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A GENERALIZATION OF M-SEPARABILITY BY NETWORKS

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ABSTRACT. All spaces are assumed to be Tychonoff. A space is M-separable if for every sequence $(D_n : n \in \omega)$ of dense subsets of X one can pick finite $F_n \subset D_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} F_n$ is dense in X. Every space having a countable base is M-separable but not every space with countable network weight is M-separable. We introduce a new Menger type property defined by networks, called *M-nw-selective property*, such that every M-nw-selective space has countable network weight and is M-separable. By analogy, we also introduce H- and R- nw-selective spaces for Hurewicz and Rothberger type properties. Several properties of the new classes of spaces are studied and some questions are posed.

Dedicated to the memory of Mikhail (Misha) Matveev

1. Introduction

Throughout this paper all spaces are assumed to be Tychonoff. In terminology, we in general follow Engelking (1989). Recall that a space X is Menger if for every sequence $(\mathscr{U}_n : n \in \omega)$ of open covers of X one can select finite $\mathscr{F}_n \subset \mathscr{U}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathscr{F}_n$ covers X (Hurewicz 1925), where Menger's property was denoted by E^* (see also Menger 1924; Engelking 1989; Just *et al.* 1996); X is Rothberger if for every sequence $(\mathscr{U}_n : n \in \omega)$ of open covers of X one can select $F_n \in \mathscr{U}_n$, $n \in \omega$, such that $\{F_n : n \in \omega\}$ covers X (Rothberger 1938), where Rothberger's property is denoted by C'' (see also Just and Miller 1988; Scheepers 1996); X is Hurewicz if for every sequence $(\mathscr{U}_n : n \in \omega)$ of open covers of X one can select finite $\mathscr{F}_n \subset \mathscr{U}_n$, $n \in \omega$, such that for every $x \in X$, $x \in \bigcup \mathscr{F}_n$ for all but finitely many n (Hurewicz 1927).

The previous definitions motivated the following ones.

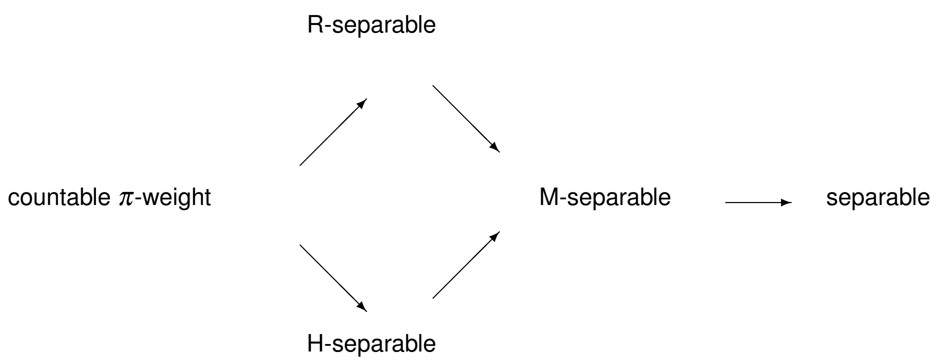
Definition 1.1. A space is

- M-separable (introduced by Scheepers (1999a) using a different terminology; Bella *et al.* (2008) called it selectively separable (see also Bella, Bonanzinga, and Matveev 2009; Bella, Matveev, and Spadaro 2012) if for every sequence $(D_n : n \in \omega)$ of dense subsets of X one can pick finite $F_n \subset D_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} F_n$ is dense in X.

- R-separable (introduced by Scheepers (1999a) using a different terminology) if for every sequence $(D_n : n \in \omega)$ of dense subsets of X one can pick $p_n \in D_n$, $n \in \omega$, such that $(p_n : n \in \omega)$ is dense in X.
- H-separable (Bella, Bonanzinga, and Matveev 2009) if for every sequence $(D_n : n \in \omega)$ of dense subsets of X one can pick finite $F_n \subset D_n$, $n \in \omega$, such that for every nonempty open set $O \subset X$, the intersection $O \cap F_n$ is nonempty for all but finitely many n.

