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THE FUNDAMENTAL PRINCIPLE FOR MEAN-PERIODIC DISTRIBUTIONS

ADELINA FABIANO

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ABSTRACT. The purpose of this paper is to provide a proof for a Fundamental Principle for convolution equations $\mu * f = 0$, with $\mu \in \mathcal{E}'(\Omega)$ and $f \in \mathcal{D}'(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is a convex open set and $\mathcal{E}'(\Omega)$, $\mathcal{D}'(\Omega)$ are the corresponding distribution spaces.

1. Introduction and Notations

The *Fundamental Principle* of L. Ehrenpreis, [3], is a sophisticated extension of the classical *Fundamental Principle* of Euler, which shows how to write every smooth solution of a linear constant coefficient O.D.E, as the sum of its "elementary" exponential polynomial solutions. Ehrenpreis' result, on the other hand, deals with the case of solutions to systems of linear constant coefficients P.D.E.'s, in a variety of spaces of "generalized functions", which he calls Analytically Uniform Spaces (*AU-spaces*). Among the main examples of Analytically Uniform Spaces (see Chapter V of [3]), we just mention the space $\mathcal{E}$ of infinitely differential functions, the space $\mathcal{O}$ of holomorphic functions and the space $\mathcal{D}'$ of the Schwartz distributions.
Ehrenpreis' Theorem is a consequence of the following refined version of the well known Hilbert's Nullstellensatz:

(1.1) Theorem ([3]) — *Let $P_1(z), \dots, P_r(z)$ be r polynomials in n complex variables, and let $I = I(P)$ be the ideal which they generate.*
Denote by

$$V = \{ z \in \mathbb{C}^n \mid P_1(z) = \cdots = P_r(z) = 0 \}$$

the variety of common zeroes.
Then there exist subvarieties $V_1, \dots, V_s$ of V and differential operators $\partial_1, \dots, \partial_s$ on these same varieties, such that a polynomial Q belong to the ideal I if and only if, for any $j = 1, \dots, s$, the polynomial $\partial_j Q$ vanishes on V_j.

We can therefore state

(1.2) Theorem (Fundamental Principle of Ehrenpreis) — *Let*

$$D = \left(-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}\right),$$

let f be an infinitely differentiable function and maintain the notation as in theorem (1.1). *The function f is a solution of*

$$P\left(-i\frac{\partial}{\partial x_1}, \dots, -i\frac{\partial}{\partial x_n}\right) f(x) = 0$$

if and only if it admits an integral representation

$$f(x) = \sum_{1 \le j \le r} \int_{V_j} \partial_j \big(\exp(-ix \cdot z)\big)\, d\nu_j(z)\,,$$

where the $d\nu_j$'s are Borel Measures, and

$$x \cdot z = x_1 z_1 + \cdots + x_n z_n\ .$$

As it is well know, convolution operators are the most general continuous operators which are invariant by translations, and so it has been quite natural to attempt to generalize Ehrenpreis' Principle to the case of (systems of) convolution equations in various spaces. Many results are known in this direction; among them we recall the following cases, which have already been explored:

- The space $\mathcal{O}(\mathbb{C}^n)$ or $\mathcal{O}(\Omega)$ of holomorphic functions on $\mathbb{C}^n$ or on an open convex set Ω in $\mathbb{C}^n$ (see [9]).
- The space $\mathcal{E}(\mathbb{R}^n)$ or $\mathcal{E}(\Omega)$ of infinitely differentiable functions on $\mathbb{R}^n$ or on an open convex set Ω in $\mathbb{R}^n$.
- The space $\mathcal{E}_\omega$ of ultradifferentiable functions in the sense of Beurling.
- The space $\mathcal{D}_\omega$ of ultradistributions in the sense of Beurling.
- The space $\mathbb{B}$ of hyperfunctions.

So far, however, no detailed proof has ever been given for a Fundamental Principle for convolution equations

$$\mu * f = 0\,,$$

with $\mu \in \mathcal{E}'(\Omega)$ and $f \in \mathcal{D}'(\Omega)$, where Ω is a convex open set in $\mathbb{R}^n$ and $\mathcal{E}'(\Omega)$, $\mathcal{D}'(\Omega)$ are the corresponding distribution spaces.
The purpose of this paper is to provide this proof.
More specifically, we shall consider "systems" of convolution equations of the form

$$\mu * f = (\mu_1 * f, \dots, \mu_r * f) = 0\,, \tag{1}$$

where $\mu_1, \dots, \mu_r \in \mathcal{E}'(\mathbb{R}^n)$, $1 \le r \le n$ and $f \in \mathcal{D}'(\mathbb{R}^n)$, so that

$$\mu * : \mathcal{D}' \longrightarrow \big(\mathcal{D}'\big)^r\ .$$

Finding solutions of (1) is equivalent to study the kernel of the operator $\mu *$.
In particular, one would like to give an integral representation to f, whose frequencies are contained in the variety

$$V = \{\, z \in \mathbb{C}^n \mid \hat{\mu}_1(z) = \cdots = \hat{\mu}_r(z) = 0 \,\} \subseteq \mathbb{C}^n\,,$$

where $\hat{\mu}_j$ denotes the Fourier transform of the compactely supported distribution μ_j and is defined by

$$\hat{\mu}_j(z) =< \mu_j, \zeta \to \exp(iz \cdot \zeta) > \ .$$

We will show that such representation indeed exists, a fact which is equivalent to show a sort of Fundamental Principle for such equations.
More precisely, we will prove

$$\ker(\mu *) \simeq \big(D(V)\big)', \tag{2}$$

where $D(V)$ is a space of holomorphic functions on V satisfying suitable growth conditions. This space will be defined in the sequel (theorem (4.3)).
In order to prove isomorphism (2), one recalls, from functional analysis, that

$$\ker(\mu *) \simeq D/I\,,$$

where I is the non principal ideal generated by $(\hat{\mu}_1, \dots, \hat{\mu}_r)$ in D.
To complete the proof of the Fundamental Principle, it will be therefore sufficient to show

$$D/I \simeq D(V)\ .$$

The proof relies on two theorems (a division one and an extension one) which we will prove in the next two sections.

