DOI: 10.1478/AAPP.1021A2

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 102, No. 1, A2 (2024)

ON THE ELECTROMAGNETIC ENERGIES AND FORCES

Tamás Matolcsi *

(communicated by Peter Ván)

Abstract. The problem of the "infinite electric energy" of a point charge is well known in connection with the Lorentz–Abraham–Dirac equation and, more significantly, in quantum electrodynamics. Though it is not stated usually, this is strongly related to the old problem of what the electromagnetic energy-momentum tensor is. Electromagnetic energies and forces will be examined thoroughly from a new point of view in the framework of Distribution Theory.

Introduction

The question of what the electromagnetic energy and electromagnetic energy flow are is as old as the theory of electromagnetism. First, the original definition of the electromagnetic energy flow, the Poynting vector, seems to be problematic (see, *e.g.*, Bergmann 1962; Lombardi 1983; Jiménez, Campos, and López-Mariño 2011). Then the Abraham-Minkowksi controversy regarding the energy-momentum tensor is an open problem still (see Leonhardt 2006, and references therein). Similarly, the question of what the electromagnetic force and the electromagnetic interaction are is an open problem. There are a number of attempts to reinterpret or replace the LAD equation (*e.g.*, Spohn 2000; Rohrlich 2007; Gralla, Harte, and Wald 2009; S. Raju and C. K. Raju 2011; Matolcsi, Fülöp, and Weiner 2017; Bild, Deckert, and Ruhl 2019). Matolcsi (2023) clarified the radiation reaction force and its role in the LAD equation by showing that not a mathematically incorrect derivation of the radiation reaction force but a physical misapprehension is the source of why the LAD equation does not work well. The role of electromagnetic energies and forces in connection with continuous media and thermodynamics is an essential question from a practical, technical point of view, too (see, *e.g.*, Maugin 1988; Ván 1998; Barnett 2010; Wang 2017). Therefore, it is important to see clearly what is right and what is not right in the usual approaches,

The present paper deals with electromagnetic energies and forces in general, it tries to reveal the roots of the problems, opening so a new perspective which can help to find a solution (if it exists at all).

First of all, note that neither continuous mass distributions nor masspoints exist in nature, they are mathematical models which work well in mechanics (the Earth is not a point, nevertheless, its motion around the Sun is well modelled in point mechanics). Similarly, neither continuous charge distributions nor point charges exist in nature, they would be mathematical models. The problem is that the elementary notions of energy and force in electromagnetism are defined by point charges but the conceptually simple notion of a point charge cannot be well modelled in every respect in the framework of usual functions. The attempts of limiting procedures from continuous charge distributions to point charges do not work. Since Dirac, some applications of Distribution Theory appeared in electromagnetism (*e.g.*, Taylor 1958; Rowe 1978; Gralla, Harte, and Wald 2009). The present paper is based upon a systematic and over all use of Distribution Theory.

The word "distribution" will appear in two different senses: 1. having its everyday meaning, 2. being a mathematical notion. To distinguish between them, I write distribution for the first one and Distribution for the second one. I use the coordinate-free formulation of spacetime expounded by Matolcsi (2020), which makes formulas shorter and more easily comprehensible. A brief summary of the fundamental notions and some special notations was given by Matolcsi (2023).

The usual setting of Distribution Theory is based on $\mathbb{R}^n$ (Gel'fand and Shilov 1964; Demidov 2001; Horváth 2012; van Dijk 2013). It is a quite simple generalization that spacetime and an observer space are taken instead of $\mathbb{R}^4$ and $\mathbb{R}^3$. Another simple generalization is that vector and tensor Distributions are included, too. All the electromagnetic quantities are considered to be Distributions, denoted by calligraphic letters, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{B}}$, *etc.*, and the guiding principle is that only formulae definable in Distribution Theory can make sense. Nevertheless, the present article can be understood without a thorough knowledge of Distributions; besides the elementary notions the only non-trivial one is **pole-taming**, described in the Appendix.

1. Basic notions of electrostatics

The usual and well-known formulae of electrostatics (de Groot and Suttorp 1972; Jackson 1998) will be reviewed from the point of view of Distributions in such a manner that $c = 1$, $\hbar = 1$ and the electric charge is measured by real numbers.

