r}, t), \quad \vec{r} \in \partial D, t \in (0, \tilde{t}], w_l^m \in \mathbb{R}, {}^m y \in Z \setminus \bar{D}$$

where

$$(36) \quad G_m^l(\vec{r}, t) \equiv {}_0\lambda F_{,j}(\vec{r} - {}^m \vec{y}, t - {}^l s) + \alpha F(\vec{r} - {}^m \vec{y}, t - {}^l s)$$

Let the set ${}^l t : m = 1, \bar{n}$ to be referred to as a test time and let ${}^k t < {}^{k+1} t$, $k = 1, \bar{p} - 1$. This allows replacing the (36) equation by the set of equations

$$(37) \quad \sum_{l=1}^p \sum_{m=1}^n w_l^m G_m^l(\vec{r}, {}^k t) = f(\vec{r}, {}^k t), \quad \vec{r} \in \partial D, {}^k t \in (0, \tilde{t}]$$

Let us assume

$$(38) \quad {}^1 s = 0, \quad {}^{l-1} t \leq {}^l s < {}^l t, \quad l = 2, \bar{p}$$

The F function vanishes for ${}^l s \geq {}^k t$ irrespective of ${}^m \vec{y}$, while for ${}^l s < {}^k t$ does not vanish in any place ${}^m \vec{y}$

$$(39) \quad \sum_{l=1}^k \sum_{m=1}^n w_l^m G_m^l(\vec{r}, {}^k t) = f(\vec{r}, {}^k t), \quad \vec{r} \in \partial D, {}^k t \in (0, \tilde{t}]$$

The w_l^m values may be determined in recurrent manner with respect to the test instants ${}^k t$

$$(40) \quad \sum_{m=1}^n w_l^m G_m^l(\vec{r}, {}^k t) = f_k(\vec{r}, {}^k t), \quad \vec{r} \in \partial D, {}^k t \in (0, \tilde{t}]$$

Here the following denotations are used

$$(41) \quad f_k(\vec{r}, {}^k t) \equiv f(\vec{r}, {}^k t) - \sum_{l=1}^{k-1} \sum_{m=1}^n w_l^m G_m^l(\vec{r}, {}^k t), \quad k = 2, \bar{p}$$

Lets cover the ∂D boundary with a mesh ${}_h \vec{r}$, $h = 1, \bar{n}$, where $N > n$. Then

$$(42) \quad \sum_{m=1}^n w_l^m B_{hm}^{kl} = f_k({}_h \vec{r}, {}^k t) \quad {}_h \vec{r} \in \partial D, {}^k t \in (0, \tilde{t}], \quad B_{hm}^{kl} \equiv G_m^l({}_h \vec{r}, {}^k t)$$

The formula (44) is a recurrence with regard to the test time. The ∂D boundary is a closed surface, while analytical explicit presentation exists only for a straight faces obtained from rectangle in result of extension, compression, and bending, without tearing off and gluing ([18], p. 95). The ∂D surface may include singular points, i.e. edges or cone vertices. Then, it is divided into regular surfaces, without singular points and, further on, into regular

faces. A set of points in the space as a unique one-to-one and continuously differentiable picture of a rectangle with its boundary is called a regular face

$$(43) \quad \vec{r} = \vec{\rho}(\omega), \quad \omega = (\omega_1, \omega_2)$$

A close rectangle in the plane of the parameters (ω_1, ω_2) is a domain of the function, of the derivatives denoted $\vec{\rho}_i, i = 1, 2$. Unit normal to the regular face for ${}_0\omega = ({}_0\omega_1, {}_0\omega_2)$ is defined by the formula

$$(44) \quad \nu_i(\vec{\rho}({}_0\omega)) = \mp (h)^{-1} h_i, \quad h_i \equiv \epsilon_{ijk} a_j b_k, \quad \vec{a} \equiv \vec{\rho}_1({}_0\omega), \quad \vec{b} \equiv \vec{\rho}_2({}_0\omega)$$

Let us choose the sign of the normal ν placed outside the regular face which is cutting out from the closed surface ([18], p. 108). For the regular face $h \neq 0$ is always valid ([18], p. 96). A parallel face located in the distance d from the regular face (44) is defined by the formula

$$(45) \quad \vec{y} = \vec{\rho}(\omega) + d\vec{\nu}(\vec{\rho}(\omega)), \quad \omega = (\omega_1, \omega_2)$$