2. Division Theorems

Let us consider the system of convolution equations

$$\mu * f = 0\,,$$

with $\mu = (\mu_1, \dots, \mu_n) \in \big(\mathcal{E}'(\mathbb{R}^n)\big)^n$ and $f \in \mathcal{D}'(\mathbb{R}^n)$.
Let

$$V = \{\, z \in \mathbb{C}^n \mid \hat{\mu}_1(z) = \cdots = \hat{\mu}_n(z) = 0 \,\} \qquad \text{and} \qquad V_i = \{\, z \in \mathbb{C}^n \mid \hat{\mu}_i(z) = 0 \,\},,$$

and define

$$d(z, V_i) = \min(1, \text{distance from } z \text{ to } V_i)\ .$$

(2.1) Definition — $\mu = (\mu_1, \dots, \mu_n)$ *is aid to be* **slowly decreasing** *if the following conditions hold:*

(i) *for any* $\varepsilon > 0$, *there exist positive constants* A, B *such that the connected components of the set*

$$S(\hat{\mu}; A, B, \varepsilon) = \{\, z \in \mathbb{C}^n \mid \text{for } i = 1, \dots, n,\ d(z, V_i) \leq A(1 + |z|)^{-B} \exp\big(-\varepsilon\, |\mathrm{Im}(z)|\big) \,\}$$

are relatively compact and their diameters are uniformely bounded;

(ii) *there exist positive constants* C, D, F *and* m *such that, for all* $z \in \mathbb{C}^n$ *and* $i = 1, \dots, n$,

$$|\hat{\mu}_i(z)| \geq C(1 + |z|)^{-D} d(z, V_i)^m \exp(F\, |\mathrm{Im}\, z|)$$

and

$$|\hat{\mu}(z)| \geq C(1 + |z|)^{-D} d(z, V)^m \exp(F\, |\mathrm{Im}\, z|)\ .$$

A key tool in what follows is the so called Jacobi's interpolation formula, [2], whose construction we now briefly recall.
Let $g = (g_1, \dots, g_n)$ be analytic functions such that the variety

$$V = \{\, z \in \mathbb{C}^n \,|\, g_1(z) = \cdots = g_n(z) = 0 \,\}$$

is discrete and let $\Omega = \Omega(\varepsilon)$ be a bounded component of the analytic polyhedron

$$P(\varepsilon) = \{\, z \in \mathbb{C}^n \,|\, \text{for } i = 1, \dots, n,\ |g_i(z)| \le \varepsilon_i \,\} ,$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ is chosen so that $P(\varepsilon)$ is nondegenerate.
Then, for all choices of $1 \le i_1 < \cdots < i_k \le n$, we have

$$dg_{i_1} \wedge \cdots \wedge dg_{i_k} \neq 0 ,$$

on those points of the boundary of $P(\varepsilon)$ where $|g_{i_l}| = \varepsilon_{i_l}$ for $l = 1, \dots, k$ and $|g_i| < \varepsilon_i$ for $i \notin \{i_1, \dots, i_k\}$.
Let

$$\Gamma = \{\, z \in \partial\Omega \,|\, \text{for } i = 1, \dots, n,\ |g_i(z)| = \varepsilon_i \,\} .$$

Suppose that the jacobian of g never vanishes on V.
It is well known, [2], that it is always possible to choose analytic functions Q_{ij} on $\overline{\Omega} \times \overline{\Omega}$ such that

$$g_i(\zeta) - g_i(z) = \sum_{1 \le j \le n} Q_{ij}(\zeta, z)(\zeta_j - z_j) \qquad \text{for} \qquad i = 1, \dots, n .$$

Let

$$H(\zeta, z) = \mathrm{Det}[Q_{ij}(\zeta, z)] .$$

Now, for $\lambda \in \mathcal{O}(\overline{\Omega})$, we construct the Jacobi interpolation formula as

$$(J\lambda)(z) = J(\lambda; z) = \frac{1}{(2i\pi)^n} \int_\Omega \frac{\lambda(\zeta) H(\zeta, z) d\zeta_1 \wedge \cdots \wedge d\zeta_n}{g_1(\zeta) \cdots g_n(\zeta)} .$$

With respect to this formula, the following results hold [2].
(2.1) Proposition — *If $\lambda \in \mathcal{O}(\overline{\Omega})$, then*

(i) $\lambda(z) - (J\lambda)(z) = \sum_{1 \le i \le n} a_i(z) g_i(z)$, *with $a_i \in \mathcal{O}(\Omega)$ for $i = 1, \dots, n$;*
(ii) *If $\lambda \in I_{\mathcal{O}(\overline{\Omega})}(g_1, \dots, g_n)$, then $(J\lambda)(z) = 0$ for any $z \in \Omega$.*

(2.1) Theorem — *Let*

$$S = \{\, z \in \Omega \,|\, \text{for } i = 1, \dots, n,\ |g_i(z)| < \frac{\varepsilon_i}{2} \,\}$$

and take

$$\lambda \in \mathcal{O}(\overline{\Omega}) \qquad \Gamma = \partial\Omega \qquad \eta = \prod_{1 \le i \le n} \varepsilon_i .$$

Assume the following bounds hold

(i) $|\lambda(z)| \le M$ *on* $\overline{\Omega}$;
(ii) $|H(z, \zeta)| \le D$ *on* $\overline{\Omega} \times S$;
(iii) $\int_\Gamma |d\zeta_1 \wedge \cdots \wedge d\zeta_n| \le L$;
(iv) $|g_i(\zeta)| \le D_1$ *on* $\overline{\Omega}$.

Then

$$|(J\lambda)(z)| \le \frac{DLM}{\eta} \qquad \text{on} \qquad S\,.$$

Moreover, if

$$\lambda(z) = f_1(z)g_1(z) + \cdots + f_n(z)g_n(z)$$

for some $f_1, \ldots, f_n \in \mathcal{O}(\overline{\Omega})$*, then it is possible to find* $\alpha_1, \ldots, \alpha_n \in \mathcal{O}(\Omega)$ *such that*

$$\lambda(z) = \alpha_1(z)g_1(z) + \cdots + \alpha_n(z)g_n(z) \qquad \text{on} \qquad \Omega$$

with

$$|\alpha_i(z)| \le \frac{CDMLD_1^{n-1}}{\eta} \qquad \text{on} \qquad S\,,$$

where the constant C *depends only on* n*.*

This result gives us a (semilocal) control on the growth of holomorphic functions. Its proof is based on the Jacobi interpolation formula which holds only for discrete varieties (we will see how to deal with the non discrete case in the sequel). The next result provides the connection between semilocal and global.

