

**A MODIFIED BELINFANTE/ROSENFELD PROCEDURE FOR
 TESTING THE COMPATIBILITY OF GENERAL-COARIANT
 CONTINUUM PHYSICS AND GENERAL RELATIVITY THEORY**

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ABSTRACT. Creating a modified Belinfante/Rosenfeld procedure, Mathisson-Papapetrou-like equations are derived by which a comparison of General-Covariant Continuum Physics (GCCP) with General Relativity Theory (GRT) becomes possible. Three cases emerge: If the energy-momentum tensor of GCCP is symmetric and divergence-free, it serves as source of Einstein's equation; if the energy-momentum tensor is non-symmetric or/and not divergence-free, GRT and GCCP are incompatible with each other, if the Mathisson-Papapetrou-like equations are not valid; if they are valid, a source of Einstein's equation can be generated which is different from the energy-momentum tensor of GCCP.

In memory of Gérard A. Maugin

Preface

General-Covariant Continuum Physics (GCCP) is characterized by the energy-momentum balance (1)₁ and by the spin balance (1)₂ (Muschik and Borzeszkowski 2009)

$$T^{ab}{}_{;a} = K^b \quad \text{and} \quad S^{cab}{}_{;c} = M^{ab}. \quad (1)$$

The energy momentum tensor T^{ab} is in general not symmetric and not divergence-free, and the spin tensor S^{cab} is anti-symmetric in the last upper indices¹

$$T^{ab} \neq T^{ba}, \quad T^{ab}{}_{;a} \neq 0 \quad \text{and} \quad S^{cab} = -S^{cba}. \quad (2)$$

These properties do not fit the General Relativity Theory (GRT) defined by Einstein's field equations (Stephani 1982)

$$R^{ab} - \frac{1}{2}g^{ab}R = \kappa\Theta^{ab} = \kappa\Theta^{ba}, \quad \Theta^{ab}{}_{;b} = 0 \quad (3)$$

(Ricci tensor R^{ab} , curvature scalar $R = R^m{}_m$, metric g^{ab} , energy-momentum tensor Θ^{ab}), because the properties of T^{ab} and Θ^{ab} are different by comparing (2)₁ with (3), inducing the following reflections:

¹for the meaning of K^b and M^{ab} see below Sect. 4.

1: Taking a special material into consideration, are there two different energy-momentum tensors, Θ^{ab} and T^{ab} , one for gravitation, the other one for material properties? The answer is “no”! There should be a common theoretical formulation of a material in a gravitational field. Consequently, one of GRT or GCCP have to change its statements (Muschik 2016).

2: A candidate for changing GRT into a theory which allows non-symmetric energy-momentum tensors is the Einstein-Cartan theory (ECT) (Hermann *et al.* 2004). Here besides the metric, an additional field, the torsion, comes into play with severe corollaries for the gravitation. Up to now, no experimental evidence for torsion in the universe is found, although spin material is present. Consequently, a better deal may to keep GRT and to change GCCP.

3: There is a well-known method making GCCP fit GRT, the Belinfante/Rosenfeld procedure (Belinfante 1940; Rosenfeld 1940) which generates from a special combination of T^{ab} and S^{cab} a symmetric Θ^{ab} which serves as source in GRT field equations, thus adapting GCCP to GRT.

4: By adapting GCCP to GRT, T^{ab} and S^{cab} remain unchanged, but the sources of their corresponding balance equations are modified, so that the original balance equations of GCCP become the Mathisson-Papapetrou equations (Mathisson 1931, 1937; Papapetrou 1951), whereas those of GRT are kept with the energy-momentum tensor generated by the Belinfante/Rosenfeld procedure.

5: Result: The structures of GRT and GCCP are untouched by the adapting procedure of both theories. The three sources of the two theories are entangled in such a way that the structures of GRT and GCCP are kept, but entangled.

The paper considers the Belinfante/Rosenfeld procedure in more detail: Are there modified versions of the Belinfante/Rosenfeld procedure for generating different (and not unequivocal) suitable sources of Einstein’s field equations, if general-covariant energy-momentum and spin balances are basic?

1. Introduction

The Belinfante/Rosenfeld procedure generates a symmetric and divergence-free tensor of second order which serves as a source of Einstein’s equations. The original procedure implemented for the special-relativistic case (Belinfante 1940; Rosenfeld 1940) was recently extended to General-Covariant Continuum Physics (GCCP) (Muschik and Borzeszkowski 2014b,c). The special-relativistic Belinfante/Rosenfeld procedure and also the general-covariant one start out with an especially defined combination of spin divergences which together with the symmetric part of the energy-momentum tensor is symmetrized and made divergence-free. Special constraints by performing the procedure do not appear in the special-relativistic case, whereas in the general-covariant case such constraints are generated: the Mathisson-Papapetrou equations which have to be satisfied as necessary conditions for the energy-momentum tensor and the spin tensor of GCCP (Muschik and Borzeszkowski 2014b,c).