Note that “M-", “R-", and “H-" were motived by analogy with Menger, Rothberger, and Hurewicz properties. Let $\delta(X) = sup\{d(Y) : Y \text{ is dense in } X\}$ (Weston and Shilleto 1976); $\delta(X) = \omega$ for every M-separable space X. If $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{d}$, then X is M-separable (a stronger version of this fact was estabilished by Scheepers (1999a, Theorem 40)); moreover, if $\delta(X) = \omega$ and $\pi w(X) < \text{cov}(\mathscr{M})$, then X is R-separable (a stronger version of this fact was estabilished by Scheepers (1999a, Theorem 29)); Bella, Bonanzinga, and Matveev (2009, Theorem 29) also showed that if $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{b}$, then X is H-separable. As a consequence of these results, it is shown that the existence of a countable M-separable space which is not H-separable is consistent with ZFC (Bella, Bonanzinga, and Matveev 2009).

The following implications are obvious.



For compact spaces, M-, R- and H- separability are equivalent to each other and to having a countable π-base (see Bella *et al.* 2008). Then, in particular, every space having a countable base is M-, R- and H- separable. However, not every space with countable network weight is M-separable: consider any countable not M-separable space (see, for example, Bella *et al.* 2008, Example 2.14).

Hence, it is natural to pose the following question.

Question 1.2. Under what conditions must a space with countable network weight be M-separable?

In this paper we introduce a Menger type property defined by network, called *M-nw-selective property*, such that every M-nw-selective space has countable network weight by definition and is M-separable. By analogy, we also introduce H- and R- nw-selective spaces for the corresponding Hurewicz and Rothberger type properties. Several properties of the new classes of spaces are studied and some questions are posed.

Terminology and preliminaries. If A is a subset of a space X and $\mathscr{B}$ is a family of subsets of X, we say that A refines $\mathscr{B}$ if A is a subset of some element of $\mathscr{B}$; in this case we write $A \prec \mathscr{B}$.

A family $\mathscr{P}$ of open sets is called a π-base for X if every nonempty open set in X contains a nonempty element of $\mathscr{P}$; $\pi w(X) = min\{|\mathscr{P}| : \mathscr{P}$ is a π-base for $X\}$ is the π-weight of X. A family $\mathscr{N}$ of sets is called a network for X if for every $x \in X$ and for every open neighbourhood U of x there exists an element N of $\mathscr{N}$ such that $x \in N \subseteq U$; $nw(X) = min\{|\mathscr{N}| : \mathscr{N}$ is a network for $X\}$ is the network weight of X; $iw(X) = min\{w(Y) : Y$ is the continuous bijective image of $X\}$, where $w(X)$ denotes the weight of the space X, is the injective weight of X. It is known that $iw(X) \leq nw(X) \leq w(X)$, and in the class of compact Hausdorff spaces $iw(X) = nw(X) = w(X)$ (see Bailey 2007).

A space X has countable (strong) fan tightness (see Sakai 1988; Arkhangel'skii 1992) if, for every sequence $(A_n : n \in \omega)$ of subspaces of X and every $x \in \overline{A_n}$ for all $n \in \omega$, one can choose finite $F_n \subset A_n$ (resp., a point $x_n \in A_n$) so that $x \in \overline{\bigcup\{F_n : n \in \omega\}}$ (resp., $x \in \overline{\{x_n : n \in \omega\}}$). X is weakly Fréchet in the strict sense if, for every sequence $(A_n : n \in \omega)$ of subspaces of X and every $x \in \overline{A_n}$ for all $n \in \omega$, there are finite $F_n \subset A_n$ such that every neighborhood of x intersects all but finitely many F_n (Sakai 2006). X is weakly Fréchet in the strict sense with respect to dense subspaces if this statement is true for A_n dense in X, that is for every sequence $(D_n : n \in \omega)$ of dense subspaces of X and every $x \in X$ there are finite $F_n \subset D_n$ such that every neighborhood of x intersects all but finitely many F_n (Bella, Bonanzinga, and Matveev 2009). Recall that for $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means that $f(n) \leq g(n)$ for all $n \in \omega$). A subset $B \subseteq \omega^\omega$ is bounded if there is $g \in \omega^\omega$ such that $f \leq^* g$ for every $f \in B$. $D \subseteq \omega^\omega$ is dominating if for each $g \in \omega^\omega$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of ω^ω is denoted by $\mathfrak{b}$, and the minimal cardinality of a dominating subset of ω^ω is denoted by $\mathfrak{d}$. The value of $\mathfrak{d}$ does not change if one considers the relation $\leq$ instead of $\leq^*$ (van Douwen 1984, Theorem 3.6). $\mathscr{M}$ denotes the family of all meager subsets of $\mathbb{R}$. $\mathrm{cov}(\mathscr{M})$ is the minimum of the cardinalities of subfamilies $\mathscr{U} \subseteq \mathscr{M}$ such that $\bigcup \mathscr{U} = \mathbb{R}$.