(2.2) Theorem — *Let* $\varepsilon > 0$ *be sufficiently small and let* $\lambda \in \mathcal{O}\big(S(\hat{\mu}; A, B, \varepsilon)\big)$*.*
Suppose that, for any $B_1 > 0$*, there exist* $A_1, C_1 > 0$ *such that*

$$|\lambda(z)| \le A_1(1+|z|)^{-B_1} \exp(C_1\,|\mathrm{Im}(z)|) \qquad \text{for} \qquad z \in S(\hat{\mu}; A, B, \varepsilon)\,.$$

Then there exists an entire function $\overset{\circ}{\lambda} \in \mathcal{O}(\mathbb{C}^n)$ *such that for any* $B_2 > 0$*, exist* $A_2, C_2 > 0$*, with*

$$\left|\overset{\circ}{\lambda}(z)\right| \le A_2(1+|z|)^{-B_2} \exp\big(C_2\,|\mathrm{Im}\, z|\big) \qquad \text{for} \qquad z \in \mathbb{C}^n$$

and there exists analytic functions $\alpha_1, \ldots \alpha_r$ *such that*

$$\overset{\circ}{\lambda}(z) = \lambda(z) + \sum_{1 \le i \le r} \alpha_i(z) i\hat{\mu}_i(z) \qquad \text{for} \qquad z \in S'(\hat{\mu}; A, B, 2\varepsilon)$$

and such that for any $B_3 > 0$ *exist* $A_3, C_3 > 0$ *with*

$$|\alpha_i(z)| \le A_3(1+|z|)^{-B_3} \exp\big(C_3\,|\mathrm{Im}\, z|\big) \qquad \text{for} \qquad z \in S'(\hat{\mu}; A, B, 2\varepsilon)\,.$$

Proof — For $i = 1, \ldots, r$, for $j = 1, \ldots, n$ and for $z \in \mathbb{C}^n$, one has

$$\left|\frac{\partial \hat{\mu}_i(z)}{\partial z_j}\right| \le A(1+|z|)^B \exp\big(C\,|\mathrm{Im}\, z|\big)\,.$$

For the definition of $S(\hat{\mu}; A, B, \varepsilon)$, we can find positive constants $\overset{\circ}{A}, A', B'$ such that the distance from $S'(\hat{\mu}; A, B, 2\varepsilon)$ to the complement of $S(\hat{\mu}; \overset{\circ}{A}, B, \varepsilon)$ is at least

$$A'(1+|z|)^{-B'} \exp\big(-\varepsilon\,|\mathrm{Im}\, z|\big)\,.$$

Consider the function $\chi \in \mathcal{C}^\infty(\mathbb{C}^n)$ such that $0 \le \chi \le 1$ and $\chi = 1$ on $S'(\hat{\mu}; A, B, 2\varepsilon)$, $\chi = 0$ on a neighborhood of the complement of $S(\hat{\mu}; A, B, \varepsilon)$. Moreover, for some $A'' > 0$, one has

$$\big|\overline{\partial}\chi(z)\big| \le A''(1+|z|)^{-B'} \exp\big(-\varepsilon\,|\mathrm{Im}\, z|\big)\,.$$

Then $\chi * \lambda \in \mathcal{C}^\infty(\mathbb{C}^n)$ and, if $w = \lambda * \overline{\partial}\chi$, on has that $w \in \mathcal{C}^\infty_{(0,1)}(\mathbb{C}^n)$ is a $\overline{\partial}$-closed form which vanishes in a neighborhood of V. Moreover there is an integer m such that

$$\int_{\mathbb{C}^n} |w(z)|^2 \, |\hat{\mu}(z)|^{-2m} \exp\bigl(-m\varepsilon \, |\mathrm{Im}\, z| - 2F \, |\mathrm{Im}\, z|\bigr) \, dz < +\infty \; .$$

This immediately follows from the bounds of $\hat{\mu}$. By following [5] one has that, for m sufficiently large, there exist $\overline{\partial}$-closed $(0,1)$-forms $w_1, \dots, w_r$ such that

$$w = w_1 \hat{\mu}_1 + \cdots + w_r \hat{\mu}_r \; .$$

Moreover, from the construction of the w_j's one has

$$\int_{\mathbb{C}^n} |w_j(z)|^2 \, |\hat{\mu}(z)|^{2k} \exp\bigl(-k\varepsilon \, |\mathrm{Im}\, z| - 2F' \, |\mathrm{Im}\, z|\bigr) \, dz < +\infty$$

for some integer k and some $F' > 0$. The usual $\overline{\partial}$-techniques show that there exist analytic functions $\alpha_1, \dots, \alpha_r$ such that

$$\overset{\circ}{\lambda} = \lambda + \sum_{1 \le i \le r} \alpha_i \hat{\mu}_i$$

and such that, for some $A_0 > 0$ one has

$$\int_S' |\alpha_i(z)|^2 \, |\hat{\mu}(z)|^{-2l} \exp\bigl(-2F'' \, |\mathrm{Im}\, z| - 2A_0 \log(1 + |z|)\bigr) \, dz < +\infty$$

for $l = m - 2(2n+1)$. Therefore, if $l \geq 0$, one deduces from [4] that the mean value of $|\alpha_i|$ on the balls

$$\{\, \zeta \in \mathbb{C}^n \mid |\zeta - z| \le 1 \,\}$$

is bounded by

$$\overline{A}(1 + |z|)^{-\overline{B}} \exp\bigl(F'' \, |\mathrm{Im}\, z|\bigr) \, ,$$

for some constants $\overline{A}, \overline{B} > 0$, a fact which proves the theorem. Let us state Ehrenpreis' Division Theorem for AU-spaces.

(2.3) Theorem — *Let $P_1, \dots, P_r$ be polynomials in $\mathbb{C}^n$ and let $F \in \hat{X}$ for X any AU-space, and where denotes a Fourier transform on X'. If there exist $\lambda_1, \dots, \lambda_r \in \mathcal{O}(\mathbb{C}^n)$ such that*

$$F = \lambda_1 P_1 + \cdots + \lambda_r P_r \, ,$$

the one can find $\overset{\circ}{\lambda}_1, \dots, \overset{\circ}{\lambda}_r \in \hat{X}$ such that

$$F = \overset{\circ}{\lambda}_1 P_1 + \cdots + \overset{\circ}{\lambda}_r P_r \; .$$

We now improve this result.