Now the question arises whether the general-covariant Belinfante/Rosenfeld procedure is unique, or if there exist different combinations of spin divergences for generating symmetric

and divergence-free tensors. This question is here investigated for the case of GCCP by introducing a family of Belinfante/Rosenfeld procedures.

The paper is organized as follows: after this introduction and recalling the general-covariant Belinfante/Rosenfeld procedure, a family of spin divergences is introduced, implementing a modified Belinfante/Rosenfeld procedure resulting in Mathisson-Papapetrou-like equations. These equations decompose into two classes: one class contains the curvature tensor for special family parameters, the other does not. The first class is chosen because we are interested in the curvature by which the special-relativistic case can be introduced easily in contrast of using a combination of spin divergences. Inserting the balance equations of energy-momentum and spin into the necessary Mathisson-Papapetrou-like equations, we obtain a system of differential equations for the spin and the metric. Finally, a remark is made about the compatibility of GCCP with GRT with regard to constitutive properties. First of all, we start out with the original general-covariant Belinfante/Rosenfeld procedure citing the well known

■ Proposition (Mathisson 1931, 1937; Muschik and Borzeszkowski 2014b,c; Papapetrou 1951): The general-covariant Belinfante/Rosenfeld procedure generates a symmetric and divergence-free tensor

$$\dagger \Theta^{ab} := T^{ab} - \frac{1}{2} [S^{cab} + S^{abc} + S^{bac}]_{;c} = \dagger \Theta^{ba}, \quad \dagger \Theta^{ab}_{;a} = 0, \quad (4)$$

if the Mathisson-Papapetrou equations

$$\frac{1}{2} S^{cab}_{;c} = T^{[ab]}, \quad T^{ab}_{;a} = \frac{1}{2} [S^{cab} + S^{abc} + S^{bac}]_{;c;a} = -\frac{1}{2} R^b_{klm} S^{klm} \quad (5)$$

are satisfied as necessarily constraints. ■

Here, T^{ab} is the in general non-symmetric and not divergence-free energy-momentum tensor of GCCP, $S^{cab} = -S^{cba}$ the spin tensor and R^b_{klm} the curvature tensor with respect to the underlying metric. The Mathisson-Papapetrou equations (5) as result of the Belinfante/Rosenfeld procedure (according to the proposition) are balance equations (1) of GCCP

$$S^{cab}_{;c} = 2T^{[ab]}, \quad T^{ab}_{;a} = -\frac{1}{2} R^b_{klm} S^{klm} \quad (6)$$

whose special-relativistic version is²

$$S^{cab}_{;c} = 2T^{[ab]}, \quad T^{ab}_{;a} = 0. \quad (7)$$

Inserting (5)₁ into (4)₁, results in

$$\dagger \Theta^{ab} = T^{(ab)} - \frac{1}{2} [S^{abc} + S^{bac}]_{;c} = T^{(ab)} - S^{(ab)c}_{;c}, \quad (8)$$

a tensor which is symmetric and divergence-free according to (4)₁ and (5)₂. Consequently, the general-covariant Belinfante/Rosenfeld procedure (BRP) transforms by use of the spin divergences a not necessary symmetric and divergence-free tensor – which appears in the

²The semicolon denotes covariant derivatives, round brackets the symmetric part of a tensor, square brackets its anti-symmetric part.

energy-momentum balance equation (1)₁ of GCCP (Muschik and Borzeszkowski 2014b,c) – into one which is symmetric and divergence-free and defined by (4)

$$T^{ab} \neq T^{ba}, T^{ab}_{;a} \neq 0 \xrightarrow{BRP} \dagger \Theta^{ab} = \dagger \Theta^{ba}, \dagger \Theta^{ab}_{;a} = 0. \quad (9)$$

Because such a mathematically produced $\dagger \Theta^{ab}$ serves as a source of Einstein's field equations³, the question arises, whether different symmetrizations are possible which do not refer to the special choice (4). That means, can the general-covariant Belinfante/Rosenfeld procedure (9) be replaced by a different proposal generating a symmetric and divergence-free tensor different from $\dagger \Theta^{ab}$? Consequently, we are looking for another possibility than (4) to use spin divergences for a symmetrization of the GCCP energy-momentum tensor.

2. A modified Belinfante/Rosenfeld procedure

2.1. A family of spin divergences. The general-covariant BRP starts out with the ad-hoc expression (4). We now replace the square bracket in (4) by an ad-hoc family of spin divergences whose family parameters are scalars (λ, μ, ν) implementing a modified Belinfante/Rosenfeld procedure. Analogously to (4), we define

$$\Theta^{ab}(\lambda, \mu, \nu) := T^{ab} - \Sigma^{cab}_{;c}(\lambda, \mu, \nu) \quad (10)$$

$$\begin{aligned} \Sigma^{cab}(\lambda, \mu, \nu) &:= \mu S^{cab} + \lambda S^{(ab)c} + \nu S^{[ab]c} = \\ &= \mu S^{cab} + \lambda \frac{1}{2}(S^{abc} + S^{bac}) + \nu \frac{1}{2}(S^{abc} - S^{bac}) = \\ &= \mu S^{cab} + \frac{1}{2}(\lambda + \nu)S^{abc} + \frac{1}{2}(\lambda - \nu)S^{bac}. \end{aligned} \quad (11)$$

