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SIMILARITY SOLUTIONS AND SPHERICAL DISCONTINUITY WAVES IN HYPERELASTIC MATERIALS SUBJECT TO A NON CONSTANT DEFORMATION

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SOMMARIO. Si fa l'analisi gruppale completa dell'equazione del secondo ordine che governa le deformazioni di un mezzo iperelastico omogeneo ed isotropo. Si caratterizzano alcune soluzioni esatte di similarità che corrispondono a classi di materiali compatibili con le condizioni di invarianza. Si studia la propagazione di onde sferiche di discontinuità in uno stato non costante caratterizzato dalle soluzioni suddette.

SUMMARY. The full group analysis is applied to the second order equation governing the deformation of a homogeneous isotropic hyperelastic material. Some exact similarity solutions are obtained corresponding to classes of materials compatible with the conditions of invariance. The propagation of spherical discontinuity waves is then considered in a non-constant state characterized by the above mentioned solutions. The occurrence of shock waves is discussed.

1. INTRODUCTION

In a recent paper [1] the propagation of finite amplitude spherical acceleration waves into an isotropic hyperelastic material subject to uniform dilatation has been considered.

The governing equation is the following

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = -\frac{1}{r}\frac{\partial \Sigma}{\partial \xi} + \left(\frac{2}{r} + \frac{\partial}{\partial r}\right)\frac{\partial \Sigma}{\partial \eta} \tag{1.1}$$

where ρ_0 is the density in the undeformed state, ξ and η are, respectively, the circumferential and the radial stretches given by

$$\xi = \frac{u}{r} \qquad \eta = \frac{\partial u}{\partial r} \tag{1.2}$$

$\Sigma(\xi, \eta)$ is the strain energy function and $u(r, t)$ is the displacement.

The main aim of the present paper is to carry out the full group analysis of the equation (1.1) in order to characterize the physical simmetrics of the model equation and the corresponding possible constitutive laws for Σ [2, 3].

As is well known, this procedure enables us to characterize particular exact solutions. The propagation of weak discontinuities is explored in a non-constant state characterized by a similarity solution using a procedure given in [2].

The results are specialized for two cases of elastic potentials: the first one permits the equation (1.1) to be invariant with respect to the dilatation group, the second one characterizes a class of rubber-like materials similar to those first considered by Ko [5, 1].

The paper concludes with a discussion on the occurrence of shock waves.

2. GROUP ANALYSIS

The equation (1.1) may be written in the form:

$$L(u) = u_{tt} + f_{\xi\eta}\frac{u}{r^2} - f_{\xi\eta}\frac{u_r}{r} - f_{\eta\eta}u_{rr} + f_\xi\frac{1}{r} - f_\eta\frac{2}{r} = 0 \tag{2.1}$$

where $f = \Sigma/\rho_0$.

To find the one parameter (ϵ) group of transformations

$t' = t'(t, r, u; \epsilon)$; $r' = r'(t, r, u; \epsilon)$; $u' = u'(t, r, u; \epsilon)$

leaving invariant the equation (2.1) we must determine the generators of the group defined by

$T = (dt'/d\epsilon)_{\epsilon=0}$; $X = (dr'/d\epsilon)_{\epsilon=0}$; $U = (du'/d\epsilon)_{\epsilon=0}$.

As (2.1) in a second order equation we introduce the twice extended operator

$$X_2 = T\partial_t + X\partial_x + U\partial_u + \tilde{u}_t\partial_{u_t} + \tilde{U}_r\partial_{u_r} + \tilde{U}_{tt}\partial_{u_{tt}} + \tilde{U}_{rr}\partial_{u_{rr}} + \tilde{U}_{rt}\partial_{u_{rt}} \tag{2.2}$$

where $\tilde{U}_t$, $\tilde{U}_r$, $\tilde{U}_{rr}$, $\tilde{U}_{rt}$, $\tilde{U}_{tt}$ have known expressions [6, 7]. The invariance condition for (2.1), $X_2(L(u)) = 0$, leads to