Statics: all the quantities are time independent in a **uniquely determined** $\boldsymbol{u}_0$ standard inertial frame; the space of that frame, denoted by S_0, is a three dimensional affine space over the Euclidean vector space $\mathbf{S}_0$. A static electric field is supposed to be a vector Distribution $\boldsymbol{\mathcal{E}}$ in S_0; its regular part is called field function and is denoted by $\boldsymbol{E}$; for the sake of simplicity, without danger of confusion, we often say field instead of field function, too. A static charge distribution is a locally finite signed measure $\mathfrak{e}$ (the quantity of charges is finite in every compact subset), a static dipole distribution is a locally finite vector measure $\boldsymbol{\mathcal{P}}$, on S_0. As Distributions, they act on test functions ψ by integration, *e.g.*

$$(\mathfrak{e} \mid \psi) := \int\limits_{\mathrm{S}_0} \psi(q)\ d\mathfrak{e}(q). \tag{1}$$

The electric field $\boldsymbol{\mathcal{E}}$ produced by a **given** static charge distribution $\mathfrak{e}$ and a **given** static dipole distribution $\boldsymbol{\mathcal{P}}$ satisfy the Maxwell equations

$$\nabla \cdot \boldsymbol{\mathcal{E}} = \mathfrak{e} - \nabla \cdot \boldsymbol{\mathcal{P}}, \qquad \nabla \times \boldsymbol{\mathcal{E}} = 0. \tag{2}$$

Special cases are when the charge distribution has a continuous density ρ, the dipole distribution has a continuously differentiable density $\boldsymbol{P}$, and the electric field is the regular Distribution corresponding to a continuously differentiable function $\boldsymbol{E}$; then

$$\nabla \cdot \boldsymbol{E} = \rho - \nabla \cdot \boldsymbol{P}, \qquad \nabla \times \boldsymbol{E} = 0. \tag{3}$$

holds as well.

It follows from Poincaré's lemma in Distribution Theory that there is a potential, a scalar Distribution $\mathcal{V}$ such that $\boldsymbol{\mathcal{E}} = -\nabla\mathcal{V}$; its regular part, denoted by V, is the potential function. Introducing

$$\boldsymbol{\mathcal{D}} := \boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{P}}, \qquad (\boldsymbol{D} := \boldsymbol{E} + \boldsymbol{P}), \tag{4}$$

the first Maxwell equation can be transformed into the most frequently used form

$$\nabla \cdot \boldsymbol{\mathcal{D}} = \mathfrak{e}, \qquad (\nabla \cdot \boldsymbol{D} = \rho). \tag{5}$$

As concerns a dipole distribution, it can exist independently of the electric field (electret) or can depend on the electric field in a medium (polarization). Since electrets – contrary to magnets – are not familiar in every-day life, mostly polarizations are taken in usual treatments, and a relation is given between $\boldsymbol{P}$ and $\boldsymbol{E}$, *e.g.* $\boldsymbol{P} = \kappa\boldsymbol{E}$ involving $\boldsymbol{D} = \epsilon\boldsymbol{E}$. By this unfortunate manipulation, one hides the dipole distribution and considers $\boldsymbol{D}$ a proper field quantity though it is not: it includes material quantities whether an independent or a polarized dipole distribution is in question. It is worth repeating: *$\boldsymbol{\mathcal{D}}$ – and $\boldsymbol{D}$ – are not proper field quantities.*

The charge distribution of a point charge e in the space point q_o is $e\delta_{q_o}$. The produced electric field function

$$\boldsymbol{E}(q) := \frac{1}{4\pi}\frac{(q-q_o)e}{|q-q_o|^3} \qquad (q \neq q_o) \tag{6}$$

has the potential function

$$V(q) := \frac{1}{4\pi}\frac{e}{q-q_o} \qquad (q \neq q_o); \tag{7}$$

both E and V are locally integrable, so the potential $\mathcal{V}$ and the electric field $\boldsymbol{\mathcal{E}}$ are the corresponding regular Distributions. The electric field produced by a **given** charge distribution $\mathfrak{e}$ is

$$\boldsymbol{E}(q) := \frac{1}{4\pi}\int\limits_{\mathrm{S}_0} \frac{(q-q')\ d\mathfrak{e}(q')}{|q-q'|^3}, \tag{8}$$

provided the integral exists for almost all q and the function $\boldsymbol{E}$ is locally integrable. Similar formula holds for the potential, too, according to the sense. The dipole distribution of a point dipole $\boldsymbol{p}$ in the space point q_o is $\boldsymbol{p}\delta_{q_o}$. The produced electric field function (Jackson 1998, p. 101)

$$\boldsymbol{E}(q) := \frac{1}{4\pi}\frac{3(q-q_o)(q-q_o)\cdot\boldsymbol{p}}{|q-q_o|^5} - \frac{\boldsymbol{p}}{|q-q_o|^3} \qquad (q \neq q_o) \tag{9}$$

has the potential function

$$V(q) = \frac{1}{4\pi}\frac{(q-q_o)\cdot\boldsymbol{p}}{|q-q_o|^3} \qquad (q \neq q_o); \tag{10}$$