2. M- R- H- nw-selective spaces and general properties

Definition 2.1. A space X is

- *M-nw-selective* if $nw(X) = \omega$ and for every sequence $(\mathscr{N}_n : n \in \omega)$ of countable networks for X one can select finite $\mathscr{F}_n \subset \mathscr{N}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathscr{F}_n$ is a network for X.
- *H-nw-selective* if $nw(X) = \omega$ and for every sequence $(\mathscr{N}_n : n \in \omega)$ of countable networks for X one can select finite $\mathscr{F}_n \subset \mathscr{N}_n$, $n \in \omega$, such that for any $x \in X$ and any open neighbourhood U of x, there exists some $\kappa \in \omega$ such that for any $n \geq \kappa$ there exists $A \in \mathscr{F}_n$ with $x \in A \subseteq U$.

- *R-nw-selective* if $nw(X) = \omega$ and for every sequence $(\mathscr{N}_n : n \in \omega)$ of countable networks for X one can pick $F_n \in \mathscr{N}_n$, $n \in \omega$, such that $\{F_n : n \in \omega\}$ is a network for X.

Note that if the networks $\mathscr{N}_n$, $n \in \omega$, in the previous definitions were uncountable then the space must be countable. Indeed, the sequence of networks consisting of all singletons witnesses that the space is not M-nw-selective.

Hurewicz (1925) proved that a basis property formulated by Menger (1924) is equivalent to Menger's property. In particular Hurewicz proved the following proposition (we give the proof for sake of completness). Then, replacing "countable network" with "base" in the definition of M-nw-selectivity, one obtains a property equivalent to the Menger property in the class of metrizable spaces.

Proposition 2.2. (Hurewicz 1925) Let X be a metrizable space. X is Menger iff for every sequence $(\mathscr{B}_n : n \in \omega)$ of bases for X one can select finite $\mathscr{F}_n \subset \mathscr{B}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathscr{F}_n$ is a base for X.

Proof. Let (X,d) be Menger and $\xi = (\mathscr{B}_n : n \in \omega)$ a sequence of bases for X. Re-enumerate ξ as $(\mathscr{B}_{n,m} : n,m \in \omega)$. We may assume that $\mathscr{B}_{n,m}$ consist of sets of diameter $< \dfrac{1}{2^n}$. For each n, pick finite $\mathscr{F}_{n,m} \subset \mathscr{B}_{n,m}$, $m \in \omega$, such that $\bigcup_{m\in\omega} \mathscr{F}_{n,m}$ is a cover of X. Then $\bigcup_{n,m\in\omega} \mathscr{F}_{n,m}$ is a base for X. Indeed, every point is contained in a set of diameter $< \dfrac{1}{2^n}$. Now let $(\mathscr{U}_n : n \in \omega)$ be a sequence of open covers of X. For every $n \in \omega$, put $\mathscr{B}_n = \{U : U$ is an open in X and $U \prec \mathscr{U}_n\}$. Then $(\mathscr{B}_n : n \in \omega)$ is a sequence of bases for X and by hypothesis we conclude the proof. □

Now we prove the following proposition.

Proposition 2.3. If X is countable second countable space, then X is R-nw-selective.

Proof. Of course, $nw(X) = \omega$. Let $X = (x_n : n \in \omega)$, $\mathscr{B} = (B_n : n \in \omega)$ be a base for X and $(\mathscr{N}_n : n \in \omega) = (\mathscr{N}_{n,m} : n,m \in \omega)$ be a sequence of countable networks for X. For each $n,m \in \omega$, if $x_n \in B_m$, then take $A_{n,m} \in \mathscr{N}_{n,m}$ such that $x_n \in A_{n,m} \subset B_m$; if $x_n \notin B_m$, then take any $A_{n,m} \in \mathscr{N}_{n,m}$. Then $(A_{n,m} : n,m \in \omega)$ is a network for X. □

Question 2.4. Are there M-nw-selective spaces which are not R-nw-selective or not H-nw-selective?