(2.4) Theorem (Division) — *Let $F \in \mathcal{D}(\mathbb{R}^n)$ and let $\hat{\mu} = (\hat{\mu}_1, \dots, \hat{\mu}_n) \in \bigl(\mathcal{E}'(\mathbb{R}^n)\bigr)^n$ be slowly decreasing. If there are entire functions $f_1, \dots, f_n$ such that for any $z \in \mathbb{C}^n$*

$$F(z) = f_1(z)\hat{\mu}_1(z) + \cdots + f_n(z)\hat{\mu}_n(z) \, ,$$

then one can find functions $\gamma_1, \dots, \gamma_n \in \mathcal{D}(\mathbb{R}^n)$ such that

$$F(z) = \gamma_1(z)\hat{\mu}_1(z) + \cdots + \gamma_n(z)\hat{\mu}_n(z) \; . \tag{3}$$

Proof — Since $\hat{\mu}$ is slowly decreasing, one can choose A, B, ε so that the set $S(\hat{\mu}; A, B, \varepsilon)$ has relatively compact connected components. Let Ω such a component, and let $\Gamma = \partial\Omega$ be its boundary. In order to apply teorem (2.1), one notices that:

(i) $F \in \mathcal{D}(\mathbb{R}^n)$ and therefore for any $B_1 > 0$, exist $A_1, C_1 > 0$ such that, for all $z \in \mathbb{C}^n$,

$$|F(z)| \leq A_1(1+|z|)^{-B_1} \exp\bigl(C_1 |\mathrm{Im}\, z|\bigr) \leq M \ ;$$

(ii) because of theorem (2.2) one has the functions $Q_{ij}(\zeta, z)$, and therefore $H(\zeta, z)$ can be chosen in $\mathcal{E}'(\mathbb{R}^n)$ and there exist $A_2, B_2, C_2 > 0$ such that

$$|H(\zeta, z)| \leq A_2(1+|z|)^{B_2} \exp\bigl(C_2 |\mathrm{Im}\, z|\bigr) \leq D \qquad \text{on} \qquad \Omega \times \{0\} \ ;$$

(iii) we have

$$\Gamma \subseteq \{\, z \in \mathbb{C}^n \mid d(z, V_i) \leq A(1+|z|)^{-B} \exp\bigl(-\varepsilon |\mathrm{Im}\, z|\bigr) \,\}$$

and therefore there are constants $A_3, B_3, C_3 > 0$ such that, for all $z \in \overline{\Omega}$,

$$\int_\Gamma |d\zeta_1 \wedge \cdots \wedge d\zeta_n| \leq A_3(1+|z|)^{B_3} \exp\bigl(C_3 |\mathrm{Im}\, z|\bigr) \leq L \ ;$$

(iv) $\hat{\mu}_i \in \mathcal{E}'(\mathbb{R}^n)$ and so exist $A_4, B_4, C_4 > 0$, such that, for all $z \in \overline{\Omega}$ one has

$$|\hat{\mu}_i(\zeta)| \leq A_4(1+|z|)^{B_4} \exp\bigl(C_4 |\mathrm{Im}\, z|\bigr) \leq D_1 \ .$$

Let now

$$\eta = A_5(1+|z|)^{B_4} \exp\bigl(C_5 |\mathrm{Im}\, z|\bigr) \ .$$

Then the are $\lambda_1, \ldots, \lambda_n \in \mathcal{O}(\Omega)$ such that, for all $z \in \Omega$, one has

$$F(z) = \lambda_1(z)\hat{\mu}_1(z) + \cdots + \lambda_n(z)\hat{\mu}_n(z)$$

and, for a constant $\overset{\circ}{E}$ depending only on n, one has

$$|\lambda_i(z)| \leq \frac{\overset{\circ}{E} DMLD_1^{n-1}}{\eta^2} \ .$$

If we now set

$$\overset{\circ}{A} = \frac{\overset{\circ}{E} A_1 A_2 A_3 (A_n)^{n-1}}{(A_5)^2} > 0 \tag{1}$$

$$\overset{\circ}{B} = B_2 + B_3 + (n-1)B_4 - 2B_5 > 0 \tag{2}$$

$$\overset{\circ}{C} = C_1 + C_2 + C_3 + (n-1)C_4 - 2C_5 > 0 \tag{3}$$

one has

$$|\lambda_i(z)| \leq \overset{\circ}{A}(1+|z|)^{\overset{\circ}{B} - B_1} \exp\bigl(\overset{\circ}{C} |\mathrm{Im}\, z|\bigr)$$

and so, for any B_1, exist $\overset{\circ}{A}, \overset{\circ}{C} > 0$ such that

$$|\lambda_i(z)| \leq \overset{\circ}{A}(1+|z|)^{B_1} \exp\bigl(\overset{\circ}{C} |\mathrm{Im}\, z|\bigr) \ .$$

This implies that $\lambda_i \in \mathcal{D}(\Omega)$, $i = 1, \dots, n$.
This argument can be repeated on every connected component, but we are interested to extend the representation (3) to all $\mathbb{C}^n$. For this purpose, consider the characteristic function

$$\chi(z) = \begin{cases} 1 & if \quad z \in S \\ 0 & if \quad z \notin S \end{cases}$$

and replace λ_i with

$$\overset{\circ}{\lambda}_i = \chi\lambda_i + (1-\chi)\frac{F\hat{\mu}_i}{\|\hat{\mu}\|^2} ,$$

where

$$\|\hat{\mu}\|^2 = |\hat{\mu}_1|^2 + \cdots + |\hat{\mu}_n|^2 \ .$$

Then

$$\sum_{1\le i\le n} \overset{\circ}{\lambda}_i \hat{\mu}_i = \sum_{1\le i\le n} \chi\lambda_i\hat{\mu}_i + \sum_{1\le i\le n}(1-\chi)\frac{F\hat{\mu}_i\hat{\mu}_i}{\|\hat{\mu}\|^2} = \chi F + (1-\chi)F = F \ .$$

Therefore (see [2]), $\overset{\circ}{\lambda}_1, \dots, \overset{\circ}{\lambda}_n \in \mathcal{C}^\infty(\mathbb{C}^n)$, $\overset{\circ}{\lambda}|_S = \lambda$ and

$$\left|\overline{\partial}\overset{\circ}{\lambda}_i(z)\right| \le A(1+|z|)^B \left|\hat{\mu}(z)\right|^{2n+1} \exp\bigl(C\left|\operatorname{Im} z\right|\bigr) \ .$$

Moreover,

$$F(z) = \overset{\circ}{\lambda}_1(z)\hat{\mu}_1(z) + \cdots + \overset{\circ}{\lambda}_n(z)\hat{\mu}_n(z)$$

and then

$$\overline{\partial}\overset{\circ}{\lambda}_1\hat{\mu}_1 + \cdots + \overline{\partial}\overset{\circ}{\lambda}_n\hat{\mu}_n = 0 \ .$$