V is locally integrable, so the potential $\mathcal{V}$ is the corresponding regular Distribution. On the contrary, $\boldsymbol{E}$ is not locally integrable, the electric field is not a regular Distribution. $\boldsymbol{E}$ has a pole at q_o which can be tamed, and it can be proved (Matolcsi 2021) that the electric field is

$$\boldsymbol{\mathcal{E}} = \mathrm{tm}\boldsymbol{E} + \frac{\boldsymbol{p}}{3}\delta_{q_o} \tag{11}$$

whose regular part is $\boldsymbol{E}$. As a consequence, the electric field $\boldsymbol{\mathcal{E}}$ produced by a **given** dipole distribution $\boldsymbol{\mathcal{P}}$ is not so straightforward.

The charges (dipoles) in a charge (dipole) distribution interact, i.e. attract and/or repulse each other; in order that the distribution be static, some non-electric force (and torque) must compensate those electric forces; we have referred to the non-electric forces by emphasizing that the distributions in question are **given**.

In some papers the total field is separated into an external field and a self-field (see, *e.g.*, Bergmann 1962; Jiménez, Campos, and López-Mariño 2011) which seems a useful tool for investigations. The field which is external to a charge, however, is a self-field for some other charge.

We introduce the following definitions. Two charge distributions are called **extraneous** to each other if their supports do not intersect. Accordingly, an electric field $\boldsymbol{\mathcal{E}}$ and a charge distribution $\mathfrak{e}$ are **extraneous** to each other if $\boldsymbol{\mathcal{E}}$ is produced by a charge distribution extraneous to $\mathfrak{e}$. A similar notion of being extraneous to each other is defined for a charge distribution and a dipole distribution, for two dipole distributions, for an electric field and a dipole distribution. The **self-field** of a charge distribution $\mathfrak{e}$ or a dipole distribution $\boldsymbol{\mathcal{P}}$ is the electric field produced by $\mathfrak{e}$ or $\boldsymbol{\mathcal{P}}$.

Now let us consider two charge distributions $\mathfrak{e}_1$ and $\mathfrak{e}_2$, extraneous to each other. Their self-fields $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$ satisfy $\nabla\cdot\boldsymbol{\mathcal{E}}_1 = \mathfrak{e}_1$ and $\nabla\cdot\boldsymbol{\mathcal{E}}_2 = \mathfrak{e}_2$; $\boldsymbol{\mathcal{E}}_1$ is extraneous to $\mathfrak{e}_2$ and $\boldsymbol{\mathcal{E}}_2$ is extraneous to $\mathfrak{e}_1$. Further, $\nabla\cdot(\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_2) = \mathfrak{e}_1 + \mathfrak{e}_2$; thus, $\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_2$ is the self-field of $\mathfrak{e}_1 + \mathfrak{e}_2$.

2. About energies in electrostatics

2.1. Extraneous energies. Analogously to the situation in a Newtonian gravitational field, it is commonly accepted that the work we do when transporting a point charge e from a space point q_1 to the space point q_2 on any path in the static electric field, **extraneous** to the charge, and having the potential function V, is the path integral of $e\boldsymbol{E}$ which results in $e\big(V(q_2)-V(q_1)\big)$ (of course, provided that the electric field is defined on the path). Such a definition, however, is somewhat incorrect. If the charge moves then the situation is not static. It is an empirical fact that an accelerated charge radiates and reacts on itself, and, contrary to the Newtonian gravitational case, the work done is modified by this reaction. Even if the path is a straight line then, at least when beginning and ending the motion, the charge must be accelerated, and more acceleration occurs necessarily if the path is not a straight line. Nevertheless, we can accept this formula as an ideal one because the less the acceleration, the less the radiation. If V tends to zero at infinity, then

$$eV(q_o), \tag{12}$$

provided that V is defined at q_o, is the work done when transporting the point charge e from infinity to the point q_o. This is considered the electric energy of the point charge e in the **extraneous** electric potential function V. From that, it is quite obvious that

$$\mathfrak{e}V \tag{13}$$

can be accepted as the static electric energy distribution of a **given** charge distribution $\mathfrak{e}$ in an **extraneous** static electric potential function V, provided that $\mathfrak{e}V$ is locally finite; even, it is expected to be finite in order that the total energy be finite. If the charge has a continuous density then

$$\rho V \tag{14}$$

can be accepted as the static electric energy density of the **given** charge density ρ in an **extraneous** static electric potential function V, provided that ρV is locally integrable; even, it is expected to be integrable in order that the total energy be finite. It is commonly accepted, too, (Jackson 1998, p. 101) that the static electric energy of a point dipole $\boldsymbol{p}$ in the space point q_o in the **extraneous** electric field $\boldsymbol{E}$ is

$$-\boldsymbol{p}\cdot\boldsymbol{E}(q_o), \tag{15}$$

provided that $\boldsymbol{E}$ is defined at q_o and

$$-\boldsymbol{P}\cdot\boldsymbol{E} \tag{16}$$

can be accepted as the static electric energy density of a **given** dipole density $\boldsymbol{P}$ in an **extraneous** static electric field function $\boldsymbol{E}$, provided that $\boldsymbol{P}\cdot\boldsymbol{E}$ is locally integrable (and even integrable).