Question 2.5. Are there uncountable M-nw-selective spaces?

Since every space with countable netweight is hereditarily Lindelöf, M-nw-selectivity is a strengthening of Lindelöf property. We also prove that M-nw-(resp., R-nw-, H-nw-) selectively is a common strengthening of Menger (resp., Rothberger and Hurewicz) property and M- (resp., R- and H-) separability.

Proposition 2.6. If X is M-nw-selective, then X is Menger.

Proof. Let X be M-nw-selective and $(\mathscr{U}_n : n \in \omega)$ a sequence of open covers of X. Fix a countable network $\mathscr{N}$ for X. For every $n \in \omega$, put $\mathscr{N}_n = \{N \in \mathscr{N} : N$ refines $\mathscr{U}_n\}$. For every $n \in \omega$, $\mathscr{N}_n$ is a countable network for X (in fact, let W be an open subset of X and

$x \in W$. Since $\mathscr{U}_n$ covers X, there exists $V \in \mathscr{U}_n$ such that $x \in V$. Then $V \cap W$ in an open set containing x. Then there exits $N \in \mathscr{N}$ such that $x \in N \subset V \cap W \subset V$. Hence $N \in \mathscr{N}_n$). Then $(\mathscr{N}_n : n \in \omega)$ is a sequence of countable networks for X. By hypothesis, there exist finite $\mathscr{F}_n \subset \mathscr{N}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathscr{F}_n$ is a network for X. For every $N \in \mathscr{F}_n$, pick $U_{N,n} \in \mathscr{U}_n$ such that $N \subset U_{N,n}$ and put $\mathscr{A}_n = \{U_{N,n} : N \in \mathscr{F}_n\}$. Then $\mathscr{A}_n$, $n \in \omega$, is a finite subfamily of $\mathscr{U}_n$ such that $\bigcup_{n\in\omega} \bigcup \mathscr{A}_n = X$. □

Proposition 2.7. If X is R-nw-selective (H-nw-selective), then X is Rothberger (Hurewicz).

Proof. The proof is similar to the proof of Proposition **2.6**. □

The converse of propositions **2.6** and **2.7** is not true as the following example shows.

Example 2.8. A Rothberger and Hurewicz space which is not M-nw-selective (hence not R-nw-, H-nw-selective).

Bella, Bonanzinga, and Matveev (2009, Example 2.14) proved the existence of a countable subspace X of $C_p(\omega^\omega)$ which is not M-separable. By next Proposition **2.12**, the space X is not M-nw-selective. Of course, $nw(X) = \omega$ and X is Rothberger and Hurewicz. △

Proposition 2.9. If X is M-nw-selective, then X has countable fan tightness.

Proof. Let X be M-nw-selective, $\mathscr{M}$ be a countable network for X, $x \in X$ and $(A_n : n \in \omega)$ be a sequence of subsets of X such that $x \in \overline{A_n}$, for every $n \in \omega$. Every space with countable network is hereditarily separable and thus has countable tightness. Then we may assume that the sets A_n are countable. Let $Y = \{x\} \cup \bigcup_{n\in\omega} A_n$. Y is a countable subset of X and by Proposition **2.26**, Y is M-nw-selective. For every $n \in \omega$, put $\mathscr{M}_n = \{\{y\} : y \in Y \setminus \{x\}\} \cup \{\{x,a\} : a \in A_n\}$. Since $x \in \overline{A_n}$ for every $n \in \omega$, $(\mathscr{M}_n : n \in \omega)$ is a sequence of countable networks for Y. Then one can select finite $\mathscr{F}_n \subset \mathscr{M}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathscr{F}_n$ is a network for Y. Put $B_n = \{a \in A_n : \{x,a\} \in \mathscr{F}_n\}$, $n \in \omega$. Then, for every $n \in \omega$, B_n is a finite subset of A_n and $x \in \overline{\bigcup\{B_n : n \in \omega\}}$. □

The converse of the previous result does not hold, as the following example shows.

