DOI:10.1478/C1S0801018

Atti dell'Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
Vol. LXXXVI, C1S0801018 (2008) - Suppl. 1

WAVES PROPAGATION IN SUPERFLUID HELIUM IN PRESENCE OF COMBINED ROTATION AND COUNTERFLOW

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ABSTRACT. Using the linear macroscopic mono-fluid model of liquid helium II, in which the fundamental fields are the density ρ, the velocity $\mathbf{v}$, the temperature T and heat flux $\mathbf{q}$ and taking into account the expression of an additional pressure tensor $\mathbf{P}_\omega$, introduced to describe phenomena linked to vortices, a complete study of wave propagation is made in the complex situation involving thermal counterflow in a rotating cylinder.

1. Introduction

The most known phenomenological model of He II, given by *Tisza* [1] and *Landau* [2] is called the two-fluid model. The basic assumption is that the liquid behaves as a mixture of two fluids: the normal component and the superfluid component. But this model is only an intuitive picture because, in reality, the fluid is not partitioned into two distinct components. Further some nonlinear effects, such as the evolution of the thermal shock profiles and the turbulence cannot be completely explained within this framework.

The quantized superfluid vortices play an important role in the hydrodynamics of the fluid and they have been the principal object of many studies. The state of the fluid in which vortices are present, is referred to as the *superfluid turbulent state*. The quantized vortices, created applying a thermal counterflow, form an irregular, spatially disorder tangle of lines. In this case, the vortex line density (length of vortex line per unit volume) is $L_H \approx \gamma^2 V^2$, where V is the absolute value of the counterflow velocity $\mathbf{V} := \mathbf{v}_n - \mathbf{v}_s$ and γ is a temperature dependent coefficient [3]. The vortex system is almost isotropic, provided that one neglects a small anisotropy induced by the imposed counterflow [5].

The creation of the vortices cannot be made only in this way, in fact the first studies of quantized vorticity involved a sample of He II rotating at constant angular velocity exceeding a certain small critical value. The results brought to an ordered array of vortices aligned along the rotation axis, whose areal number density is given by Feynman's rule $L_R = \frac{2\Omega}{\kappa}$, where $\kappa = 9.97\ 10^{-4}\ \mathrm{cm}^2/\mathrm{sec}$ is the quantum of vorticity.

Now, an important question arises: what does it happen when vortices are created by both rotation and counterflow? There has been only one experiment we are aware [6], in which the formation of vortices in combined situation of rotation and counterflow along the rotational axis is experimented. The experimental observations show that the two effects (thermal counterflow and rotation) are not merely additive, in fact for high values of V the measured values of L are always less than $L_H + L_R$. These experiments show that the anisotropy of the tangle is important in combined situations.

In this work the propagation of longitudinal and transversal waves in the combined situation of a cylindrical container in presence of thermal counterflow is studied. Our aim is to obtain more information on the vortex tangle by measurements on the wave propagation.

Under combined influence of rotation and counterflow, the vortex tangle cannot be assumed isotropic, in contrast with the usual situation in pure counterflow. Then, we are interested to find not only the vortex line density L but also the geometrical characterization, which requires to consider wave propagation in several different directions. In this analysis the parameters which characterize the vorticity on the propagation of the waves is put in evidence.

The plane of this paper is the following: Sect.**2** is concerned with a recent mono-fluid model for helium II, in which the use of a pressure tensor associated to some parameters describing the vorticity has been considered; in Sect.**3** we study the wave propagation in the case of rotation and counterflow, analyzing two different situations about the relative direction of propagation with respect to the rotation vector.

2. Evolution equations

A linear macroscopic mono-fluid model of liquid helium II, based on Extended Thermodynamics [7, 8], has been formulated in a previous work [9]. This model is able to describe the laminar flow of the superfluid both in the presence and in absence of dissipative phenomena and to predict the propagation of the two sounds in bulk liquid helium II and of the fourth sound in liquid helium flowing in a porous medium [9], [10]-[13], in agreement with microscopic and experimental data.

In order to describe phenomena such as the formation of vortices in rotating helium II, in superfluid turbulence or in combined situation of rotation and thermal counterflow, the use of a further additional pressure tensor $\mathbf{P}_\omega$, associated to some parameters describing the vorticity, is necessary. Neglecting, in a first study, the evolution of the vortex lines, the constitutive relation for $\mathbf{P}_\omega$ and its influence on the dynamics of the heat flux has been studied [13]-[17].