The above notions of energy in an extraneous field is a consequence that only an **action** *is taken into account instead of an interaction: a field acts on charges and dipoles. The field, however, is produced by some other charges and/or dipoles, and charges and dipoles* **interact** *through fields.*

2.2. Self-energies, charges. The charges inside a static charge distribution **interact** with each other which results in an electric **self-energy** of the distribution in question. The usual way to get this self-energy runs as follows (see Jackson 1998). Let the point charges $e_1,\dots,e_n$ rest at the different space points $q_1,\dots,q_n$. Their self-energy is defined as the work done when all the charges are transported from infinity to their places.

First step. No work is done for the first charge because it is carried in an empty space; no action, no interaction, the electric self-energy of a single point charge is zero.

Second step. The charge e_2 is transported in the electric field of e_1. The work done is (see (7) and (12)) $e_2\frac{e_1}{4\pi|q_2-q_1|} = \frac{e_1e_2}{4\pi|q_2-q_1|} = \frac{1}{2}\big(e_1V_2(q_1)+e_2V_1(q_2)\big)$. And so on, the total work done, *i.e.* the self-energy is

$$\frac{1}{2}\sum_{k=1}^{n} e_k \left(\sum_{k\neq i=1}^{n} V_i(q_k)\right). \tag{17}$$

Comment. For later convenience, we make now a little detour from the usual way. The charge distribution in question is

$$\mathfrak{e} := \sum_{k=1}^{n} e_k\delta_{q_k} \tag{18}$$

and instead of the total self-energy (17) we consider the self-energy distribution

$$\frac{1}{2}\sum_{k=1}^{n} e_k \left(\sum_{k\neq i=1}^{n} V_i\delta_{q_k}\right). \tag{19}$$

According to the first step, the self energy of a single point charge is zero, thus, we can define without ambiguity the originally senseless $e_kV_k\delta q_k$ "self-energy" to be zero. Then adding zero to the above sum, we get

$$\frac{1}{2}\sum_{k=1}^{n} e_k \left(\sum_{k\neq i=1}^{n} V_i\delta_{q_k}\right) + \frac{1}{2}\sum_{k=1}^{n} e_kV_k\delta_{q_k} = \frac{1}{2}\sum_{k=1}^{n} e_kV\delta_{q_k} = \frac{1}{2}\mathfrak{e}V \tag{20}$$

where V is the potential produced by all the charges.

Third step. Putting together more and more particles with smaller and smaller charges, one arrives at a continuous charge distribution, thus,

$$\frac{1}{2}\rho V \tag{21}$$

is accepted as the self-energy density of a static charge density ρ in its own potential function V.

Comments.

(1) Though the transition from point charges to a continuous charge distribution has not a mathematically exact sense, we can accept without a serious objection that (21) is *the self-energy density of a* **given** *static charge density ρ in its own potential function V*, provided that ρV is locally integrable, *i.e.* it defines a regular Distribution; even, it is expected to be integrable in order that the total self-energy be finite.

(2) It is worth examining the relation between extraneous energy and self-energy. Let us take two charge densities ρ_1 and ρ_2, extraneous to each other,

producing the potential functions V_1 and V_2, respectively. Then the charge distributions together have the self-energy[1]

$$\frac{1}{2}\int\limits_{\mathsf{S}_0}(\rho_1+\rho_2)(V_1+V_2) = \frac{1}{2}\int\limits_{\mathsf{S}_0}\rho_1 V_1 + \frac{1}{2}\int \rho_1 V_2 + \frac{1}{2}\int\limits_{\mathsf{S}_0}\rho_2 V_1 + \frac{1}{2}\int\limits_{\mathsf{S}_0}\rho_2 V_2. \tag{22}$$

The first and the last term are the corresponding self-energies. The other two terms give the energy of interaction; note that

$$\int\limits_{\mathsf{S}_0}\rho_1 V_2 = \int\limits_{\mathsf{S}_0}\rho_2 V_1 = \frac{1}{4\pi}\int\limits_{\mathsf{S}_0}\int\limits_{\mathsf{S}_0}\frac{\rho_1(q_1)\rho_2(q_2)}{|q_1-q_2|}\ dq_1\ dq_2. \tag{23}$$