Finally (see theorem (4.2)), this yields functions $\gamma_i \in \mathcal{D}(\mathbb{R}^n)$ such that

$$F = \gamma_1\hat{\mu}_1 + \cdots + \gamma_n\hat{\mu}_n \ .$$

(2.1) Remark — An analogous result has been proved by Berenstein-Taylor in [2] for the case $F \in \mathcal{E}'(\mathbb{R}^n)$. They proved, indeed, that is possible to determine some $\lambda_i \in \mathcal{E}'(\mathbb{R}^n)$ such that

$$F(z) = \lambda_1(z)\hat{\mu}_1(z) + \cdots + \lambda_n(z)\hat{\mu}_n(z) \ . \tag{4}$$

Although $\mathcal{D} \subset \mathcal{E}'$, the two results are not directly comparable, because both the thesis and the hypothesis of theorem (2.2) are "stronger" than (4). Let now $K, \tilde{K}, T$ be compact convex subsets of $\mathbb{R}^n$, with $\tilde{K} = K + T$.

(2.2) Definition — $\mu = (\mu_1, \dots, \mu_n) \in \bigl(\mathcal{E}'(T)\bigr)^n$ *is* T**-slowly decreasing** *if the following conditions are satisfied:*

(i) *for any $\varepsilon > 0$, there exist positive constants A, B such that the connected components of the set $S(\hat{\mu}; A, B, \varepsilon)$ are relatively compact with uniformely bounded diameters;*

(ii) *there exist positive constants C, D and m such that, for all $z \in \mathbb{C}^n$ and $i = 1, \dots, n$,*

$$|\hat{\mu}_j(z)| \ge C(1+|z|)^{-D} d(z, V_i)^m \exp\bigl(H_T(\operatorname{Im} z)\bigr)$$

and

$$|\hat{\mu}(z)| \geq C(1+|z|)^{-D} d(z,V)^m \exp\big(H_T(\operatorname{Im} z)\big) ,$$

where H_T denotes the support function of T.

(2.1) Corollary — *Let $K, \tilde{K}, T$ be as above. Let $F \in \mathcal{D}(\tilde{K})$ and $\hat{\mu} \in \big(\mathcal{E}'(T)\big)^n$ be T-slowly decreasing.*
Then, if there exist entire functions $f_1, \dots, f_n \in \mathcal{O}(\mathbb{C}^n)$ such that

$$F(z) = f_1(z)\hat{\mu}_1(z) + \dots + f_n\hat{\mu}_n(z) \qquad \text{for} \qquad z \in \mathbb{C}^n ,$$

it is possible to find entire functions $\lambda_1, \dots, \lambda_n \in \mathcal{D}(K_\varepsilon)$ such that

$$F(z) = \lambda_1(z)\hat{\mu}_1(z) + \dots + \lambda_n\hat{\mu}_n(z) \qquad \text{for} \qquad z \in \mathbb{C}^n .$$

The proof of this corollary is similar to that of theorem (2.4) by replacing its constants with the following

$$\begin{aligned} M &= A'(1+|z|)^{-B'} \exp\big(H_{\tilde{K}}(\operatorname{Im} z)\big) \\ D &= A''(1+|z|)^{-B''} \exp\big(H_{nT}(\operatorname{Im} z)\big) \\ L &= A_1(1+|z|)^{B_1} \exp\big(\varepsilon\,|\operatorname{Im} z|\big) \\ D_1 &= \overline{A}(1+|z|)^{\overline{B}} \exp\big(H_T(\operatorname{Im} z) - \varepsilon\,|\operatorname{Im} z|\big) \\ \eta &= A(1+|z|)^{-B} \exp\big(H_{nT}(\operatorname{Im} z) - nm\varepsilon\,|\operatorname{Im} z|\big) . \end{aligned}$$

3. Extension Theorems

We now describe the main ideas which are needed to generalize our division theorem to the non discrete case and, at the same time, to prove a general extension theorem.
As we already mentioned, the Jacobi interpolation formula (and therefore theorem (2.1)) only hold for a discrete variety. Therefore, if we want to study the ideal generated by $\hat{\mu}_1 \dots, \hat{\mu}_r$, $1 \leq r \leq n$, $\hat{\mu}_i \in \mathcal{E}'(\mathbb{R}^n)$, we need to reduce the problem to a discrete situation as in [1, 2, 8]. The idea consist in "cutting" the (complete intersection) variety

$$V = \{\, z \in \mathbb{C}^n \mid \hat{\mu}_1(z) = \dots = \hat{\mu}_r(z) = 0 \,\}$$

with a family $\mathcal{L} = \{L\}$ of complex r-dimensional affine spaces, in order to be able to apply the Jacobi interpolation formula to each "section". Some conditions have to be imposed on the family $\mathcal{L}$ and on the behaviour of the $\hat{\mu}_i$'s on $\mathcal{L}$ itself. Such conditions are summarized by the following definitions:

(3.1) Definition — *Let $\mu_1, \dots, \mu_r \in \mathcal{E}'(\mathbb{R}^n)$ have compact support $K \subset \mathbb{R}^n$. We say that $\mu = (\mu_1, \dots, \mu_n)$ is K-**slowly decreasing** if there exists a family $\mathcal{L} = \{L\}$ of r-dimensional complex affine spaces, which cover $\mathbb{C}^n$, such that, for suitable positive constants $C, D, F, m > 0$, for any $L \in \mathcal{L}$ and any $z \in L$ one has*

$$\begin{aligned} |\hat{\mu}_i(z)| &\geq C(1+|z|)^{-D} d(z, V_i \cap L)^m \exp\big(F\,|\operatorname{Im} z|\big) \\ \big|\hat{\mu}_(z)\big| &\geq C(1+|z|)^{-D} d(z, V \cap L)^m \exp\big(F\,|\operatorname{Im} z|\big) . \end{aligned}$$

Moreover, for any $\varepsilon > 0$, there exist constant $A, B > 0$ such that, for all $L \in \mathcal{L}$, the sets

$$O_L(\hat{\mu}; A, B, \varepsilon) = \{\, z \in L \mid \text{for } i = 1, \dots, r,\ d(z, V_i) \leq A(1+|z|)^{-B} \exp\big(-\varepsilon\,|\operatorname{Im} z|\big) \,\}$$

have relatively compact connected components with uniform bounded diameters.
Let $L \in \mathcal{L}, \varepsilon, A, B$ be fixed so that every connected component G of $O_L(\hat{\mu}; A, B, \varepsilon)$ has a diameter bounded by a fixed constant. An open set $\Omega \subseteq \mathbb{C}^n$ is said to be *good* if, for some positive constants α, β, it is of the type

$$\Omega = \{\, z \in \mathbb{C}^n \mid \text{exist } \zeta \text{ with } |z-\zeta| \leq \alpha(1+|\zeta|)^{-\beta} \exp\bigl(-F\,|\mathrm{Im}\, z|\bigr)\,\}\ .$$