$$\begin{aligned}
&\tilde{U}_{tt} - 2f_{\xi\eta}\frac{uX}{r^3} + f_{\xi\eta}\frac{U}{r^2} + f_{\xi\xi\eta}\frac{uU}{r^3} - f_{\xi\xi\eta}\frac{u^2X}{r^4} + \\
&- f_{\xi\eta\eta}\frac{U}{r}u_{rr} + f_{\xi\eta\eta}\frac{uu_{rr}X}{r^2} - f_{\eta\eta\eta}\tilde{U}_r u_{rr} - f_\xi\frac{X}{r^2} + \\
&+ f_{\xi\eta\eta}\frac{u\tilde{U}_r}{r^2} - f_{\xi\eta\eta}\frac{u_rU}{r^2} + f_{\xi\xi\eta}\frac{Xuu_r}{r^3} - f_{\xi\eta\eta}\frac{u_r\tilde{U}_r}{r} - \\
&- f_{\eta\eta}\tilde{U}_{rr} + f_{\xi\xi}\frac{U}{r^2} - f_{\xi\xi}\frac{uX}{r^3} + f_{\xi\eta}\frac{\tilde{U}_r}{r} + \\
&+ 2f_\eta\frac{X}{r^2} - 2f_{\xi\eta}\frac{U}{r^2} + 2f_{\xi\eta}\frac{uX}{r^3} - 2f_{\eta\eta}\frac{\tilde{U}_r}{r} = \\
&= \Lambda\, u_{tt}\left(+f_{\xi\eta}\frac{u}{r^2} - f_{\xi\eta}\frac{u_r}{r} - f_{\eta\eta}u_{rr} + \frac{1}{r}f_\xi - \frac{2}{r}f_\eta\right)
\end{aligned} \tag{2.3}$$

where Λ is an arbitrary parameter.

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Requiring (2.3) to be identically satisfied we obtain some conditions involving T, X, U and $f(\xi, \eta)$. After simple but tedious calculations, only two possibilities arise, which are analyzed separately.

i) $f(\xi, \eta)$ arbitrary function of its arguments. In this case the only possible generators of the group are

$$\begin{aligned} T &= c_1 t + c_2 \\ X &= c_1 r \qquad (2.4) \\ U &= c_1 u \end{aligned}$$

where $c_1 = -\Lambda$ and c_2 are arbitrary constants.

ii) $f(\xi, \eta)$ satisfies the equation (2.5)

$$f_\xi (c_3 \xi - c_4) + f_\eta (c_3 \eta - c_4) + 2f(\Lambda + c_1) = \rho(\xi)\eta + \Omega(\xi)$$

with the condition

$$\Omega' = 2\rho - \xi\rho' \qquad (2.6)$$

In this case we obtain

$$\begin{aligned} T &= c_1 t + c_2 \\ X &= (\Lambda + 2c_1 + c_3) r \qquad (2.7) \\ U &= (\Lambda + 2c_1) u + c_4 r \end{aligned}$$

where $\Lambda, c_1, c_2, c_3, c_4$ are arbitrary constants.

The relations (2.5) and (2.6) characterize possible classes of strain energy functions which render the equation (2.1) invariant with respect to the groups of transformations having generators of the form (2.7). Furthermore we shall be interested in the following two particular cases:

A) $c_2 = 0,\ c_4 = 0,\ \rho = 0,\ \Omega = 0$

this implies the invariance of (2.1) with respect to the stretching group of transformations. Consequently we have:

$$f = \eta^{-\frac{2(\Lambda + c_1)}{c_3}} \Phi(\eta/\xi) \qquad (2.8)$$

where Φ is an arbitrary function.

B) $\rho = k\xi^2,\ \Omega = k_1,\ c_4 = 0$ (k and k_1 constants)

In this case we have again invariance with respect to the stretching group but we recover the strain energy function for Ko materials [5] characterized by:

$$f_\eta = \mu_0 (\xi^2 - \eta^{-3})$$

$$\mu_0 = \frac{k}{5c_3}$$

$$f_\xi = 2\mu_0 \xi\eta$$

3. SPECIAL SIMILARITY SOLUTIONS

The second order equation (2.1) may be written as a first order quasi-linear system of the form

$$\nu_t - \frac{\partial}{\partial r} f_n = \frac{2}{r} f_\eta - \frac{1}{r} f_\epsilon$$

$$\eta_t - \nu_r = 0 \qquad (3.1)$$

$$\xi_t = \nu/r \qquad (3.1)$$

Let us consider the following transformation of variables:

$$\tau = \ln t \qquad \sigma = r/t^{a+2-b} \qquad \nu = t^{a+1} V \qquad (3.2)$$

$$\eta = t^b \epsilon \quad \xi = t^b \mathcal{M}$$

where $a = \Lambda/c_1$, $b = -c_3/c_1$.