The fundamental fields of a mono-fluid model for helium II are density ρ, velocity $\mathbf{v}$, absolute temperature T and heat flux $\mathbf{q}$. The linearized system of field equations for liquid helium II, in a non inertial frame and in absence of external force, is [14]:

$$\tag{1} \begin{cases} \frac{\partial \rho}{\partial t} + \rho \frac{\partial v_j}{\partial x_j} = 0 \\ \rho \frac{\partial v_i}{\partial t} + \frac{\partial p}{\partial x_i} + \mathbf{i}_i^0 + 2\rho \left(\mathbf{\Omega} \wedge \mathbf{v} \right)_i = 0 \\ \frac{\partial T}{\partial t} + \frac{1}{\rho c_V} \frac{\partial q_j}{\partial x_j} = 0 \\ \frac{\partial q_i}{\partial t} + \zeta \frac{\partial T}{\partial x_i} + 2 \left(\mathbf{\Omega} \wedge \mathbf{q} \right)_i = \left(\sigma_\omega \right)_i = - \left(\mathbf{P}_\omega \cdot \mathbf{q} \right)_i . \end{cases}$$

where $\mathbf{i}^0 + 2\rho \left(\mathbf{\Omega} \wedge \mathbf{v} \right)$ is the inertial force, ζ is a positive coefficient linked to the second sound velocity, p is the thermostatic pressure and c_V is the specific heat. The effect of vortices is described by a source term proportional to $\mathbf{P}_\omega \cdot \mathbf{q}$ in the evolution equation of the heat flux. In this equation $\mathbf{P}_\omega$ is a vorticity tensor associated to vortex lines, whose explicit form is [14]:

$$\tag{2} \mathbf{P}_\omega = \kappa L \left(\gamma < \mathbf{U} - \mathbf{s}'\mathbf{s}' > + \gamma' < \mathbf{W} \cdot \mathbf{s}' > \right)$$

where L is the vortex line density and $\mathbf{s}'$ is the unit vector along the vortex line at a given point. Here, $\mathbf{s} = \mathbf{s}(\xi, t)$ is a function describing the vortex line (being ξ the arc-length measured along the curve of vortex filament), the prime symbol in $\mathbf{s}'$ denotes the partial derivative with respect to the length ξ. The angular brackets denote the macroscopic averages made by integration along all vortices in the sample volume, i.e. $< \mathbf{W} \cdot \mathbf{s}' >:= \frac{1}{\Lambda L} \int \mathbf{W} \cdot \mathbf{s}' d\xi$.

3. Waves propagation in the combined situation of rotation and counterflow

The combined situation of rotation and heat flux, as shown in Fig.1, is a relatively new area of investigation [16]-[20].

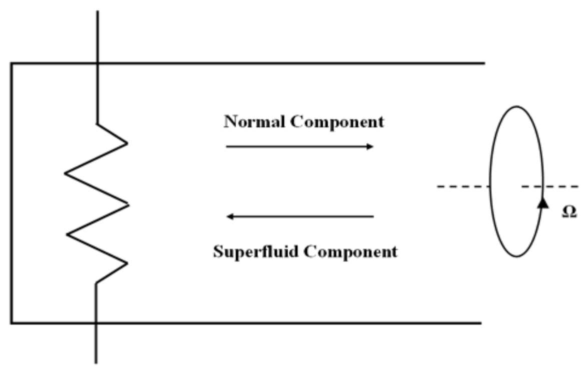

Fig.1 Rotating counterflow container configuration

Some experiments shown that the vortex tangle is anisotropic so that, assuming that the rotation is along the x direction $\mathbf{\Omega} = (|\Omega|, 0, 0)$ and isotropy is in the transversal $(y - z)$ plane, for the vorticity tensor $\mathbf{P}_\omega$, in combined counterflow and rotation, the following explicit expression has been taken [13]:

$$\mathbf{P}_\omega = \gamma\kappa L \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix} \right\} \tag{3}$$