Fourth step. From the Maxwell equation $\nabla\cdot\boldsymbol{E} = \rho$ and $\boldsymbol{E} = -\nabla V$, the self-energy density can be written in the form

$$\frac{1}{2}\rho V = \frac{1}{2}\big(\nabla\cdot(\boldsymbol{E}V) + |\boldsymbol{E}|^2\big). \tag{24}$$

Applying Gauss' theorem for a ball around an arbitrary space point, one obtains that the integral of $\nabla\cdot(\boldsymbol{E}V)$ is zero, thus

$$\frac{1}{2}\int\limits_{\mathsf{S}_0}\rho V = \frac{1}{2}\int\limits_{\mathsf{S}_0}|\boldsymbol{E}|^2. \tag{25}$$

Fifth step. From (25) one concludes that $\frac{1}{2}|\boldsymbol{E}|^2$ is the electric energy density of the electric field $\boldsymbol{E}$ (Jackson 1998, p. 21).

Comments.

(1) Equality (25) is all right, provided that the self-energy density is integrable and V and $\boldsymbol{E}$ are sufficiently smooth and tend to zero at infinity in a convenient order (which is satisfied, *e.g.*, if ρ has compact support).

(2) The integral of $\frac{1}{2}|\boldsymbol{E}|^2$ equals the integral of $\frac{1}{2}\rho V$, but the functions themselves are different; in particular, ρV is zero where ρ is zero but $|\boldsymbol{E}|^2$ is not zero there. Therefore, *it is unjustified* to consider $\frac{1}{2}|\boldsymbol{E}|^2$, instead of $\frac{1}{2}\rho V$, the electric self-energy density |it of the charge distribution ρ.

(3) *It is more unjustified* that $\frac{1}{2}|\boldsymbol{E}|^2$, instead of the self-energy density of the charge distribution, is considered the energy density *of the electric field*; this mis-switching is due to the fact that $\frac{1}{2}|\boldsymbol{E}|^2$ refers to the electric field only (does not refer to any charges).

(4) For an electric field $\boldsymbol{\mathcal{E}}$, the product $|\boldsymbol{\mathcal{E}}|^2 = \boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}$, as the product of two Distributions, makes no sense, in general. For a field function $\boldsymbol{E}$ the following cases can occur:

A. $\frac{1}{2}|\boldsymbol{E}|^2$ is integrable; then it defines a regular Distribution. Nevertheless, it cannot be considered the self-energy density of a charge distribution; *it has the only physical meaning* that its integral over all the space – *i.e.*, as a regular Distribution, applied to the constant function 1 – equals

[1]We use a simplified notation: when integrating by the Lebesgue measure λ_{S_0} (volume form) of S_0, the measure is not written in the integrals.

the total electrostatic self-energy of the charge distribution producing $\boldsymbol{E}$.

B. $\frac{1}{2}|\boldsymbol{E}|^2$ is not integrable but is locally integrable; then it defines a regular Distribution. Nevertheless, it cannot be considered a (self-)energy density and *it is questionable* whether it has a physical meaning.

C. $\frac{1}{2}|\boldsymbol{E}|^2$ is not locally integrable; then it does not define a Distribution without further ado.

Sixth step. The electric self-energy of a single point charge is infinite because the "integral of the energy density" $\frac{1}{2}|\boldsymbol{E}|^2$ of a point charge is infinite.

Comment. *It is amazing and shocking that from the first step "the electric self-energy of a single point charge is zero", one arrives at the conclusion "the electric self-energy of a single point charge is infinite".*

Infinite energy is a result of a three times incorrect argumentation:

- $\frac{1}{2}\rho V$ makes no sense for a point charge,
- Gauss' theorem for $\nabla \cdot (\boldsymbol{E}V)$ cannot be applied in the case of a point charge because of the singularity,
- $\frac{1}{2}|\boldsymbol{E}|^2$, even if it is integrable, is not the energy density even for a continuous charge density,

2.3. Point charge, self-energy. Infinite electric energy is a nonsense; there are two usual ways to eliminate it, based on the mass-energy equivalence.

(1) The actual finite mass of a charged point particle is the sum of a negative infinite mechanical mass and the positive infinite electric mass (energy).

(2) A charged particle is not point-like; then considering it a continuously charged ball, its classical radius is determined from its known electric energy (mass).