Example 2.10. A space having countable fan tightness which is not M-nw-selective.

Consider the space $C_p(I)$, where $I = [0,1]$. Since a space $C_p(X)$ is Menger iff X is finite (Arkhangel'skii 1992), by Proposition **2.6**, we have that $C_p(I)$ is not M-nw-selective. Arkhangel'skii (1986, 1992, Theorem 2.2.2) proved that $C_p(X)$ has countable fan tightness iff all finite powers of X are Menger. Then $C_p(I)$ has countable fan tightness.

Recall the following

Proposition 2.11. (Bella *et al.* 2008, Proposition 2.3) Every separable space having countable fan tightness is M-separable.

Then, by Proposition **2.9**, we obtain

Proposition 2.12. If X is M-nw-selective, then X is M-separable.

The converse of the previous proposition is not true: the space $C_p(I)$, where $I = [0,1]$ is M-separable (Bella *et al.* 2008, Example 2.14) and we have proved that it is not M-nw-selective.

Inspired by R-separability, Scheepers (1999a) introduced the following game on a space X: Players ONE and TWO play an inning per $n \in \omega$. In the n-th inning ONE selects a set D_n dense in X, after TWO selects a point $p_n \in D_n$. A play $(D_0, p_0, D_1, p_1, ...)$ is won by TWO if $(p_n : n \in \omega)$ is dense in X; otherwise ONE wins. Denote by G_1 this game. Of course, if ONE has no winning strategy in the game G_1 on X, then X is R-separable. Scheepers (1999a) showed the following result.

Theorem 2.13. (Scheepers 1999a, Theorem 3) The following statements are equivalent:

(1) TWO has a winning strategy in the game G_1 on X.
(2) X has countable π-weight.

By the following example we show, in particular, that a countable M-separable space need not be M-nw-selective.

Example 2.14. A countable space X such that TWO has a winning strategy in the game G_1 on X (hence X is R-separable and then M-separable), which is not M-nw-selective space.

Let us take the product of the usual convergent sequence $\omega + 1$ with the discrete space ω. The quotient space of it obtained by identifying all non isolated points is called Fréchet-Urysohn fan space; it is usually denoted by S_ω. This space is a typical example of a countable space with only one non-isolated point the fan-tightness of which is not countable. Then, by Proposition **2.9**, S_ω is not M-nw-selective. Since, obviously, S_ω has countable π-weight, then TWO has a winning strategy in the game G_1 on S_ω. △

We can prove that

Proposition 2.15. (Scheepers 2014, Lemma 30) Every separable space having countable strong fan tightness is R-separable.

Proposition 2.16. If X is R-nw-selective, then X has countable strong fan tightness.

Proof. The proof is similar to the proof of Proposition **2.9**. □

The converse of the previous result is not true as the following example shows.

Example 2.17. A space having countable strong fan tightness which is not M-nw-selective, hence not R-nw-selective.

Consider the space Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1)$. It is known that a compact space is Rothberger iff it is scattered (see, for a proof, Bonanzinga, Cammaroto, and Matveev 2010, Proposition 34). Since a $C_p(X)$ space has countable strong fan tightness iff all finite powers of X are Rothberger (Sakai 1988), we have that $C_p(T)$ has countable strong fan tightness. However $nw(C_p(T)) > \omega$, hence $C_p(T)$ is not M-nw-selective. Note that, since $C_p(X)$ is R-separable iff $iw(X) = \omega$ and all finite powers of X are Rothberger (Bella, Bonanzinga, and Matveev 2009, Theorem 57), in fact $C_p(T)$ is not R-separable.

By the previous proposition, we have the following result.

Proposition 2.18. If X is R-nw-selective, then X is R-separable.

Example 2.19. A countable space X such that TWO has a winning strategy in the game G_1 on X (hence X is R-separable), which is not R-nw-selective space.

Consider the spaces S_ω (see Example **2.14**). △

Proposition 2.20. If X is H-nw-selective, then X is weakly Fréchet in the strict sense.

Proof. The proof is similar to the proof of Proposition **2.9**. □

The converse of the previous result does not hold, as the following example shows.

Example 2.21. A weakly Fréchet in the strict sense space which is not H-nw-selective.