where $a = \langle {s'}_x^2 \rangle$, $b = \langle {s'}_y^2 \rangle = \langle {s'}_z^2 \rangle$, and the coefficient c, linked to the antisymmetric tensor $\mathbf{W}$, is proportional to $\langle {s'}_x \rangle$, i.e. it characterizes the percent of vortices which are parallel to the rotation axis [17].
Choosing $\mathbf{q} = (q_1, q_2, q_3)$, the production term assumes the following expression:

$$\begin{aligned} \mathbf{P}_\omega \cdot \mathbf{q} &= \gamma\kappa L \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix} \right\} \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \\ &= \gamma\kappa L \left\{ (1-a)\, q_1 \mathbf{c_1} + [(1-b)\, q_2 + c q_3]\, \mathbf{c_2} + [(1-b)\, q_3 - c q_2]\, \mathbf{c_3} \right\} \end{aligned} \tag{4}$$

where $\mathbf{c}_1$, $\mathbf{c}_2$ and $\mathbf{c}_3$ are the unit vectors of the non-inertial frame of the rotating container.

Substituting the expression (4) into the linearized set of field equations (1), it assumes the following form:

$$(5)\qquad \begin{cases} \frac{\partial\rho}{\partial t}+\rho\frac{\partial v_j}{\partial x_j}=0\\ \rho\frac{\partial v_i}{\partial t}+\frac{\partial p}{\partial x_i}+2\rho|\Omega|v_j\epsilon_{3ji}=0\\ \frac{\partial T}{\partial t}+\frac{1}{\rho c_V}\frac{\partial q_j}{\partial x_j}=0\\ \frac{\partial q_i}{\partial t}+\zeta\frac{\partial T}{\partial x_i}+2|\Omega|q_j\epsilon_{1ji}=\\ \qquad -\gamma\kappa L\left\{(1-a)\,q_1\delta_{1i}+\left[(1-b)\,q_2+cq_3\right]\delta_{2i}+\left[(1-b)\,q_3-cq_2\right]\delta_{3i}\right\} \end{cases}$$

A stationary solution of this system is:

$$\rho=\rho_0,\ \ \dot{\mathbf{v}}=\mathbf{0},\ \ \mathbf{q}=\mathbf{q}_0\equiv(q_{01},q_{02},q_{03})$$
$$T=T(x_i)=T_0-\frac{2|\Omega|}{\zeta}q_{0j}\epsilon_{1ji}x_i+$$
$$-\frac{\gamma\kappa L}{\zeta}\left\{(1-a)\,q_{01}\delta_{1i}+\left[(1-b)\,q_{02}+cq_{03}\right]\delta_{2i}+\left[(1-b)\,q_{03}-cq_{02}\right]\delta_{3i}\right\}x_i.$$

In order to study the propagation of harmonic plane waves, we look for solutions of the following form:

$$(6)\qquad V=V_0+\tilde{V}e^{i(Kn_jx_j-\omega t)},$$

where $V_0=(\rho_0,\,0,\,T(x_i),\,\mathbf{q}_0)$ and $T(x_i)$ is a linear function of x_i. Substituting the expression (6) in the system (5) an algebraic system for the small amplitudes $\tilde{V}$ is obtained, whose latter equation reads as:

$$(7)\qquad \begin{aligned}&-\omega\tilde{q}_i+\zeta K\tilde{T}n_i-2i|\Omega|\tilde{q}_j\epsilon_{1ji}=\\ &i\gamma\kappa L\left\{(1-a)\,\tilde{q}_1\delta_{1i}+\left[(1-b)\,\tilde{q}_2+c\tilde{q}_3\right]\delta_{2i}+\left[(1-b)\,\tilde{q}_3-c\tilde{q}_2\right]\delta_{3i}\right\}\end{aligned}$$