Then we can put the question: all these do not apply for a neutral particle; what about its mass and radius? Now recall what has been said in the Introduction: point charges, as well as continuous charge distributions, are models. Considering a material object point-like, we do not assert that it is a point (the Earth is not a point because its orbit around the Sun is calculated in this way). Let me emphasize: the nonsense of infinite electric energy of a point charge is the result of a three times incorrect reasoning; *we have to get rid of this reasoning itself* by some correct mathematical formulae. According to the first step, *it can be stated unmistakably that the self-energy of a point charge is zero.*

The same result can be derived in a correct way by taming (see Section **??**) the fictitious "self-energy density"

$$\frac{1}{2}|\boldsymbol{E}(q)|^2 = \frac{1}{2}\frac{1}{16\pi^2}\frac{e^2}{|q-q_o|^4} \qquad (q \neq q_o) \tag{26}$$

of a point charge e resting in the space point q_o which has a pole at q_o and is not locally integrable. Introducing $\boldsymbol{q} := q - q_o$, we have

$$(\mathrm{tm}|\boldsymbol{E}|^2 \mid \psi) = \int\limits_{\mathbf{S}_0} \boldsymbol{E}(q_o + \boldsymbol{q})|^2(\psi(\boldsymbol{q}) - \psi(\mathbf{0}))\ d\boldsymbol{q}\ !! \tag{27}$$

for all test functions ψ. The quantity $\frac{1}{2}\mathrm{tm}|\boldsymbol{E}|^2$ is a mathematical object; it is not a measure, and even if it were, it could not be considered the self-energy distribution (see cases A. and B.). Its only physical meaning could be (see case A.) that its "integral" makes sense, *i.e.* it can be applied to the constant function 1 and then the result gives the self-energy of the point charge.

Proposition 2.1.

$$\left(\frac{1}{2}\mathrm{tm}|\boldsymbol{E}|^2 \,\middle|\, 1\right) = 0. \tag{28}$$

Proof. For a given $0 < a$ let us take the series $\omega_n := \omega_{na,(n+1)a}$ defined by (118). Then we have

$$\big(\mathrm{tm}|\boldsymbol{E}|^2 \mid \omega_n\big) = \int\limits_{\mathbf{S}} |\boldsymbol{E}(q_o + \boldsymbol{q})|^2\big(\omega_n(\boldsymbol{q}) - \omega_n(0)\big)\ d\boldsymbol{q}\ !! =$$

$$= 4\pi\frac{e^2}{16\pi^2}\left(\int\limits_{na}^{(n+1)a} \frac{\omega_n(r) - 1}{r^2}\ dr - \int\limits_{(n+1)a}^{\infty} \frac{1}{r^2}\ dr\right).$$

Since $|\omega_n - 1| \le 1$, the absolute value of the first integral above is majorized by the integral of $\frac{1}{r^2}$ which is $\frac{1}{na} - \frac{1}{(n+1)a}$; thus, $\lim\limits_{n\to\infty}\big(\mathrm{tm}|\boldsymbol{E}|^2 \mid \omega_n\big) = 0$. ■

As a consequence, not without any reservations, $\frac{1}{2}\mathrm{tm}|\boldsymbol{E}|^2$ can be considered the *Distribution of self-energy* (which is not a self-energy distribution!) of a point charge, keeping in mind its only physical meaning: *the self-energy is zero.*

Remark. In the not too simple theory of Colombeau, $\frac{1}{2}|\boldsymbol{\mathcal{E}}|^2$ as a product of Distributions can be defined, and it is obtained (see Gsponer 2008) that the energy is infinite but is located at the position of the point charge rather than spread over the whole space; this corresponds to the view of "infinite electric mass" which we want to eliminate.

2.4. Self-energies, charges and dipoles. For point dipoles, we can copy the formulae obtained for charges. Let the point dipoles $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$ rest at the different space points $q_1, \ldots, q_n$. The self-energy distribution of the dipole distribution

$$\boldsymbol{\mathcal{P}} := \sum_{k=1}^{n} \boldsymbol{p}_k \delta_{q_k} \tag{29}$$

is

$$-\frac{1}{2}\sum_{k=1}^{n} \boldsymbol{p}_k \cdot \left(\sum_{k\neq i=1}^{n} \boldsymbol{E}_i \delta_{q_k}\right). \tag{30}$$