According to its definition (11)₁, the symmetric and anti-symmetric parts of $\Sigma^{cab}(\lambda, \mu, \nu)$ are

$$\Sigma^{c(ab)}(\lambda) = \lambda S^{(ab)c}, \quad (12)$$

$$\Sigma^{c[ab]}(\mu, \nu) = \mu S^{cab} + \nu S^{[ab]c}, \quad (13)$$

by taking the anti-symmetry of the GCCP spin tensor (2)₂ into account. We obtain from (11)₃ by changing $c \leftrightarrow a$ and using (2)₂

$$\begin{aligned} \Sigma^{acb}(\lambda, \mu, \nu) &= \mu S^{acb} + \frac{1}{2}(\lambda + \nu)S^{cba} + \frac{1}{2}(\lambda - \nu)S^{bca} = \\ &= -\mu S^{abc} - \frac{1}{2}(\lambda + \nu)S^{cab} - \frac{1}{2}(\lambda - \nu)S^{bac}. \end{aligned} \quad (14)$$

Addition of (14)₂ with (11)₃ results in

$$\Sigma^{cab}(\lambda, \mu, \nu) + \Sigma^{acb}(\lambda, \mu, \nu) = \left[\mu - \frac{1}{2}(\lambda + \nu) \right] S^{cab} + \left[\frac{1}{2}(\lambda + \nu) - \mu \right] S^{abc}. \quad (15)$$

³thus obtaining physical meaning.

Consequently, we obtain a condition for the family parameters generating the anti-symmetry of $\Sigma^{cab}(\lambda, \mu, \nu)$ in the first two indices⁴ which is necessary for introducing the curvature below

$$\text{if } 2\mu \stackrel{\bullet}{=} \lambda + \nu \longrightarrow \Sigma^{cab}(2\mu - \nu, \mu, \nu) = -\Sigma^{acb}(2\mu - \nu, \mu, \nu). \quad (16)$$

According to (15), no anti-symmetry of $\Sigma^{cab}(\lambda, \mu, \nu)$ in the first two indices exists, if the family parameters do not obey (16)₁.

2.2. Symmetrization procedure. Starting out with (10), we demand

$$\Theta^{[ab]}(\lambda, \mu, \nu) = T^{[ab]} - \Sigma^{c[ab]}{}_{;c}(\lambda, \mu, \nu) \stackrel{\bullet}{=} 0, \quad (17)$$

$$\Theta^{ab}{}_{;a}(\lambda, \mu, \nu) = T^{ab}{}_{;a} - \Sigma^{cab}{}_{;c;a}(\lambda, \mu, \nu) \stackrel{\bullet}{=} 0, \quad (18)$$

generating a symmetric and divergence-free tensor Θ^{ab} . Consequently, we obtain from (17)₂ and (13)

$$T^{[ab]} = \Sigma^{c[ab]}{}_{;c}(\mu, \nu) = \mu S^{cab}{}_{;c} + \nu S^{[ab]c}{}_{;c}. \quad (19)$$

Taking (12) and (18)₂ into account, (10) results in

$$\Theta^{ab}(\lambda) = T^{(ab)} - \Sigma^{c(ab)}{}_{;c}(\lambda) = T^{(ab)} - \lambda S^{(ab)c}{}_{;c}, \quad (20)$$

$$\longrightarrow \Theta^{ab}{}_{;a}(\lambda) = 0 = T^{(ab)}{}_{;a} - \lambda S^{(ab)c}{}_{;c;a}. \quad (21)$$

2.3. Mathisson-Papapetrou-like equations and curvature. Taking (11)₃ into account, (18) becomes

$$T^{ab}{}_{;a} = \Sigma^{cab}{}_{;c;a}(\lambda, \mu, \nu) = \left[\mu S^{cab} + \frac{1}{2}(\lambda + \nu)S^{abc} + \frac{1}{2}(\lambda - \nu)S^{bac} \right]_{;c;a}. \quad (22)$$

Similar to (5), (19) and (22) are Mathisson-Papapetrou-like equations which become the original ones, if the family parameters are

$$\lambda = 1, \quad \mu = 1/2, \quad \nu = 0. \quad (23)$$

This combination of the family parameters satisfies (16)₁ so that the anti-symmetry

$$\Sigma^{cab}(1, 1/2, 0) = -\Sigma^{acb}(1, 1/2, 0) \quad (24)$$

is valid in this case. We now remember the

■ Proposition(Muschik and Borzeszkowski 2014b,c): If Σ^{cab} is anti-symmetric in the first two indices

$$\Sigma^{cab}(2\mu - \nu, \mu, \nu) = -\Sigma^{acb}(2\mu - \nu, \mu, \nu) \quad (25)$$

according to (16), the second derivatives in (22) can be replaced by the curvature tensor $R^b{}_{klm}$

$$\begin{aligned} T^{ab}{}_{;a} &= \Sigma^{cab}{}_{;c;a}(2\mu - \nu, \mu, \nu) = \\ &= \left[\mu S^{cab} + \mu S^{abc} + (\mu - \nu)S^{bac} \right]_{;c;a} = -R^b{}_{klm}\Sigma^{k[lm]}(\mu, \nu). \end{aligned} \quad (26)$$

⁴The sign $\stackrel{\bullet}{=}$ stands for a setting.