In the case of a strain energy function given by (2.8) we obtain

$$f = \eta^{\frac{2(a+1)}{b}-1} f_1 = t^{2(a+1)-b} \epsilon^{\frac{2(a+1)}{b}-1} f_1(\epsilon/\mathcal{M}) = t^{2(a+1)-b}\, \Psi$$

$$f = \xi^{\frac{2(a+1)}{b}-1} f_2 = t^{2(a+1)-b} \epsilon^{\frac{2(a+1)}{b}-1} f_2(\epsilon/\mathcal{M}) = t^{2(a+1)-b}\, G \qquad (3.3)$$

where

$$f_1 = \frac{2(a+1)}{b} \Phi + \frac{\eta}{\xi} \Phi'$$

$$f_2 = -(\eta/\xi)^2 \Phi' \qquad (3.4)$$

By means of the variable transformation (3.2), after (3.3), the system (3.1) assumes the form

$$U_\tau + F_\sigma - \sigma(a + 2 - b)\, U_\sigma = B \qquad (3.5)$$

where

$$U = \begin{pmatrix} V \\ \epsilon \\ \mathcal{M} \end{pmatrix} \quad F = \begin{pmatrix} -\Psi \\ -V \\ 0 \end{pmatrix} \quad B = \begin{pmatrix} 2\Psi - G - (a+1)V \\ -b\epsilon \\ V/\sigma - b\mathcal{M} \end{pmatrix} \qquad (3.6)$$

This system turns out to be invariant with respect to the following group (3.7)

$$\tau^* = \tau;\ \sigma^* = \omega\sigma;\ V^* = \omega^\alpha V,\ \epsilon^* = \omega^\beta \epsilon,\ \mathcal{M}^* = \omega^\beta \mathcal{M} \qquad \omega \in \mathrm{R} - \{0\}$$

and

$$\alpha = \beta + 1 \qquad \beta = b/(b - a - 1) \qquad (3.8)$$

Such invariance suggests a change of variables [4]:

$$\zeta = \ln\sigma,\quad \tau = \tau,\ V = \sigma^\alpha W,\ \epsilon = \sigma^\beta E,\ \mathcal{M} = \sigma^\beta M \qquad (3.9)$$

which premit us to write the system (3.5) in the form

$$\mathbf{w}_\tau - (a + 2 - b)\, \mathbf{w}_\zeta + \hat{\mathbf{F}}_\zeta = \mathbf{B} \qquad (3.10)$$

$$\mathbf{w} = \begin{pmatrix} W \\ E \\ M \end{pmatrix},\ \hat{\mathbf{F}} = \begin{pmatrix} -\hat{\Psi} \\ -W \\ 0 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} \beta\hat{\Psi} - G + \{(a+2-b) - (a+1)\}W \\ (a+2)E + \alpha W \\ W + (a+2)M \end{pmatrix}$$

where the independent variables ζ and τ do not appear explicitly among the components of B. The symbol ˆ means that a quantity has been evaluated by formally substituting ϵ with E and $\mathcal{M}$ with M.

The system (3.10) admits the constant solution

$$E_0 = -\frac{\alpha}{a+2} W_0 \qquad M_0 = -\frac{1}{a+2} W_0 \qquad (3.11)$$

and W_0 satisfying the relation:

$$-\left(\frac{\alpha}{a+2}\right)^{\frac{2(a+1)}{b}-1} W_0^{-2/\beta}(\beta f_1(\alpha)-f_2(\alpha))+$$

$$+(\alpha(a+2-b)-(a+1))=0$$

In the original variables this corresponds to a similarity solution given by

$$\xi=r^{b/(b-a-1)}t^{b}M_0;\ \eta=r^{b/(b-a-1)}t^{b}E_0;\ \nu=(r/t)^{\frac{2b-a-1}{b-a-1}}t^{2b}W_0 \tag{3.12}$$

In the case of Ko materials characterized by (2.9) the equation (1.1) reduces simply to the following second order equation:

$$u_{tt}-3\mu_0(u_r)^{-4}u_{rr}+2(\mu_0/r)u_r^{-3}=0 \tag{3.13}$$

which is equivalent to the first order system:

$$\begin{aligned} &v_t-3\mu_0^{-4}\,\eta\eta_r=-2(\mu_0/r)\eta^{-3}\\ &\eta_t-v_r=0 \end{aligned} \tag{3.14}$$