The self energy of a single point dipole is zero, thus, we can define without ambiguity the originally senseless $\boldsymbol{p}_k \cdot \boldsymbol{E}_k \delta_{q_k}$ "self-energy" to be zero. Then adding zero to the above sum we get

$$-\frac{1}{2}\sum_{k=1}^{n} \boldsymbol{p}_k \cdot \left(\sum_{k\neq i=1}^{n} \boldsymbol{E}_i \delta_{q_k} \right) - \frac{1}{2}\sum_{k=1}^{n} \boldsymbol{p}_k \cdot \boldsymbol{E}_k \delta_{q_k} = -\frac{1}{2}\sum_{k=1}^{n} \boldsymbol{p}_k \cdot \boldsymbol{E} \delta_{q_k} = -\frac{1}{2}\boldsymbol{\mathcal{P}} \cdot \boldsymbol{E} \quad (31)$$

where $\boldsymbol{E}$ is the electric field produced by all the dipoles. Putting together more and more, smaller and smaller dipoles, we arrive – not in a mathematically exact sense – at a continuous dipole distribution, thus, without a serious objection,

$$-\frac{1}{2}\boldsymbol{P} \cdot \boldsymbol{E} \quad (32)$$

can be accepted as the self-energy density of a **given** *static dipole density* $\boldsymbol{P}$ *in its own field* $\boldsymbol{E}$, provided that $\boldsymbol{P} \cdot \boldsymbol{E}$ is locally integrable (and even integrable).

Let us now consider a system consisting of point charges and point dipoles described by the distribution

$$\sum_{k=1}^{m} e_k \delta_{q_k} + \sum_{k=m+1}^{n} \boldsymbol{p}_k \delta_{q_k}. \quad (33)$$

The energy of the charge e_i in the potential V_j of the dipole $\boldsymbol{p}_j$ is $e_i V_j(q_i) = -\frac{1}{4\pi}\frac{e_i(q_i-q_j)\cdot \boldsymbol{p}_j}{|q_i-q_j|^3}$; the energy of the dipole $\boldsymbol{p}_j$ in the electric field $\boldsymbol{E}_i$ of the charge e_i is $-\boldsymbol{p}_j \cdot \boldsymbol{E}_i(q_j) = -\frac{1}{4\pi}\frac{e_i(q_i-q_j)\cdot \boldsymbol{p}_j}{|q_i-q_j|^3}$; thus. $e_i V_j(q_i) = -\boldsymbol{p}_j \cdot \boldsymbol{E}_i(q_j)$. Then the self-energy distribution of this system of charges and dipoles is

$$\frac{1}{2}\sum_{k=1}^{m} e_k \left(\sum_{k\neq i=1}^{m} V_i + \sum_{j=m+1}^{n} V_j \right) \delta_{q_k} - \\ -\frac{1}{2}\sum_{j=m+1}^{n} \boldsymbol{p}_j \cdot \left(\sum_{i=1}^{m} \boldsymbol{E}_i + \sum_{j\neq l=m+1}^{n} \boldsymbol{E}_l \right) \delta_{q_j}. \quad (34)$$

As previously, adding to it formal expressions defined to be zero and going over to continuous distributions, *it can be accepted that*

$$\frac{1}{2}(\rho V - \boldsymbol{P} \cdot \boldsymbol{E}) \quad (35)$$

is the self-energy density of the **given** *static charge density* ρ *and the* **given** *static dipole density* $\boldsymbol{P}$ *together in the potential and field produced by them*, provided that it is locally integrable (and even integrable).

Now two manipulations, similar to the one resulting in (24), can be made. From the Maxwell equation $\nabla \cdot \boldsymbol{E} = \rho - \nabla \cdot \boldsymbol{P}$ and $\boldsymbol{E} = -\nabla V$, one has

(1) $\rho V = \nabla \cdot (\boldsymbol{E}+\boldsymbol{P})V = \nabla \cdot ((\boldsymbol{E}+\boldsymbol{P})V) + (\boldsymbol{E}+\boldsymbol{P}) \cdot \boldsymbol{E}$; the integral of the divergence (with convenient integrability and differentiability conditions) over

all the space will be zero, thus

$$\frac{1}{2}\int\limits_{S_0}(\rho V - \boldsymbol{P}\cdot\boldsymbol{E}) = \frac{1}{2}\int\limits_{S_0}|\boldsymbol{E}|^2. \tag{36}$$

(2) $-\boldsymbol{P}\cdot\boldsymbol{E} = \boldsymbol{P}\cdot\nabla V = \nabla\cdot(\boldsymbol{P}V) - (\nabla\cdot\boldsymbol{P})V = \nabla\cdot(\boldsymbol{P}V) + (\nabla\cdot\boldsymbol{E} - \rho)V = \nabla\cdot((\boldsymbol{P}+\boldsymbol{E})V) - \rho V + |\boldsymbol{E}|^2$; integrating (35) over all the space, the same result (36) is obtained. Then one could have the erroneous conclusion that $\frac{1}{2}|\boldsymbol{E}|^2$ is the energy density of the electric field regardless of its source and we can repeat, according to the sense, what we said at the and of Subsection **2.2**.