This replacement is only possible, if (25) is valid, otherwise the second derivatives in (22) cannot be replaced by the curvature tensor. ■

Inserting (13), we obtain from (26) and (19) the Mathisson-Papapetrou-like equations

$$T^{ab}{}_{;a} = -R^b{}_{klm} \left(\mu S^{klm} + \nu S^{[lm]k} \right), \quad T^{[ab]} = \left[\mu S^{cab} + \nu S^{[ab]c} \right]{}_{;c}. \quad (27)$$

These Mathisson-Papapetrou-like equations represent the energy-momentum balance equation (27)₁ and the spin balance equation (27)₂ of GCCP both of which are necessary for generating the symmetric and divergence-free tensor (20), if using (25) in (10). The curvature tensor $R^b{}_{klm}$ is determined by the space-time geometry which is fixed by a background metric or by corresponding field equations⁵. The corresponding symmetric and divergence-free tensor (20) is according to (16)₁

$$\Theta^{ab}(2\mu - \nu) = T^{(ab)} - (2\mu - \nu)S^{(ab)c}{}_{;c}. \quad (28)$$

Choosing the parameters (23) results in the Mathisson-Papapetrou equations (5). For generating the Mathisson-Papapetrou equations two facts are necessary: One has to know the relation between the anti-symmetric part of the energy-momentum tensor and the spin divergences (27)₂⁶, and (25) has to be valid, because otherwise the curvature tensor cannot be introduced to the Mathisson-Papapetrou-like equation (27)₁. Consequently, the applicability of the Belinfante/Rosenfeld procedure depends on knowledge from the outside of the procedure: the scalars μ and ν have to be determined by external facts, such as (23) which are Lagrangian-based.

We now consider the case for which the GCCP energy-momentum tensor itself is symmetric and divergence-free

$$T^{[ab]} = 0, \quad T^{ab}{}_{;a} = 0, \quad (29)$$

the case for which the Belinfante/Rosenfeld procedure is dispensable because the energy-momentum tensor itself satisfies the result of this procedure. Consequently, if (29) is a fixpoint for arbitrary BRPs, according to (27)₂, we obtain

$$\mu S^{cab}{}_{;c} + \nu S^{[ab]c}{}_{;c} = 0, \quad (30)$$

and according to (21)₂

$$\lambda S^{(ab)c}{}_{;c;a} = 0. \quad (31)$$

Fixpoint means, (30) and (31) are valid for arbitrary (λ, μ, ν) , resulting in the following conditions for the spin divergences

$$S^{cab}{}_{;c} = 0, \quad S^{[ab]c}{}_{;c} = 0, \quad S^{(ab)c}{}_{;c;a} = 0. \quad (32)$$

According to (20)₂ and (29)₁ also

$$\Theta^{ab}(\lambda) = T^{(ab)} = T^{ab} \longrightarrow S^{(ab)c}{}_{;c} = 0 \quad (33)$$

has to be additionally satisfied that (29) are fix-points of the BRP, even if the spin is different from zero.

⁵Especially here, we will choose Einstein's equations in Sect. 3.

⁶a knowledge from outside the Belinfante/Rosenfeld procedure which establish the factor 1/2 in (4).

3. Belinfante/Rosenfeld procedure and GRT

For the sequel, we now consider the case (16) which causes that the curvature tensor appears in the energy-momentum balance equation (27)₁ which is determined by the space-time geometry. Here, we are interested in the Riemannian space-time which is described by Einstein's equations (3)

$$R^{ab} - \frac{1}{2}g^{ab}R = \kappa\Theta^{\dagger ab} \implies \Theta^{\dagger ab} = \Theta^{\dagger ba}, \quad \Theta^{\dagger ab}{}_{;a} = 0 \quad (34)$$

We remind that the following proceeding is standard⁷: if the energy-momentum tensor is not symmetric⁸ or not divergence-free⁹, one has to generate a symmetric and divergence-free tensor (28) by use of the BRP. This tensor serves as a source of (34)₁

$$\Theta^{\dagger ab} \stackrel{\bullet}{\equiv} \Theta^{ab}(2\mu - \nu) = T^{(ab)} - (2\mu - \nu)S^{(ab)c}{}_{;c}. \quad (35)$$

The scalars μ and ν stem from the spin balance equation (27)₂ whose source is the anti-symmetric part of the energy-momentum tensor. According to the setting (35)₁, the two scalars fix the contribution of the spin to the source of Einstein's equations, this means that, the spin's contribution to gravitation. Using Einstein's equations and the spin balance (27)₂, we obtain a representation of Einstein's equations

$$T^{ab} = \frac{1}{\kappa} \left(R^{ab} - \frac{1}{2}g^{ab}R \right) + \left[\mu S^{cab} + \mu S^{abc} + (\mu - \nu)S^{bac} \right]_{;c} \quad (36)$$

which is compatible with (26)₂ according to (34)₃.