If we fix $\hat{\mu}, L, \varepsilon, A, B, \alpha, \beta, F$, we obtain a *good family* of sets, which we indicate by $\mathcal{C}$. Finally, a *good refinement* $\mathcal{C}'$ of $\mathcal{C}$ is a good family obtained by decreasing A and α and by increasing B, β, ε, F. Note that for every good refinement $\mathcal{C}'$ of $\mathcal{C}$, is defined a natural refinement map

$$\rho : \mathcal{C}' \longrightarrow \mathcal{C}\ .$$

(3.2) Definition — *The family $\mathcal{L}$ is said* **almost parallel** *if, given a good family $\mathcal{C}$, there exists a refinement $\mathcal{C}'$ of $\mathcal{C}$ such that, if $\Omega_0, \Omega_1 \in \mathcal{C}'$, one has*

$$\Omega_0 \cap \Omega_1 \neq \emptyset \quad \Longrightarrow \quad \Omega_0 \cup \Omega_1 \subseteq \rho(\Omega_0) \cap \rho(\Omega_1)\ .$$

(3.3) Definition — *The family $\mathcal{L}$ is said* **analytic** *if there is a good family $\mathcal{C}'$ associated with $\mathcal{L}$ with the following property: given $\Omega \in \mathcal{C}$ associated with the spaces $L \in \mathcal{L}$, there exist local analytic coordinates (s,t) on Ω such that*

$$\Omega \cap \{(s,t) : t = 0\} = \Omega \cap L$$

and

$$\Omega \cap \{(s,t) : t = const\} = \Omega \cap L_i \quad \textit{for some} \quad L_i \in \mathcal{L}\ .$$

By applying the Jacobi formula to these discrete intersections we obtain local extensions, which we now need to patch-up toghether. To this purpose, it becomes indispensable to use the Koszul complex, which we briefly recall.
Denote with $\mathcal{A}'$ the space of analytic s-cochains defined with respect to a good family $\mathcal{C}$ and which satisfy the growth conditions of $\mathcal{E}$: this means that $\gamma \in \mathcal{A}'$ if and only if γ is an alternating function

$$\begin{array}{rcl} \gamma : \mathcal{C}^{s+1} & \longrightarrow & \mathit{Div}()\bigl(\widehat{\mathcal{E}}'(\Omega_0 \cap \cdots \cap \Omega_s)\bigr) \\ (\Omega_0, \ldots, \Omega_s) & \longmapsto & f_{01}, \ldots, f_{0s} \in \mathcal{O}(\Omega_0 \cap \cdots \cap \Omega_s) \end{array}$$

such that

$$|f_{01}(z), \ldots, f_{0s}(z)| \leq A(1+|z|)^B \exp\bigl(A\,|\mathrm{Im}\, z|\bigr) \qquad \text{for} \qquad z \in \Omega_0 \cap \cdots \cap \Omega_s\,,$$

with $A, B > 0$ which depends only on γ.
Define now two operators.
Let s be an integer such that $1 \leq s \leq r$, let $q = 0, 1, 2, \ldots$ and set

$$\mathcal{A}^s_q = \mathcal{A}^s(\mathcal{C}) \otimes \Lambda^q \mathbb{C}'\ .$$

The family $(\mathcal{A}_q^s)$ defines the following double complex

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \\ \to & & \mathcal{A}_q^s & \xrightarrow{\delta} & \mathcal{A}_q^{s+1} & \to & \\ & & P\downarrow & & \downarrow P & & . \\ \to & & \mathcal{A}_{q-1}^s & \xrightarrow{\delta} & \mathcal{A}_{q-1}^{s+1} & \to & \\ & & \downarrow & & \downarrow & & \end{array}$$

In the diagram, δ is the coboundary operator associated with a family of sets, while P is the Koszul operator associated with the index q and the functions $\mu_1, \ldots, \mu_r$, defined as follows

$$\begin{array}{rcl} P : \mathcal{A}_q^s & \longrightarrow & \mathcal{A}_{q-1}^s \\ \omega = \omega_I^J & \longmapsto & P(\omega) = P(\omega)_K^J = \sum_{1\le i\le r} \omega_{K\cup\{i\}}^J \hat{\mu}_i \, , \end{array}$$

with $|J| = s, |I| = q, |K| = q - 1$.
We refer the reader to [2] for further details on this double complex. We now state a theorem from [6] which we will need in the sequel.

(3.1) Theorem — *Let $\mathcal{C}$ be a good family. Let $\omega \in \mathcal{A}_q^{s+1}(\mathcal{C})$ be such that $\delta(\omega) = 0$. Then there are a good refinement $\mathcal{C}'$ of $\mathcal{C}$ and $\eta \in \mathcal{A}_q^s(\mathcal{C}')$ such that $\rho(\omega) = \delta(\eta)$.*

(3.2) Theorem — *Let $\mathcal{C}$ be a good family and let $\{L\}$ be an almost parallel family.*

(i) *Let $q \geq 1, s \geq 0$. Then there exists a good refinement $\mathcal{C}'$ of $\mathcal{C}$ such that, for any $\omega \in \mathcal{A}_q^s(\mathcal{C})$, one has $P(\omega) = 0$; there exists also $\eta \in \mathcal{A}_{q+1}^s(\mathcal{C}')$ such that $\rho(\omega) = P(\eta)$.*
Moreover, if for some $A, B > 0$ and for any $N > 0$, we have

$$|\omega_i(z)| \leq A(1 + |z|)^{-N} \exp\bigl(B\,|\mathrm{Im}\, z|\bigr)\,,$$

i.e. $\omega_i \in \mathcal{D}(\mathbb{R}^n)$ for any i, the one can choose η so that, forsome $\mathring{A}, \mathring{B} > 0$ and for any $\mathring{N} \geq 0$ which depends on A, B, N and μ, one has

$$|\eta(z)| \leq \mathring{A}(1 + |z|)^{-\mathring{N}} \exp\bigl(\mathring{B}\,|\mathrm{Im}\, z|\bigr)\ .$$