3.1. First case: $\mathbf{n}$ parallel to $\mathbf{\Omega}$. In this subsection we analyze the case in which the unit vector $\mathbf{n}$, orthogonal to the wave front, is parallel to the direction of the rotation, i.e. $\mathbf{n}=(1,0,0)$. Letting $\mathbf{t}_1=(0,1,0)$ and $\mathbf{t}_2=(0,0,1)$ as unit vectors tangent to the wave front, the system for the small amplitudes becomes:

$$(8)\qquad \begin{cases} -\omega\tilde{\rho}+K\rho\tilde{v}_1=0\\ -\omega\tilde{v}_1+K\frac{p_\rho}{\rho}\tilde{\rho}=0\\ -\omega\tilde{T}+\frac{K}{\rho c_V}\tilde{q}_1=0\\ \left[-\omega-i\gamma\kappa L(1-a)\right]\tilde{q}_1+\zeta K\tilde{T}=0\\ \\ -\omega\tilde{v}_2+2i|\Omega|\tilde{v}_3=0\\ \left[-\omega-i\gamma\kappa L\,(1-b)\right]\tilde{q}_2+\left(2i|\Omega|-i\gamma\kappa Lc\right)\tilde{q}_3=0\\ \\ -\omega\tilde{v}_3-2i|\Omega|\tilde{v}_2=0\\ \left[-\omega-i\gamma\kappa L\,(1-b)\right]\tilde{q}_3-\left(2i|\Omega|-i\gamma\kappa Lc\right)\tilde{q}_2=0 \end{cases}$$

This system (8) can be written in the following matricial form:

$$\begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \\ \tilde{U}_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

where

$$A = \begin{pmatrix} -\omega & \rho K \\ \frac{p_\rho}{\rho} K & -\omega \end{pmatrix}, \quad B = \begin{pmatrix} -\omega & \frac{K}{\rho c_V} \\ \zeta K & -\omega - i\gamma\kappa L(1-a) \end{pmatrix},$$
$$C = \begin{pmatrix} -\omega & 2i|\Omega| \\ -2i|\Omega| & -\omega \end{pmatrix}, \quad D = \begin{pmatrix} -\omega - i\gamma\kappa L(1-b) & (2i|\Omega| - i\gamma\kappa Lc) \\ -(2i|\Omega| - i\gamma\kappa Lc) & -\omega - i\gamma\kappa L(1-b) \end{pmatrix},$$
$$\tilde{U}_1 = \begin{pmatrix} \tilde{\rho} & \tilde{v}_1 \end{pmatrix}^t, \quad \tilde{U}_2 = \begin{pmatrix} \tilde{T} & \tilde{q}_1 \end{pmatrix}^t, \tilde{U}_3 = \begin{pmatrix} \tilde{v}_2 & \tilde{v}_3 \end{pmatrix}^t \quad \text{and} \quad \tilde{U}_4 = \begin{pmatrix} \tilde{q}_2 & \tilde{q}_3 \end{pmatrix}^t.$$

In this case the longitudinal and transversal modes evolve independently. In particular, we observe that the first sound, given by the study of the subsystem $A\tilde{U}_1 = 0$, is not influenced by the combined situation:

$$\boxed{\omega_{1,2} = \pm k_r V_1}$$
$$\boxed{\begin{array}{l} \tilde{\rho} = \psi \\ \tilde{v}_1 = \frac{V_1}{\rho}\psi \end{array}}$$

while the second sound suffers an extra attenuation due to the combined situation. In fact, studying the following subsystem $B\tilde{U}_2 = 0$,

$$\begin{cases} -\omega\tilde{T} + \frac{K}{\rho c_V}\tilde{q}_1 = 0 \\ \left[-\omega - i\gamma\kappa L(1-a)\right]\tilde{q}_1 + \zeta K\tilde{T} = 0 \end{cases} \tag{10}$$

it admits non trivial solutions if and only if its determinant vanishes, obtaining the following dispersion relation:

$$\omega^2 + i\gamma\kappa L(1-a)\omega - K^2 V_2^2 = 0. \tag{11}$$

Supposing that ω is real and $K = k_r + ik_s$ is complex, the dispersion relation (11) has the following solutions for the phase velocity and for the attenuation coefficient:

$$w_2^2 := \frac{\omega^2}{k_r^2} = V_2^2 \frac{2}{1 + \sqrt{1 + \frac{\gamma^2\kappa^2 L^2 (1-a)^2}{\omega^2}}}, \tag{12}$$

$$k_s = \frac{\gamma\kappa L(1-a) w_2}{2V_2^2}. \tag{13}$$

Approximating these solutions to second order in $L(1-a)$, we obtain:

$$w_2 \simeq V_2, \tag{14}$$

$$k_s \simeq \frac{1}{2w_2}\gamma\kappa L(1-a). \tag{15}$$

Now, we study the transversal modes. Let us consider the following subsystem $C\tilde{U}_3 = 0$:

$$\begin{cases} -\omega\tilde{v}_2 + 2i|\Omega|\tilde{v}_3 = 0 \\ -\omega\tilde{v}_3 - 2i|\Omega|\tilde{v}_2 = 0 \end{cases} \tag{16}$$

which admits nontrivial solutions if and only if its determinant vanishes; this yields:

$$\omega^2 - 4|\Omega|^2 = 0. \tag{17}$$

The solutions of this equation are $\omega_{5,6} = \pm 2|\Omega|$, to which the following modes correspond:

$$\boxed{\omega_{5,6} = \pm 2|\Omega|}$$
$$\boxed{\begin{array}{l} \tilde{v}_3 = \psi \\ \tilde{v}_2 = \pm i\psi \end{array}}$$

These modes refers to extremely slow phenomena, which tend to stationary modes when $|\Omega| \to 0$.

Finally, we consider the subsystem $D\tilde{U}_4 = 0$, whose dispersion relation is:

$$\omega^2 + 2i\gamma\kappa L(1-b)\omega + \left[-\left(\gamma\kappa L(1-b)\right)^2 - 4|\Omega|^2 + 4\gamma|\Omega|\kappa Lc - (\gamma\kappa Lc)^2\right] = 0. \tag{18}$$

The equation (18) admits the following exact complex solutions:

$$\omega_{7,8} = \pm\left(2|\Omega| - \gamma\kappa Lc\right) - i\gamma\kappa L\left(1-b\right), \tag{19}$$

whose corresponding modes are:

$$\boxed{\omega_{7,8} = \pm\left(2|\Omega| - \gamma\kappa Lc\right) - i\gamma\kappa L\left(1-b\right)}$$
$$\boxed{\begin{array}{l} \tilde{q}_3 = \psi \\ \tilde{q}_2 = \pm i\psi \end{array}}$$

From (12), (13) and (19) one may obtain the following quantities L, b and c:

$$L = \frac{-\omega_s w_2 + V_2^2 k_s}{\gamma\kappa w_2}, \quad b = \frac{V_2^2 k_s}{-\omega_s w_2 + V_2^2 k_s}, \quad c = \frac{-\omega_r w_2 + 2|\Omega| w_2}{-\omega_s w_2 + V_2^2 k_s} \tag{20}$$

where we have put $\omega_7 = \omega_r + i\omega_s$.

From the physical point of view this implies that measurements in a single direction are enough to give information on all the variables describing the vortex tangle. This is not an immediate intuitive result.

3.2. Second case: $\mathbf{n}$ orthogonal to $\boldsymbol{\Omega}$. Now we assume that the direction of the wave propagation is orthogonal to the rotation axis, i.e. for example $\mathbf{n} = (0, 1, 0)$. The unit vectors tangent to the wave front are $\mathbf{t}_1 = (1, 0, 0)$ and $\mathbf{t}_2 = (0, 0, 1)$. In these assumptions, the homogeneous algebraic linear system for the small amplitudes is:

$$
\begin{cases}
-\omega\tilde{\rho} + K\rho\tilde{v}_2 = 0 \\
-\omega\tilde{v}_2 + K\frac{p_\rho}{\rho}\tilde{\rho} + 2i|\Omega|\tilde{v}_3 = 0 \\
-\omega\tilde{T} + \frac{K}{\rho c_V}\tilde{q}_2 = 0 \\
-\omega\tilde{q}_2 + \zeta K\tilde{T} + 2i|\Omega|\tilde{q}_3 = i\gamma\kappa L\left[(1-b)\tilde{q}_2 + c\tilde{q}_3\right] \\
\\
-\omega\tilde{v}_1 = 0 \\
\left[-\omega - i\gamma\kappa L(1-a)\right]\tilde{q}_1 = 0 \\
\\
-\omega\tilde{v}_3 - 2i|\Omega|\tilde{v}_2 = 0 \\
-\omega\tilde{q}_3 - 2i|\Omega|\tilde{q}_2 = i\gamma\kappa L\left[(1-b)\tilde{q}_3 - c\tilde{q}_2\right]
\end{cases}
\tag{21}
$$

In this case the longitudinal and the transversal modes not evolve independently. In particular, the first sound is coupled with one of the two transversal modes in which velocity vibrates, while the second sound is coupled with a transversal mode in which heat flux vibrates.