One of the ways of approximate satisfaction of boundary conditions is provided by the method of boundary collocation ([19], p. 8). We shall apply a modification of this method named the method of boundary collocation in the least-squares approach ([19], p. 13). For a fixed test instant ${}^k t \in (0, \bar{t}]$ a set of faces parallel to the selected regular faces on the boundary is defined. Let us defined the collocation nodes ${}_h \vec{r}$, $h = 1, \bar{1}N$ of the first face (44), cut out from the regular surface on ∂D . The fictitious places ${}_m \vec{y}$, $h = 1, \bar{1}n$, ${}_1 n \leq {}_1 N$ shall be determined on the part of the first parallel face (46) in the distance ${}_1 d$, that does not intersect the boundary. Let us defined the collocation nodes ${}_h \vec{r}$, $h = 2, \bar{2}N$ on the second face (44) cut out from the regular surface on the boundary. A mesh ${}_m \vec{y}$, $h = 1, \bar{1}n$, ${}_1 n \leq {}_1 N$ corresponding to it will be selected at the part of the second parallel face (46) in the distance ${}_1 d$, that does not intersect the surface ∂D nor the first parallel face. All the collocation nodes ${}_h \vec{r}$, $h = 1, \bar{N}$ and the fictitious location $({}_m \vec{y}, h = 1, \bar{n})$ may be obtained by a recurrence with regard to the other regular faces on ∂D , assuming fictitious locations for the parts of parallel faces that do not intersect the boundary nor previous parallel faces. Formula (43) for the test instant ${}^1 t$ provides

$$(46) \quad \sum_{m=1}^n w_1^m B_{hm}^{11} = f({}_h \vec{r}, {}^1 t) \quad h = 1, \bar{N}$$

The system of linear equations may be solved with the least-squares method [20]. Values \tilde{w}_1^m , $m = 1, \bar{n}$ minimize the variance of summarized error at the first regular face for the test instant ${}^1 t$

$$(47) \quad {}^1 H(N, n) \equiv \sum_{h=1}^N [f({}_h \vec{r}, {}^1 t) - \sum_{m=1}^n w_1^m B_{hm}^{11}]^2 = \min$$

A condition necessary for extremum of the variance ${}^1 H$ with respect to w_1^m gives normal equation

$$(48) \quad \sum_{m=1}^n w_1^m \sum_{h=1}^N B_{hk}^{11} B_{hm}^{11} = \sum_{h=1}^N f({}_h \vec{r}, {}^1 t) B_{hk}^{11}, \quad k = 1, \bar{n}$$

Solution \tilde{w}_1^m , $m = 1, \bar{n}$ represents the temperature capacities for the fictitious instant 1s . Measure of the error is given by standard deviation

$$(49) \quad {}^1S(N, n) \equiv \sqrt{{}^1H(N, n) / (N - n)}$$

The measure may be applied if the values $f({}_h\vec{r}, {}^1t)$ of the definition (31) are statistically correlated. Let us assume that such a correlation really exists. The \tilde{w}_1^m , $m = 1, \bar{n}$ values are accepted provided that the standard deviation ${}^1S(N, n)$ for the whole boundary ∂D in the test instant 1t is smaller than given ϵ . The formula (42) is recursively applied until the test instant 2t

$$(50) \quad \sum_{m=1}^n w_2^m B_{hm}^{22} = f({}_h\vec{r}, {}^2t) \quad h = 1, \bar{N}$$

Values \tilde{w}_2^m , $m = 1, \bar{n}$ minimize the variance of summarized error for the test instant 2t

$$(51) \quad {}^2H(N, n) \equiv \sum_{h=1}^N [f_2({}_h\vec{r}, {}^2t) - \sum_{m=1}^n w_2^m B_{hm}^{22}]^2 = \min$$

A condition necessary for extremum of the variance 2H with respect to w_2^m leads to the system

$$(52) \quad \sum_{m=1}^n w_2^m \sum_{h=1}^N B_{hk}^{22} B_{hm}^{22} = \sum_{h=1}^N f_2({}_h\vec{r}, {}^2t) B_{hk}^{22}, \quad k = 1, \bar{n}$$