The curvature tensor in the energy-momentum balance equation (27) has to be match with the metric g^{ab} in Einstein's equations (34)₁, that means, it has to obey the Bianchi identities as integrability conditions (Stephani 1982)

$$R^b{}_{klm;j} + R^b{}_{kmj;l} + R^b{}_{kjl;m} = 0. \quad (37)$$

Consequently, (27)₁ becomes with $j \stackrel{\bullet}{=} b$

$$\begin{aligned} T^{ab}{}_{;a;b} &= \left[R^b{}_{kmb;l} + R^b{}_{kbl;m} \right] \left(\mu S^{klm} + \nu S^{[lm]k} \right) - R^b{}_{klm} \left(\mu S^{klm} + \nu S^{[lm]k} \right)_{;b} = \\ &= \left[-R_{km;l} + R_{kl;m} \right] \left(\mu S^{klm} + \nu S^{[lm]k} \right) - R^b{}_{klm} \left(\mu S^{klm} + \nu S^{[lm]k} \right)_{;b}. \end{aligned} \quad (38)$$

In the next section, we investigate how the BRP works with regard to GCCP.

4. GCCP and Belinfante/Rosenfeld procedure

Usually, a Lagrange formalism is not available for GCCP. Consequently, we have to start with the balance equations of energy-momentum and spin which are ad-hoc equations in the sense of a theory which is supported by a variational problem. In a curved space-time, these balance equations are (Muschik and Borzeszkowski 2009)

⁷standard, but not evident.

⁸e.g., that is the case if the stress tensor of a material is not symmetric, for liquid crystals, spin materials (Hermann *et al.* 2004), *etc.*

⁹if, e.g., in General-Covariant Physics $T^{ab}{}_{;a} \neq 0$ is valid in general.

$$T^{ab}{}_{;a} = K^b = G^b + k^b, \quad T^{ab} \neq T^{ba}, \quad S^{cab}{}_{;c} = M^{ab} = H^{ab} + m^{ab}, \quad (39)$$

$$\text{with } S^{cab} = -S^{cba}, \quad m^{ab} = -m^{ba}, \quad H^{ab} = -H^{ba}. \quad (40)$$

The G^b and H^{ab} are internal source terms – the *Geo-SMEC-terms* (Geometry-Spin-Momentum-Energy-Coupling) (Muschik and Borzeszkowski 2009) – which are caused by the choice of a special space-time geometry and by a possible coupling between energy-momentum, spin and geometry. For non-isolated systems, $k^b \neq 0$ denotes an external force density, and $m^{ab} \neq 0$ is an external momentum density. In particular, one finds such a situation in special-relativistic continuum thermodynamics (Eckart 1940; Israel and Steward 1980; Neugebauer 1980). Inserting the balance equations (39), the Mathisson-Papapetrou-like equations (27) result in

$$G^b + k^b = -R_{klm}^b \left(\mu S^{klm} + \nu S^{[lm]k} \right), \quad T^{[ab]} = \mu \left(H^{ab} + m^{ab} \right) + \nu S^{[ab]c}{}_{;c}. \quad (41)$$

The sources of the energy-momentum and spin balance equations (39) have to satisfy (41) for performing a modified Belinfante/Rosenfeld procedure. Because the source of the spin balance equation (39)₃ determines the anti-symmetric part of the energy-momentum tensor, we demand taking (41)₂ into account

$$\left(H^{ab} + m^{ab} = 0 \quad \overset{\bullet}{\rightarrow} \quad T^{[ab]} = 0 \right) \quad \rightarrow \quad \nu S^{[ab]c}{}_{;c} = 0, \quad (42)$$

thus

$$\left(\nu = 0 \quad \vee \quad S^{[ab]c}{}_{;c} = 0 \right) \quad (43)$$

follows adapting the Belinfante/Rosenfeld procedure to GCCP. Consequently, the Mathisson-Papapetrou-like equations (41) result in

$$T^{ab}{}_{;a} = G^b + k^b = -\mu R_{klm}^b S^{klm}, \quad T^{[ab]} = \mu \left(H^{ab} + m^{ab} \right) = \mu S^{cab}{}_{;c}, \quad (44)$$