Consider the space $C_p(I)$. Recall that $C_p(X)$ is weakly Fréchet in the strict sense iff all finite powers of X are Hurewicz (Kočinac and Scheepers 2002), as stated by Sakai (2006). Then $C_p(I)$ is weakly Fréchet in the strict sense but is is not M-nw-selective (cfr. Example **2.10**), hence not H-nw-selective.

Proposition 2.22. (Bella, Bonanzinga, and Matveev 2009, Proposition 35) A separable space is H-separable iff it is weakly Fréchet in the strict sense with respect to dense subspaces.

Corollary 2.23. Every separable weakly Fréchet in the strict sense is H-separable.

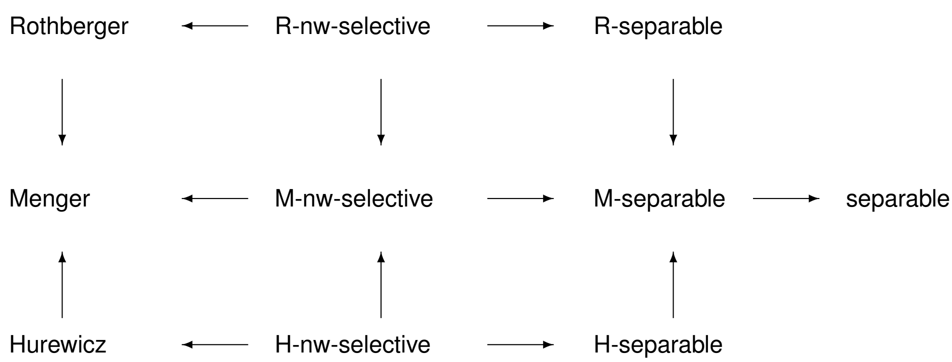
Then, by Proposition **2.20**, we have the following.

Proposition 2.24. If X is H-nw-selective, then X is H-separable.

Example 2.25. A countable H-separable non H-nw-selective space.

Consider the spaces S_ω (see Example **2.14**). We have proved that S_ω is not M-nw-selective and then S_ω is not H-nw-selective. Since it has countable π-weight, it is R-separable. △

The following diagram sums up some implications obtained above.



Recall that M-, R- and H- separability are not preserved by arbitrary subspaces, but they are preserved by open subspaces, and by dense subspaces (see Bella *et al.* 2008, for M-separability).

We prove that M-mw-, R-nw- and H-nw- separability are preserved by arbitrary subspaces.

Proposition 2.26. M-nw-separability is a hereditary property.

Proof. Let X be M-nw-selective and $Y \subset X$. In particular, $nw(X) = \omega$ and then $nw(Y) = \omega$. Let $\mathscr{M}$ be a countable network for X. Then $(M \cap Y : M \in \mathscr{M})$ is a countable network for Y. Let $(\mathscr{N}_n : n \in \omega)$ be a sequence of countable networks for Y. For every $n \in \omega$, put $\mathscr{M}_n = \mathscr{N}_n \cup \{M \setminus Y : M \in \mathscr{M}\}$. Then $(\mathscr{M}_n : n \in \omega)$ is a sequence of countable networks for X. By hypothesis, one can select finite $\mathscr{H}_n \subset \mathscr{M}_n$, $n \in \omega$ such that $\bigcup_{n\in\omega} \mathscr{H}_n$ is a network for X. For every $n \in \omega$, put $\mathscr{F}_n = \mathscr{H}_n \cap \mathscr{N}_n$. Then $\mathscr{F}_n$ is a finite subset of $\mathscr{N}_n$ and $\bigcup_{n\in\omega} \mathscr{F}_n$ is a network for Y. □

Proposition 2.27. R-nw-selective and H-nw-selective are hereditary property.

Proof. The proof is similar to the proof of Proposition **2.26**. □

Recall that a space is “analytic” if it is a continuous image of the spaces of irrationals (Kechris 1995).

Proposition 2.28. (Arkhangel’skii 1986) Every analytic Menger space is σ-compact.

Now we prove the following.

Proposition 2.29. Every analytic subset of a M-nw-selective space is countable.