(ii) *Let $s \geq 0, q = 0$. Then there exists a good refinement $\mathcal{C}'$ of $\mathcal{C}$ such that, for all $\omega = (\omega') \in \mathcal{A}_q^s(\mathcal{C})$ with the property that ω' belongs to the ideal generated by $\hat{\mu}_1, \ldots, \hat{\mu}_r$ in $\mathcal{O}(\Omega_{j_0} \cap \cdots \cap \Omega_{j_s})$, $\Omega_{j_i} \in \mathcal{C}$, there exists $\eta \in \mathcal{A}_q^s(\mathcal{C}')$ such that*

$$\rho(\omega) = P(\eta)\ .$$

Finally, for $N > 0$, there exist $A, B > 0$ such that

$$|\omega_i(z)| \leq A(1 + |z|)^{-N} \exp\bigl(B\,|\mathrm{Im}\, z|\bigr)\,,$$

one can also choose η such that, for any $\mathring{N} > 0$, there exist $\mathring{A}, \mathring{B} > 0$, we have

$$|\eta(z)| \leq \mathring{A}(1 + |z|)^{-\mathring{N}} \exp\bigl(\mathring{B}\,|\mathrm{Im}\, z|\bigr)\ .$$

Proof — It can be shown (see [2]) that this theorem follows if we can prove it for $s = 0$. Its proof is partially contained in theorem (5.3) of [2], with the exception of the bounds. Let us straighten up this point.
Suppose, first, $q = 0$. Then $\omega \in \mathcal{A}^0_0(\mathcal{C})$ determines an analytic function $\lambda(z)$ on each open set $\Omega \in \mathcal{C}$ with

$$|\lambda(z)| \leq A(1 + |z|)^{-N} \exp\bigl(B\,|\mathrm{Im}\, z|\bigr)\ . \tag{5}$$

Since, by hypothesis, $\lambda \in I_{\mathcal{O}(\Omega)}(\hat{\mu}_1, \ldots, \hat{\mu}_r)$, (5) follows from theorem (2.1) (local interpolation). From the definition of "good open sets", one has

$$\lambda(z) = \sum_{1\leq i\leq r} \alpha_i(z)\hat{\mu}_i(z)$$

for $\alpha_i \in \mathcal{O}(\Omega')$, with $\Omega' \subseteq \Omega$ and $\Omega' \in \mathcal{C}'$, a good refinement of $\mathcal{C}$ such that

$$|\alpha_i(z)| \leq A'(1 + |z|)^{-N'} \exp\bigl(B'\,|\mathrm{Im}\, z|\bigr)\ .$$

Consider now the case $q = 1$ (for $q \geq 2$, see [2] with the necessary modifications). Let $e_1, \ldots, e_r$ be a basis for $\Lambda^1(\mathbb{C}^r)$ and let $\omega \in \mathcal{A}^0_1(\mathcal{C})$. On each open set $\Omega \in \mathcal{C}$, one has

$$\omega = \omega_1 e_1 + \cdots + \omega_r e_r\ ,$$

with $\omega_j \in \mathcal{O}(\Omega)$. But $P(\omega) = 0$ means that

$$\omega_1\hat{\mu}_1 + \cdots + \omega_r\hat{\mu}_r = 0\,,$$

that is (under the hypothesis on $\mathrm{codim}(V)$) $\omega_1 \in I_{\mathcal{O}(\Omega)}(\hat{\mu}_1, \ldots, \hat{\mu}_r)$.
Therefore, from the case $q = 0$, there are $\mathcal{C}'$ and

$$\overset{\circ}{\eta} = \eta_2 e_2 + \cdots + \eta_r e_r \in \mathcal{A}'_\infty(\mathcal{C}')$$

such that

$$\omega_1 = \eta_2\mu_2 + \cdots + \eta_r\mu_r\ .$$

Moreover, if

$$|\omega_1(z)| \leq A(1 + |z|)^{-N} \exp\bigl(B\,|\mathrm{Im}\, z|\bigr)\,,$$

then also

$$\left|\overset{\circ}{\eta}(z)\right| \leq \overset{\circ}{A}(1 + |z|)^{-\overset{\circ}{N}} \exp\bigl(\overset{\circ}{B}\,|\mathrm{Im}\, z|\bigr)\ .$$

Define $\eta := -e_1 \wedge \overset{\circ}{\eta} \in \mathcal{A}^0_2(\mathcal{C}')$ and set $\gamma = \omega - P(\eta)$. Then

$$\gamma_1 = 0 \qquad P(\gamma) = 0$$
$$|\gamma_i(z)| \leq A(1 + |z|)^{-N} \exp\bigl(B\,|\mathrm{Im}\, z|\bigr)\ .$$

Repeating this procedure $(1, 1)$-times, we prove the theorem.

(3.3) Theorem — *Let $\mu = (\mu_1, \ldots, \mu_r)$ be slowly decreasing with respect to an analytic almost parallel family $\mathcal{L}$. Let V be the multiplicity variety associated with μ*

$$V = \{\, z \in \mathbb{C}^n \mid \hat{\mu}_1(z) = \cdots = \hat{\mu}_r(z) = 0 \,\}$$

and let λ be analytic on V, $\lambda \in \mathcal{O}(V)$.
Suppose that, for any $N > 0$, there are constants $A, B > 0$ and, for any $L \in \mathcal{L}$, there is a function $\overset{\circ}{\lambda}_L$ analytic on L, which verifies the following conditions:

(i) $\left|\overset{\circ}{\lambda}_L(z)\right| \leq A(1+|z|)^{-N}\exp\big(B\,|\mathrm{Im}\,z|\big)$ *for* $z \in L$;

(ii) *the restriction (with multiplicity) of* $\overset{\circ}{\lambda}_L$ *to* $V \cap L$ *equals the restricion of* λ *to* $V \cap L$, *i.e.*

$$\overset{\circ}{\lambda}_L|_{V\cap L} = \lambda|_{V\cap L}\ .$$

Then there exists an entire function $\overset{\circ}{\lambda} \in \mathcal{O}(\mathbb{C}^n)$ *such that its restriction to* V *coincides with* λ, $\lambda|_V = \lambda$ *and which satisfies, for some* $A', B' > 0$ *and any* $N' > 0$

$$\left|\overset{\circ}{\lambda}(z)\right| \leq A'(1+|z|)^{-N'}\exp\big(B'\,|\mathrm{Im}\,z|\big) \qquad \text{for} \qquad z \in \mathbb{C}^n\ .$$