This system can be written in the following matricial form:

$$
\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}
\begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{pmatrix} =
\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\tag{22}
$$

where

$$
A = \begin{pmatrix} -\omega & \rho K & 0 \\ \frac{p_\rho}{\rho}K & -\omega & 2i|\Omega| \\ 0 & -2i|\Omega| & -\omega \end{pmatrix}, \quad
C = \begin{pmatrix} -\omega & 0 \\ 0 & -\omega - i\gamma\kappa L(1-a) \end{pmatrix},
$$

$$
B = \begin{pmatrix} -\omega & \frac{K}{\rho c_V} & 0 \\ \zeta K & -\omega - i\gamma\kappa L(1-b) & 2i|\Omega| - i\gamma\kappa Lc \\ 0 & -2i|\Omega| + i\gamma\kappa Lc & -\omega - i\gamma\kappa L(1-b) \end{pmatrix},
$$

$$
\tilde{U}_1 = \begin{pmatrix} \tilde{\rho} & \tilde{v}_2 & \tilde{v}_3 \end{pmatrix}^t, \quad \tilde{U}_2 = \begin{pmatrix} \tilde{T} & \tilde{q}_2 & \tilde{q}_3 \end{pmatrix}^t, \quad \tilde{U}_3 = \begin{pmatrix} \tilde{v}_1 & \tilde{q}_1 \end{pmatrix}^t.
$$

The subsystem $A\tilde{U}_1 = 0$ admits non trivial solutions if and only if its determinant vanishes, namely:

$$
-\omega\left[\omega^2 - 4|\Omega|^2 - K^2 p_\rho\right] = 0.
\tag{23}
$$

The modes corresponding to the frequencies $\omega_{1,2} \simeq \pm k_r V_1 + O(|\Omega|^2)$ are inattenuated waves, as shown below,

$\omega_{1,2} \simeq \pm k_r V_1 + O(\lvert\Omega\rvert^2)$	$\omega_3 = 0$
$\tilde{\rho} = \psi$	$\tilde{\rho} = 0$
$\tilde{v}_2 = \frac{\pm V_1}{\rho}\psi$	$\tilde{v}_2 = 0$
$\tilde{v}_3 = -\frac{2i\lvert\Omega\rvert}{\rho K}\psi$	$\tilde{v}_3 = \psi$

which correspond to a pressure wave, i.e to the first sound, when $\mathbf{\Omega} = 0$.

The dispersion relation of the subsystem $B\tilde{U}_2 = 0$ is:

$$(-\omega - i\gamma\kappa L(1-b))\left[\omega\left(-\omega - i\gamma\kappa L(1-b)\right) + K^2 V_2^2\right] + \\ +\omega\left(2i|\Omega| - i\gamma\kappa Lc\right)^2 = 0. \tag{24}$$

Assuming that $\omega \in \Re$ and $K = k_r + ik_s$, one obtains the following two equations:

$$-\omega^3 + \gamma^2\kappa^2 L^2(1-b)^2\omega + 4|\Omega|^2\omega + \gamma^2\kappa^2 L^2 c^2\omega - 4\gamma\kappa Lc|\Omega|\omega + \\ +k_r^2 V_2^2\omega - k_s^2 V_2^2\omega - 2\gamma\kappa L(1-b)k_r k_s V_2^2 = 0, \tag{25}$$

$$-2\gamma\kappa L(1-b)\omega^2 + 2k_r k_s V_2^2\omega + \gamma\kappa L(1-b)(k_r^2 - k_s^2)V_2^2 = 0. \tag{26}$$

In the hypothesis of small dissipation ($k_r^2 \gg k_s^2$), from (26) one obtains:

$$k_s = \gamma\kappa L(1-b)\left(\frac{2w_2^2 - V_2^2}{2w_2 V_2^2}\right), \tag{27}$$

which substitutes in (25), yields:

$$\omega^4 - \left[(2|\Omega| - \gamma\kappa Lc)^2 - \gamma^2\kappa^2 L^2(1-b)^2\right]\omega^2 + \\ -k_r^2 V_2^2\omega^2 - \gamma^2\kappa^2 L^2(1-b)^2 V_2^2 k_r^2 = 0. \tag{28}$$