Proof. By contradiction, assume there exists a M-nw-selective space having an uncountable analytic subspace Y. By Proposition **2.26**, Y is M-nw-selective and then, by Proposition **2.12**, it is Menger. So, by Proposition **2.28**, Y is σ-compact and then Y contains an uncountable compact space H. By hypothesis and compactness of H, we have that $w(H) = nw(H) = \omega$. Then, by Aleksandroff-Urysohn metrization’s theorem, H is metrizable. Hence, since any uncountable compact metrizable space contains a copy of the space of irrationals, we have that Y contains a copy of the space of irrationals. Since irrationals are not Menger, hence by Proposition **2.12**, not M-nw-selective, we conclude that Y is not M-nw-selective; a contradiction. □

Corollary 2.30. Every analytic subset of an R-nw- or H-nw- selective space is countable.

Proof. The proof is similar to the proof of Proposition **2.29** using respectively Proposition **2.18** and Proposition **2.24** instead of Proposition **2.12**, and Proposition **2.27** instead of Proposition **2.26**. □

Recall the following result.

Theorem 2.31. (Arkhangel’skii 1992, Proposition II.2.11) If X is a compact space of countable weight, then $C_p(X)$ is an analytic space.

Corollary 2.32. *If X is a compact space of countable weight, then $C_p(X)$ is not M-nw-selective.*

Using the previous result we can say that, for example $C_p(2^\omega)$ and $C_p(I)$ are not M-nw-selective. Recall that $C_p(2^\omega)$ is H-separable.

3. Operations

It is well-known that Menger, Rothberger and Hurewicz properties are preserved by countable unions. Gruenhage and Sakai (2011) proved that M- and R- separability is preserved by finite unions; it is an open question if H-separability is preserved by finite unions.

We will prove the following results.

Theorem 3.1. Let $X = \bigoplus_{n\in\omega} X_n$, where $\bigoplus$ denotes the direct sum. If X_n is a M-nw-selective space for every $n \in \omega$, then X is a M-nw-selective space.

Proof. Of course, countable network is preserved by countable direct sums. Let $(\mathscr{N}_k : k \in \omega)$ be a sequence of countable networks for X. For every $n \in \omega$ consider $\{N \cap X_n : N \in \mathscr{N}_k \text{ and } k \geq n\}$ that is a sequence of countable networks for X_n. Since X_n is M-nw-selective for every $n \in \omega$, there exists $(\mathscr{F}_{n,k} : k \geq n)$ with $\mathscr{F}_{n,k}$ a finite subfamily of $\mathscr{N}_k$ for every $k \geq n$ such that $\bigcup_{k\geq n}\{F \cap X_n : F \in \mathscr{F}_{n,k}\}$ is a network for X_n. We put $\mathscr{F}_k = \bigcup\{\mathscr{F}_{n,k} : k \geq n\}$ that is a finite subfamily of $\mathscr{N}_k$ for every $k \in \omega$. We can easily see that $\bigcup_{k\in\omega} \mathscr{F}_k$ is a network for X. This means X is M-nw-selective. □

Theorem 3.2. Let $X = \bigoplus_{n\in\omega} X_n$. If X_n is a R-nw-selective space for every $n \in \omega$, then X is a R-nw-selective space.

Proof. Of course, countable network is preserved by countable direct sums. Let $(\mathscr{N}_k : k \in \omega)$ be a countable sequence of networks for X. Divide the sequence of networks into countably many pairwise disjoint sequences of countable networks $(\mathscr{M}_{k,n} : k,n \in \omega)$. For every $n \in \omega$, $(\mathscr{M}_{k,n} : k \in \omega)$ is a sequence of countable networks of X_n. Since X_n, $n \in \omega$ is R-nw-selective, there exist $F_{k,n} \in \mathscr{M}_{k,n}$, $n \in \omega$, such that $\{F_{k,n} : n \in \omega\}$ is a network for X_n. Then $\{F_{k,n} : k,n \in \omega\}$ is a network for X. □

Question 3.3. Is the countable (or finite) direct sum of H-nw-selective spaces H-nw-selective?

Scheepers (1999b) proved that Menger property is not finitely productive. Gruenhage and Sakai (2011) proved that under CH there is a countable regular maximal space X which is R-separable but X^2 is not M-separable.

Question 3.4. Is the product of two M-nw-selective spaces M-nw-selective? (or, at least, Menger or M-separable?)

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