These equations are similar to those generated by the general-covariant Belinfante/Rosenfeld procedure (5), if the balance equations of GCCP (39) are inserted. The derivation of (44) indicates that two conditions for the validity of (5) must hold: firstly the dependence of the anti-symmetric part of the energy-momentum tensor on the source of the spin balance equation represented by (43) and secondly that the factor 1/2 in (5) is introduced from knowledge beyond the Belinfante/Rosenfeld procedure¹⁰. From (35)₂ and (34)₃ it follows that

$$T^{(ab)}{}_{;a} = 2\mu S^{(ab)c}{}_{;c;a}. \quad (45)$$

Taking (44), (39)₁ and (45) into account, we obtain a differential equation for the spin

$$T^{ab}{}_{;a} = T^{[ab]}{}_{;a} + T^{(ab)}{}_{;a} = -\mu R_{klm}^b S^{klm} = \mu \left(H^{ab} + m^{ab} \right)_{;a} + 2\mu S^{(ab)c}{}_{;c;a}, \quad (46)$$

$$S^{(ab)c}{}_{;c;a} = -\frac{1}{2} R_{klm}^b S^{klm} - \frac{1}{2} \left(H^{ab} + m^{ab} \right)_{;a}. \quad (47)$$

¹⁰For all special models which we know up to now, $\mu = 1/2$ is valid, so that with (43) and (16)₁ we come back to the usual covariant Belinfante procedure starting with (4).

Here (47) is independent of the scalar μ . That is not the case for the Belinfante/Rosenfeld generated symmetric and divergence-free tensor (28)

$$\frac{1}{\kappa} \left(R^{ab} - \frac{1}{2} g^{ab} R \right) = \Theta^{ab}(2\mu) = T^{(ab)} - 2\mu S^{(ab)c}{}_{;c}. \tag{48}$$

Also in the special case of GCCP (43), the contribution of the spin to gravitation is determined by knowledge of μ beyond the Belinfante/Rosenfeld procedure. The same situation holds true for the general-covariant procedure which starts out with the definition (4) resulting in (23). The Belinfante/Rosenfeld procedure itself cannot generate the symmetric and divergence-free “mutant” (48) of the energy-momentum tensor: the scalar μ has to be known in order to solve (47) and (48) together. The complete formulation of GCCP contains beside the balance equations (39) those of particle number and entropy density¹¹

$$N^k{}_{;k} = 0, \quad S^k{}_{;k} = \sigma + \varphi \tag{49}$$

(N^k particle flux density, S^k entropy 4-vector, σ entropy production, φ entropy supply). The Second Law of Thermodynamics is taken into account by the demand that the entropy production has to be non-negative at each event and for arbitrary materials after having inserted the constitutive equations into the expression of the entropy production (Muschik and Borzeszkowski 2014a)

$$\sigma \geq 0. \tag{50}$$

Particle number and entropy density are here out of scope because the Belinfante/Rosenfeld procedure (9) does not involve particle number and entropy. That means, N^k and S^k do not have an effect on gravitation concerning GRT, because BRP generates a second rank tensor for transforming T^{AB} into Θ^{AB} .

5. Material, GCCP and Bel/Ros procedure

We now presuppose that μ in (48) is known from the outside, so that the Belinfante/Rosenfeld procedure can be performed. Consequently, we consider three tensors in $\text{GCCP} \cap \text{GRT}$

$$\Theta^{ab}(2\mu), \quad T^{ab}, \quad S^{cab}, \tag{51}$$

the source of Einstein’s equations (48), the energy-momentum tensor and the spin tensor satisfying the Mathisson-Papapetrou-like equations (44), the energy-momentum balance (44)₁ and the spin balance (44)₂. Because the Mathisson-Papapetrou-like equations are necessary for performing the Belinfante/Rosenfeld procedure (Muschik and Borzeszkowski 2014b,c), all energy-momentum balances and spin balances (39) whose sources do not satisfy (44) are not suitable for applying the Belinfante/Rosenfeld procedure, and therefore also not suitable for a GRT description because a source of Einstein’s equations cannot be generated. If the Mathisson-Papapetrou-like equations (44) are satisfied, an energy-momentum “mutant” Θ^{ab} , (48), exists beyond the energy-momentum tensor T^{ab} , and the question arises, which tensor describes the constitutive properties of the considered

¹¹For this continuum theory of irreversible processes, see Kluitenberg *et al.* (1953a,b) and Neugebauer (1980), and the contributions by Anile and Choquet-Bruhat (1989).

system ? Evident is that in GCCP the in general non-symmetric and not divergence-free energy-momentum tensor T^{ab} whose (3+1)-decomposition is

$$T^{ab} = \frac{1}{c^4} e u^a u^b + \frac{1}{c^2} u^a p^b + \frac{1}{c^2} q^a u^b + t^{ab} \quad (52)$$