In this case, by (2.7) taking into account that $c_4 = 0$, we are led to consider the following transformation of variables

$$\tau=\ln(\gamma t+T_0),\ \sigma=r/(\gamma t+T_0)^{1/\gamma},\ v=(\gamma t+T_0)^{(1-\gamma)/2\gamma}V(\sigma,\tau),$$

$$\eta=(\gamma t+T)^{(\gamma-1)/2\gamma}\epsilon(\sigma,\tau),\ \gamma=(a+2-b)^{-1}. \tag{3.15}$$

By (3.15) the system (3.14) becomes:

$$\begin{aligned} &V_\tau-\sigma V_\sigma-3\epsilon^{-4}\epsilon_\sigma=-2(\mu_0/\sigma)\epsilon^{-3}-(1-\gamma)V/2\\ &\epsilon_\tau-\sigma\epsilon_\sigma-V_\sigma=(1-\gamma)\epsilon/2. \end{aligned} \tag{3.16}$$

By the transformation of variables

$$\zeta=\ln\sigma,\ \tau=\tau,\ V=\sigma^{1/2}W\ \ \epsilon=\sigma^{-1/2}E \tag{3.17}$$

(3.16) reduces to

$$\begin{aligned} &W_\tau-W_\zeta-3\mu_0E^{-4}E_\zeta=(\gamma/2)W-(7/2)\mu_0E^{-3}\\ &E_\tau-E_\zeta-W_\zeta=(1/2)W-(\gamma/2)E \end{aligned} \tag{3.18}$$

which has the constant solution

$$E_0^2=(7\mu_0)^{1/2}/\gamma;\ W_0=\gamma E_0. \tag{3.19}$$

For this solution to be meaningfull it must be $\gamma > 0$. Using the original variables we obtain the similarity solution

$$v=\left(\frac{r}{\gamma t+T_0}\right)^{1/2}W_0;\ \eta=\left(\frac{r}{\gamma t+T_0}\right)^{-1/2}E_0 \tag{3.20}$$

The exponent γ of the similarity variable σ is determined by giving a boundary condition of the type

$$\nu(r_0,t)=\nu_0(pt+q)^{-1/2}$$

ν_0, p, q given constants.

4. EVOLUTION OF DISCONTINUITIES

The evolution of weak discontinuities in the field variables compatible with the equation (1.1) and propagating in a constant state has been considered in [1]. Now we discuss the case of a weak discontinuity across a curve $\varphi(r,t)=0$ which propagates into a non-constant state characterized by a known exact solution. Specifically we assume the medium ahead of the wave front to be perturbed and the perturbation is either characterized by the solution (3.12) or (3.20).

This problem amounts to studying the propagation in a constant state when the transformation of variables (3.2) or (3.15) is considered. The discontinuity line will be

$$\hat\varphi(\tau,\zeta)=\varphi(r(\tau,\zeta),t(\tau))=0.$$

Let us first to consider the system (3.10) which has the constant solution (3.11). The characteristic velocities associated with the system (3.10) are solutions of

$$\det(\hat A-(\hat\lambda+\gamma^{-1})I)=0 \tag{4.1}$$

where

$$\hat\lambda=-\varphi_\tau/\varphi_\zeta \qquad \hat A=\nabla_w\hat{\mathrm F}.$$

By solving (4.1) we obtain

$$\hat\lambda_1+\gamma^{-1}=(\hat\psi_E)^{1/2},\ \hat\lambda_2+\gamma^{-1}=-(\hat\psi_E)^{1/2},\ \hat\lambda_3+\gamma^{-1}=0. \tag{4.2}$$

The characteristic velocities in the original variables are obtained by taking into account that [4]:

$$\lambda=\frac{r}{t}(\hat\lambda+\gamma^{-1}),\qquad \lambda=-\varphi_t/\varphi_r. \tag{4.3}$$

In order for the governing system to be hyperbolic we have to require $\hat\psi_E > 0$ which puts a restriction on the constitutive law (2.8).