A solution of this set of equations gives the values \tilde{w}_2^m , $m = 1, \bar{n}$ for the fictitious instant 2s . The solution is accepted when ${}^2S(N, n) < \epsilon$. The measure may be applied if the values $f_2({}_h\vec{r}, {}^2t)$ of the definition (41) are statistically correlated. Let us assume that such a correlation really exists. The procedure is repeated for the next indexes $l = 3, \bar{p}$ in order to determine \tilde{w}_l^m values in fictitious location $\{{}_m\vec{y} : m = 1, \bar{n}\}$ for the fictitious time $\{{}^l s : m = 1, \bar{n}\}$. Recurrence procedure performed with respect to time from the previous fictitious instances enable determining the capacities for the next fictitious instant. Therefore, the approximate sources are derived as

$$(53) \quad {}_f^a\tilde{W}(\vec{z}, t) \equiv \sum_{l=1}^p \sum_{m=1}^n w_l^m F(\vec{z} - {}_m\vec{y}, t - {}^l s), \quad \vec{z} \in \bar{D}, t \in (0, \tilde{t}]$$

The approximate integral distribution may be obtained by substituting the function ${}_f^a\tilde{W}$ into ${}_f^aW$

$$(54) \quad \tilde{T}(\vec{z}, t) \equiv \frac{\kappa}{0\lambda} W(D \times (0, t)) * F(\vec{z}, t) + g(D) * F(\vec{z}, t) + {}_f^a\tilde{W}(\vec{z}, t)$$

For $t > \tilde{t}$ boundary condition is not satisfied and the formula (55) is not even defined. Nevertheless, the final instant may be freely late, hence, the function \tilde{T} may be considered as approximate temperature field.

8. Approximate solution

Distribution (55) enables determining approximate solution to the heat conduction problem

$$(55) \quad {}^a T(\vec{x}, t) \equiv \frac{\kappa}{{}_0\lambda} W(D \times (0, t)) * F(\vec{x}, t) + g(D) * F(\vec{x}, t) + {}_f^a \tilde{W}(\vec{x}, t)$$

The expression (56) represents an approximate solution to the problem (7)-(9). It is characterized by local dependence of the heat generated per unit surface and time and surrounding medium temperature, the dependence being intermediated by the boundary condition. The solution meets the boundary conditions only in collocation nodes and in test instants. The Fourier law allows to determine approximate form of the heat flux vector

$$(56) \quad {}^a q_i(\vec{x}, t) = -\kappa W(D \times (0, t)) * F_{,i}(\vec{x}, t) - {}_0\lambda g(D) * F_{,i}(\vec{x}, t) - {}_0\lambda_f^a \tilde{W}_{,i}(\vec{x}, t)$$

In (57) formula the global dependence of the heat source capacity, and the initial state is of direct character. On the other hand, the indirect local dependence on the boundary condition adjusted by the capacities of approximate temperature sources enables the surroundings temperature and the boundary heat source to affect the distribution heat flux vector in the region in any time instant. Thermal flow through a foil of the surface determined by a normal versor is

$$(57) \quad {}^\nu q(\vec{x}, t) = -\kappa W(D \times (0, t)) * F_{,\nu}(\vec{x}, t) - {}_0\lambda g(D) * F_{,\nu}(\vec{x}, t) - {}_0\lambda_f^a \tilde{W}_{,\nu}(\vec{x}, t)$$

A full description of thermal phenomena by means of the temperature is determined (approximately) by the formula (57) in each place inside the body in the time. The thermal fields (58)-(59) only provide more extensive illustration of the course of the considered process that is useful for further application.

9. Conclusions

The paper presents a method of solving the quasi-static problems of heat conduction named the method of temperature sources. The new distributions of the initial state and heat generated per unit volume and time enabled solving the space problem in a fictitious unlimited medium. Convolution products of thermal loads by fundamental solution to the partial differential operator have been obtained in result. Components of this products, called fictitious sources, have been used to verification of the boundary condition in the finite time range with a method of boundary collocation in least-squares approach. In order to locate the fictitious sources the regular surface faces have been applied. This enables using a least squares procedure with respect to collocation places at the boundary, in order to determine capacity values of approximate temperature sources. The solution is continuously differentiable, thus enabling determining the distribution of heat flux vector.

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