Putting $\tilde{A} = -\left[(2|\Omega| - \gamma\kappa Lc)^2 - \gamma^2\kappa^2 L^2(1-b)^2\right]$ and $\tilde{B} = -\gamma^2\kappa^2 L^2(1-b)^2$ and taking into account that $w_2 = \frac{\omega}{k_r}$, the equation (28) becomes:

$$w_2^2\left[w_2^2\left(1 + \frac{\tilde{A}}{\omega^2}\right) - V_2^2\left(1 - \frac{\tilde{B}}{\omega^2}\right)\right] = 0 \tag{29}$$

whose solutions are

$$w_2^2 = 0, \quad \text{and}$$

$$w_2^2 = V_2^2\frac{\left(\omega^2 - \tilde{B}\right)}{\left(\omega^2 + \tilde{A}\right)} = V_2^2\frac{1}{1 - \frac{(2|\Omega| - \gamma\kappa Lc)^2}{\omega^2 + \gamma^2\kappa^2 L^2(1-b)^2}}. \tag{30}$$

We can remark that the coefficient $\tilde{A}$ and $\tilde{B}$ are negative quantities and that $w_2^2 \geq V_2^2$ because $\omega^2 + \tilde{A} \leq \omega^2 - \tilde{B}$; in particular, we note that $w_2^2 = V_2^2$ for $|\Omega| = \frac{\gamma\kappa Lc}{2}$. Now, studying the transversal modes, i.e. the subsystem $C\tilde{U}_3 = 0$, we obtain $\omega_7 = 0$, which corresponds to a stationary mode and

$$\omega_8 = -i\gamma\kappa L(1-a). \tag{31}$$

Summarizing, also in this case measurements in a single direction are enough to give information on all the variables describing the vortex tangle, namely L, b and c, whose explicit forms are obtained in a trivial way from equations (27), (30) and (31):

$$L = \frac{4k_s w_2 V_2^2 - \omega_s \left(2w_2 - V_2^2\right)}{2\left(2w_2^2 - V_2^2\right)\gamma\kappa}, \tag{32}$$

$$b = -\frac{\omega_s \left(2w_2^2 - V_2^2\right)}{4k_s w_2 V_2^2 - \omega_s \left(2w_2 - V_2^2\right)}, \tag{33}$$

$$c = \frac{4|\Omega|(2w_2^2 - V_2^2) - \sqrt{(1 - V_2^2)(4k_r^2(2w_2^2 - V_2^2)^2 + 16k_s^2 V_2^4)}}{4k_s w_2 V_2^2 - \omega_s(2w_2^2 - V_2^2)} \tag{34}$$

where we have put $\omega_8 = i\omega_s$.

4. Conclusions

In this work we have studied the propagation of waves in turbulent superfluid helium in the complex situation involving thermal counterflow in a rotating frame. From the physical point of view it is interesting to note that our detailed analysis in Sect.**3** shows that, in contrast to which one intuitively expectes, measurements in a single direction are enough to give information on all the variables describing the vortex tangle, namely L, a and c, for instance, using three expressions of (12)-(13) and (19) or of (27)-(30) and (31). This is not an immediate intuitive result. Future topics along this direction could be, for instance, assuming that $\mathbf{\Omega}$ and $\mathbf{V}$ have arbitrary directions, i.e. they are not parallel to each other. In such case (4) would not be sufficient to describe the vortex tangle, because no isotropy in the $y - z$ plane could be assumed.

Acknowledgments

We acknowledge Prof. D. Jou (Department of Physics, University Autònoma of Barcelona) and Prof. M.S. Mongiovì (Dipartimento di Metodi e Modelli Matematici, Università di Palermo) for the enlightening discussions useful to the deepening of these arguments. This work is supported by MIUR of Italy under "Azioni Integrate Italia-Spagna, anno 2005". R.A. P. is supported by an "Assegno di ricerca dell'Istituto Nazionale di Alta Matematica '*F. Severi*' (INdAM)" of Italy and M.S. is supported by an "Assegno di ricerca MIUR" of Italy.

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Presented: September 29, 2005
Published on line: February 01, 2008