If we denote by

$$\pi=(\partial\mathrm w/\partial\varphi)_{\varphi=0^+}-(\partial\mathrm w/\partial\hat\varphi)_{\hat\varphi=0^-}$$

the jump in the first order derivatives across $\hat\varphi(\zeta,\tau)=0$, we have

$$\pi=\pi d(w_0) \tag{4.5}$$

where d is the right eigenvector of the matrix $\hat A$ evaluated for $\mathrm w = \mathrm w_0$ given by (3.11). The amplitude of the discontinuity, in the case we are considering, is given by [4]:

$$\pi=\pi_0\,\frac{(t/t_0)^{p_\circ}}{1+\pi_0(a_0/b_0)((t/t_0)^{p_\circ}-1)} \tag{4.6}$$

where

$$\pi_0=\pi(t_0),\quad a_0=(\nabla_w\hat\lambda\cdot d)_0,\quad b_0=(\nabla_w(\mathfrak{l}\cdot\mathrm B)d)_0$$

$\mathfrak{l}$ is the left eigenvector for $\hat A$ corresponding to the eigenvalue $\hat\lambda$ and $\circ$ means that a quantity is evaluated for $\mathrm w = \mathrm w_0$.

If we consider the wave propagating with velocity $\hat\lambda_1$ we obtain

$$a_0=\frac{1}{2}(\hat\psi_{0E})^{1/2}\cdot\hat\psi_{0EE} \tag{4.7}$$

$$b_0=(\alpha+\beta)\hat\psi_{0E}-\hat G_{0E}-(\hat\psi_{0E})^{1/2}(\alpha(a+2-b)+1-\hat\psi_{0M}) \tag{4.8}$$

The discontinuity in the original variables $\mathrm u_L=(v,\zeta,\eta)$ is related to π by

$$\delta\mathrm u_L=(\partial\mathrm u_L/\partial\varphi)_{\varphi=0^+}-(\partial\mathrm u_L/\partial\varphi)_{\varphi=0^-}=r^{\alpha_L}t^{\beta_L}\mathrm d_{0L}\pi \tag{4.9}$$

where α_L and β_L are given by (3.12) comparing the exponents of the same variables.

In (4.9) we must take into account that, according to [4],

$$r/r_0 = (t/t_0)^{\hat{\lambda}_0 + \gamma^{-1}}. \tag{4.10}$$

From (4.6) the time t_c when the discontinuity wave may evolve into a shock wave is given by

$$t_c = t_0 \left(\frac{a_0 \pi_0 - b_0}{a_0 \pi_0}\right)^{b_0^{-1}} \tag{4.11}$$

Correspondingly π goes to infinity.

If $\pi_0 > 0$ then we distinguish two cases:

i) $f_{\eta\eta\eta} > 0$, that is, $a > 0$
we have a shock provided $\pi_0 > (b_0/a_0)$.

ii) $f_{\eta\eta\eta} < 0$: then the shock appears if $\pi_0 < (b_0/a_0)$.
Similar results may be obtained for $\pi_0 < 0$.

Finally we consider the case of Ko materials. From the system (3.18) we obtain for the characteristic velocities

$$\hat{\lambda}_1 + 1 = (3\mu_0)^{1/2} E^{-2} \quad \hat{\lambda}_2 + 1 = -(3\mu_0) E^{-2} \tag{4.12}$$

the corresponding velocities in the original variables are obtained by taking into account that

$$\lambda = \frac{r}{\gamma t + T_0} (\hat{\lambda} + 1). \tag{4.13}$$

The amplitude of the discontinuity in this case is given by

$$\pi = \pi_0 \frac{(\gamma t + T_0)^{b_0}}{1 + \pi_0 (a_0/b_0)((\gamma t + T_0)^{b_0} - 1)}, \tag{4.14}$$

with

$$a_0 = -2(3\mu_0)^{1/2} E_0^{-3}, \; b_0 = 12 \mu_0 E_0^{-4}. \tag{4.15}$$

The possible critical time occurs when

$$\gamma t_c + T_0 = \left(\frac{\pi_0 a_0 - b_0}{\pi_0 a_0}\right)^{b_0^{-1}} \tag{4.16}$$

As $a_0 < 0$ and $b_0 > 0$, we distinguish two cases:
i) $\pi_0 > 0$; $\gamma t_c + T_0 = (1 + 2(3\mu_0)^{1/2}/(\pi_0 E_0))^{b_0^{-1}}$
ii) $\pi_0 < 0$; then the shock appears if

$$|\pi|_0 > \frac{2(3\mu_0)^{1/2}}{E_0}$$

and we have:

$$\gamma t_c + T_0 = \left(1 - \frac{2(3\mu_0)^{1/2}}{\pi_0 E_0}\right)^{b_0^{-1}}$$

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