contains constitutive properties such as

$$\text{energy density: } e = u_k u_l T^{(kl)}, \quad (53)$$

$$\text{momentum flux density: } p^k = h_l^k u_m T^{ml}, \quad (54)$$

$$\text{energy flux density: } q^k = h_l^k u_m T^{lm}, \quad (55)$$

$$\text{stress tensor: } t^{kl} = h_a^k h_b^l T^{ab}. \quad (56)$$

Here u_k is the 4-velocity of the material and

$$h_k^i = \delta_k^i - \frac{1}{c^2} u^i u_k, \quad (57)$$

the projector perpendicular to the 4-velocities u^k resp. u_i . Presupposing the special case that the source of the energy-momentum balance equation (39)₁ satisfies the Mathisson-Papapetrou-like equation (44)₁, the RHS of Einstein's equations is (48) describing the constitutive influence of the material to gravitation. If (44)₁ is not valid, GCCP and GRT do not fit together. Clearly, real constitutive properties do not change by performing a formal Belinfante/Rosenfeld procedure. Consequently, it is evident that the energy-momentum tensor T^{ab} remains a carrier of constitutive properties also after having performed a BRP: a material equipped with a non-symmetric stress tensor (56) cannot be described by a symmetric $\Theta^{ab}(2\mu)$, (48).

There is one special case for which constitutive properties are transferred to the source of Einstein's equations, if T^{ab} is a fix-point of the Belinfante/Rosenfeld procedure. Starting out with (48) and taking the fix-point property into account, we obtain by use of (45)

$$T^{(ab)} - 2\mu S^{(ab)c}{}_{;c} = \Theta^{ab}(2\mu) \stackrel{\cdot}{=} T^{ab} = T^{(ab)} + T^{[ab]}, \quad (58)$$

$$\longrightarrow -2\mu S^{(ab)c}{}_{;c} = T^{[ab]} = 0 \longrightarrow T^{ab}{}_{;a} = 0. \quad (59)$$

The result is as expected: fix-points of the Belinfante/Rosenfeld procedure are symmetric and divergence-free energy-momentum tensors. Consequently, the symmetric and divergence-free "mutant" is in this case the energy-momentum tensor itself equipped with constitutive properties.

Summarizing, we have three different situations:

- I. The Mathisson-Papapetrou-like equation (44)₁ is not satisfied. In this case, a GCCP-induced source of Einstein's equations cannot be generated.
- II. If the Mathisson-Papapetrou-like equation (44)₁ is valid, the source of Einstein's equations is (48) which is different from the energy-momentum tensor, the GCCP carrier of constitutive properties .

- III. If the energy-momentum tensor is symmetric and divergence-free, the source of Einstein's equations is identical to this energy-momentum tensor, and the Belinfante procedure is dispensable.

Indisputable, the case III correspond to the demand of GRT that all constitutive properties are tied down to the source of Einstein's equations. This case points out that GRT is a special constitutive theory restricted to materials of symmetric and divergence-free energy-momentum tensors. Evident is, that in case I GCCP and GRT do not fit together: GCCP cannot be adapted to GRT. Really interesting is the case II: parts of the energy-momentum tensor are considered as non-generating gravitation. We obtain this part by the Belinfante/Rosenfeld procedure from (48) and (44)₂

$$T^{ab} - \Theta^{ab}(2\mu) = T^{[ab]} + 2\mu S^{(ab)c}{}_{;c} = \mu \left[S^{cab} + S^{abc} + S^{bac} \right]{}_{;c}. \quad (60)$$

Because the formal Belinfante/Rosenfeld procedure does not change constitutive properties, these are still described by the in general non-symmetric and non-divergence-free energy-momentum tensor T^{ab} . If erroneously the constitutive properties in case II are bound up with the symmetric and divergence-free "mutant" $\Theta^{ab}(2\mu)$, the original material of GCCP is replaced by another one which now fit into GRT, but which is not the original one anymore. If one reject the fact of case II that constitutive properties and gravitational influence are described by different tensors, we obtain the following

■ Statement: Not accepting that two different tensors are necessary for describing space-time and constitutive properties, GCCP and GRT only fit together for special materials of symmetric and divergence-free energy-momentum tensor. ■

This statement has two consequences:

- A) Concerning GCCP, the Belinfante/Rosenfeld procedure is dispensable (case III).
- B) A non-restricted general relativistic constitutive theory needs a theory of gravitation which accepts non-symmetric and non-divergence-free energy-momentum tensors (case I and II)¹².

6. Discussion

The energy-momentum and the spin balance equations of General-Covariant Continuum Physics (GCCP) include covariant derivatives which are determined by the space-time geometry. Either the space-time geometry is given or the balance equations have to be solved together with the field equations which determine the space-time geometry. The energy-momentum tensor of GCCP is in general non-symmetric and not divergence-free and is a carrier of constitutive properties such as energy density, energy flux density, momentum flux density and stress tensor.

If we are interested in the conditions under which GCCP and General Relativity Theory (GRT) fit together, the first well known problem arising is, that the source of Einstein's equations is a symmetric and divergence-free tensor. The usual tool for symmetrizing the energy-momentum tensor is the general-covariant Belinfante/Rosenfeld procedure which

¹²e.g., the Einstein-Cartan space-time (Herrmann *et al.* 2014).

generates a symmetric and divergence-free tensor which can be used as source of Einstein's equations (Muschik and Borzeszkowski 2014b).