Proof — Let $\mathcal{C}$ be a good family of sets: consider $\Omega \in \mathcal{C}$ associated with space $L \in \mathcal{L}$ and a component G of $O_L(\hat{\mu}; A, B, \varepsilon)$. Fix $\zeta \in G$; then, for any $z \in G$, one has

$$A(1+|z|)^{-B}\exp\big(\varepsilon\,|\mathrm{Im}\,z|\big) \leq A'(1+|\zeta|)^{-B}\exp\big(\varepsilon\,|\mathrm{Im}\,z|\big) = 4\sigma\ .$$

Choose local coordinates $z = (s,t)$ on Ω, centered at ζ, so that

$$L = \{(s,t) : t = 0\}\ .$$

Since $\hat{\mu}$ and its derivatives satisfy

$$|\hat{\mu}(z)| \leq A_1(1+|z|)^{-B_1}\exp\big(B_1\,|\mathrm{Im}\,z|\big)\ ,$$

by the mean value theorem, an by a suitable choice of $\overline{\alpha}, \overline{\beta} > 0$, one has that, if

$$\tau = \overline{\alpha}(1+|\zeta|)^{-\overline{\beta}}\exp\big(\varepsilon\,|\mathrm{Im}\,z|\big)\ ,$$

then

$$|z-w| < 2\tau\ \Rightarrow\ |\hat{\mu}(z) - \hat{\mu}(w)| < \sigma \qquad \text{for} \qquad z \in G \quad \text{and} \quad w \in \Omega\ .$$

Choose now $A', B', \varepsilon, \alpha', \beta'$ so that a good refinement

$$\rho : \mathcal{C}' \longrightarrow \mathcal{C}\ ,$$

associated with these constants, satisfies the following condition:

$$\Omega' \subseteq \{z = (s,t) : |t| < \tau, s \in G_\sigma\} \subseteq \Omega$$

for

$$G_\sigma = \{\, z \in G \,|\, \text{for } i = 1, \ldots, r,\ d(z, V_i \cap L \leq \sigma)\,\}\ .$$

By using the local interpolation formula, and by its analytical dependency on parameters, we can define a function $(J\lambda)(s,t)$ which is analytic on Ω', satisfies the growth conditions of $\mathcal{D}$

$$|(J\lambda)(z)| \leq A(1+|z|)^{-N}\exp\big(B\,|\mathrm{Im}\,z|\big)$$

and coincides with λ on $V \cap \Omega$.

In this way, we construct a collection of $J\lambda$ which determines an element in $\mathcal{A}^0_0(\mathcal{C}')$, which we indicate with γ. Set $\omega = \delta(\gamma)$ and let Ω'_0, Ω'_1 be in $\mathcal{C}'$; ω defines, on their intersection, an analytic function

$$\omega_{0,1}(z) := (J\lambda)_0(z) - (J\lambda)_1(z)\ ,$$

where $(J\lambda)_i$, $i = 0, 1$, indicates the restriction of $J\lambda$ to Ω'_i; $\omega_{0,1}$ satisfies the growth condition of $\mathcal{D}$:

$$|\,\omega_{0,1}(z)| \leq A(1+|z|)^{-N} \exp\bigl(B\,|\mathrm{Im}\,z|\bigr)\ .$$

Moreover $\omega_{0,1} \in I(\hat{\mu}_1, \ldots, \hat{\mu}_r)$ on $\Omega'_0 \cap \Omega'_1$.
As $\delta(\omega) = 0$, by theorem (3.2), there exist a good refinement $\mathcal{C}''$ of $\mathcal{C}'$ and $\eta \in \mathcal{A}^1_1(\mathcal{C}'')$ with $P(\eta) = \rho(\omega)$.
Moreover

$$|\,\eta(z)| \leq A(1+|z|)^{-N} \exp\bigl(B\,|\mathrm{Im}\,z|\bigr)\ .$$

Again by theorem (3.2), there exist a good refinement $\mathcal{C}'''$ of $\mathcal{C}''$ and $\theta \in \mathcal{A}^0_1(\mathcal{C}''')$, satisfying the same growth conditions as η, such that $\delta(\theta) = \rho(\eta)$.
We finally define a global analytic function belonging to $\mathcal{D}$ by

$$\overset{\circ}{\gamma} := \rho(\gamma) - P(\theta)\ .$$

This definition is well posed, since

$$\delta(\overset{\circ}{\gamma}) = \delta\bigl(\rho(\gamma)\bigr) - \delta\bigl(P(\theta)\bigr) = \rho\bigl(\delta(\gamma)\bigr) - P\bigl(\delta(\theta)\bigr) = \rho(\omega) - P\bigl(\delta(\theta)\bigr) = 0\ .$$

Now, by the theorem (2.2), we conclude the proof.

(3.1) Remark — Theorem (4.3) give, implicitely a “description” of $\mathcal{D}(V)$. We now conclude this paper with the representation theorem, whose proof follows immediately from the previous theorems.

(3.4) Theorem (Representation) — *Suppose that $\mu_1, \ldots, \mu_r \in \mathcal{E}'(\mathbb{R}^n)$ are slowly decreasing compactly supported distributions and let $f \in \mathcal{D}$. Then f is a solution of*

$$\mu_1 * f = \cdots = \mu_r * f = 0$$

if and only if there exists a finite partition J_k of indexes such that

$$f(x) = \sum_{k\in\mathbb{N}} \Biggl(\sum_{j \in J_k} \int_{V_j} \partial_j\bigl(\exp(-ix\cdot z)\bigr)\, d\nu_j(z) \Biggr)\,,$$

where $\{V_j\ :\ j \in J\}$ is a locally finite family of closed sets, the ∂_j’s are differential operators on V_j and the $d\nu_j$’ are Borel measures.
Let us finally suppose that $f \in \mathcal{D}'(\Omega + K)$, with $\Omega \subseteq \mathbb{R}^n$ an open convex set and $K \subset \mathbb{R}^n$ a compact convex set. If the support of $\mu_1, \ldots, \mu_r$ is contained in K, then the same results previously obtained for the equation $\mu * f = 0$. But more is necessary as far as the bounds are concerned, because the structure of the weights which describe $\mathcal{D}'(\Omega + K)$ and $\mathcal{D}'(\Omega)$ is more delicate that the one of $\mathcal{D}'(\mathbb{R}^n)$. As a consequence, stronger restrictions on the $\hat{\mu}_j$’ will be needed by introducing the support function on K.

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Adelina Fabiano
Università della Calabria
Dipartimento di Matematica
Arcavacata di Rende (CS), Italy
* **E-mail**: fabiano@unical.it

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