The general-covariant Belinfante/Rosenfeld procedure uses an ad-hoc combination of spin divergences for achieving the symmetrization of the energy-momentum tensor. It was found out that only such energy-momentum tensors can be symmetrized which satisfy the Mathisson-Papapetrou equations (Muschik and Borzeszkowski 2014b). That means, only those materials which satisfy the constraints imposed by the necessary Mathisson-Papapetrou equations can be described by GRT. Only for those materials, a symmetric and divergence-free "mutant" of the GCCP-energy-momentum tensor can be generated by the Belinfante/Rosenfeld procedure.

If the above mentioned ad-hoc combination of spin divergences is replaced by a more general family of spin divergences, Mathisson-Papapetrou-like equations appear as necessary constraints for the Belinfante/Rosenfeld convertible energy-momentum tensors. These Mathisson-Papapetrou-like equations decay into two classes: one class contains the curvature tensor, the other does not. Here, the first class is considered, because we are interested in the relation between GCCP and GRT. The introduction of the family of spin divergences demonstrates that there are symmetrization procedures beyond the conventional Belinfante/Rosenfeld proceeding. In any case, conventional or modified Belinfante/Rosenfeld procedure, the symmetrization needs some knowledge from beyond the BRP, because otherwise the result of the Belinfante/Rosenfeld symmetrization is not unique¹³.

As expected, symmetric and divergence-free energy-momentum tensors are fix-points of the Belinfante/Rosenfeld procedures¹⁴, even if the spin is different from zero. That is the only case for which the following problem does not appear: constitutive properties of GCCP are tied down to the corresponding non-symmetric and not divergence-free energy-momentum tensor of GCCP, whereas the properties of the space-time geometry are generated by the Belinfante/Rosenfeld-transformed "mutant". The formal Belinfante/Rosenfeld procedure (9) does not transfer constitutive properties of the energy-momentum tensor to the source of Einstein's equations¹⁵. This splitting into two tensors, the material-dependent energy-momentum tensor and the Belinfante/Rosenfeld-transformed one determining the geometry contradicts the spirit of GRT which demands that the source of Einstein's equations should contain all constitutive properties concerning the energy-momentum tensor. That is the case if both these tensors are equal, that means, the energy-momentum tensor is already symmetric and divergence-free and the symmetrization procedure is dispensable. Concerning GCCP, GRT is a theory which is restricted to the class of materials of symmetric and divergence-free energy-momentum tensors. Abolishing of this restriction makes an extension of the field equations beyond GRT necessary. A possible candidate for replacing GRT with regard to GCCP may be the Einstein-Cartan space-time geometry.

The question concerning the compatibility of GCCP and GRT can be answered as follows: as expected, the compatibility exists only for materials with symmetric and divergence-free energy-momentum tensors, but only if the above mentioned spirit of GRT is accepted.

¹³and suspicion arises, if the BRP is a suitable proceeding.

¹⁴this means that, a Belinfante/Rosenfeld procedure is dispensable for such energy-momentum tensors.

¹⁵e.g., if momentum flux density and energy flux density are different in GCCP, they become equal in GRT after having performed the Belinfante/Rosenfeld procedure, implying a change to another material.

Epilogue

In the special-relativistic Belinfante-Rosenfeld symmetrization procedure of a divergenceless non-symmetric energy-momentum tensor T^{ab} , the following ansatz

$$\dagger \Theta^{ab} := T^{ab} - \frac{1}{2} \left[S^{cab} + S^{abc} + S^{bac} \right]_{,c} \quad (61)$$

is assumed. Here the antisymmetry of S^{cab} in the last two indices a and b guaranties that

$$\Sigma^{cab} := S^{cab} + S^{(ab)c} + S^{[ab]c}$$

is anti-symmetric in the first two indices c and a such that, due to the commutativity of partial derivatives, T^{ab} can be symmetrized by (61). In the general-covariant derivation of the Mathisson-Papapetrou equations given by Muschik and Borzeszkowski (2014b), the starting point is the symmetrization of a non-symmetric energy-momentum tensor, whose divergence does not vanish. To this end, the general-covariantly generalized ansatz (4) is assumed what, to some extent, is justified by the special-relativistic Belinfante-Rosenfeld procedure. But, the strategy of the present paper is the following: we do not refer to our knowledge of special relativity, but ask

- (i) whether there are other general-covariant algebraic combinations of the tensor S^{cab} (now specified as the spin tensor of continuum physics according to (39)₃) which symmetrize T^{ab} and
- (ii) how other combinations of the spin tensor modify the Mathisson-Papapetrou equations.

The result is that for the class of ansatzes (11), up to a factor μ , (4) is reproduced, if additional assumptions are made. This corroborates our knowledge from special relativity that only the symmetry properties of Σ^{cab} are relevant for the symmetrization procedure. While these properties are not changed by μ , the Mathisson-Papapetrou-like equations (44)₂ contain this arbitrary factor which has to be determined otherwise.

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