Remark. Another formula for the energy density, $\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}$, obtained by a method far from being exact, can be found in the literature (see, *e.g.*, Jackson 1998, p. 166), too. For charges and dipoles together, (36) is obtained by omitting terms with divergence as it is usual for charges. Who accepts this manipulation for charges, must accept it for the general case, too, and must arrive at $\frac{1}{2}|\boldsymbol{E}|^2$ which contradicts the other formula.

3. About forces in electrostatics

3.1. Extraneous forces. The electric force acting on a point charge e resting at the space point q_o in an **extraneous** static electric field $\boldsymbol{E}$ is $e\boldsymbol{E}(q_o)$, provided that $\boldsymbol{E}$ is defined at q_o. From that, it is quite obvious that

$$\mathfrak{e}\boldsymbol{E} \tag{37}$$

can be accepted as the static electric energy distribution of a **given** charge distribution $\mathfrak{e}$ in an **extraneous** static electric field function $\mathfrak{e}$, provided that $\mathfrak{e}\boldsymbol{E}$ is locally finite. If the charge distribution has a continuous density then

$$\rho\boldsymbol{E} \tag{38}$$

can be accepted as the static electric force density acting on a **given** charge density ρ in an **extraneous** static electric field $\boldsymbol{E}$, provided that $\rho\boldsymbol{E}$ is locally integrable. The electric force acting on a point dipole $\boldsymbol{p}$ at the space point q_o in an **extraneous** static electric field $\boldsymbol{E}$ is $\boldsymbol{p}\cdot\nabla\boldsymbol{E}(q_o)$ (Jackson 1998), provided that $\boldsymbol{E}$ is defined at q_o. Then by Maxwell equation $\nabla\times\boldsymbol{E} = 0$,

$$\boldsymbol{P}\cdot\nabla\boldsymbol{E} = (\nabla\boldsymbol{E})\cdot\boldsymbol{P} \tag{39}$$

can be accepted as the static electric force density acting on a **given** dipole density $\boldsymbol{P}$ in an **extraneous** static electric field $\boldsymbol{E}$, provided that $\boldsymbol{P}\cdot\nabla\boldsymbol{E} = (\nabla\boldsymbol{E})\cdot\boldsymbol{P}$ is locally integrable. The notion of forces in an extraneous field is a consequence of the fact that only an action is taken into account instead of an interaction.

3.2. Self-forces, charges. We follow a way similar to self-energies.

First step. An inertial point charge does not act on itself: its self-force is zero.

Second step. Taking the charge distribution (18), the self-force distribution is

$$\sum_{k=1}^{n} e_k \sum_{k\neq i}^{n} \boldsymbol{E}_i \delta_{q_k}. \tag{40}$$

Defining the originally senseless $e_k\boldsymbol{E}_k\delta_{q_k}$ "self-force" to be zero, we get

$$\sum_{k=1}^{n} e_k \sum_{k\neq i}^{n} \boldsymbol{E}_i\delta_{q_k} + \sum_{k=1}^{n} e_k\boldsymbol{E}_k\delta q_k = \sum_{k=1}^{n} e_k\boldsymbol{E}\delta_{q_k} \tag{41}$$

where $\boldsymbol{E}$ is the field produced by all the charges.

Third step. Putting together more and more particles with smaller and smaller charges, we get that

$$\rho\boldsymbol{E} \tag{42}$$

can be accepted, without a serious objection, as the static electric self-force density of the **given** *charge density ρ in its own electric field* $\boldsymbol{E}$, provided that $\rho\boldsymbol{E}$ is locally integrable.

Remarks.

(1) The formula of self-force density equals the formula of extraneous force density: both are $\rho\boldsymbol{E}$ with different physical meanings. This warns us that it is worth keeping in mind the physical meaning of an actual formula, not to be confused.

(2) Recall what has been said at the end of Section **1**: both in the discrete case and in the continuous case, some other non-electric (*e.g.*, mechanical) force distribution, opposite to the electric one, must be applied in order that the charges do not move from their given places,

3.3. Stress tensor, charges. Fourth step. One introduces

$$\boldsymbol{L} := -\boldsymbol{E}\otimes\boldsymbol{E} + \frac{1}{2}|\boldsymbol{E}|^2\mathbf{1} \tag{43}$$

($\mathbf{1}$ is the identity map of $\mathbf{S}_0$) for an electric field function $\boldsymbol{E}$ and using the Maxwell equation $\nabla\cdot\boldsymbol{E} = \rho$ one has

$$-\nabla\cdot\boldsymbol{L} = \rho\boldsymbol{E}; \tag{44}$$

here $\rho\boldsymbol{E}$ is the **self-force density** of the **given** charge density ρ because $\boldsymbol{E}$ is produced by ρ. On the analogy of continuum mechanics where the force density is the negative divergence of a stress tensor one considers $\boldsymbol{L}$ the stress tensor of the electric field $\boldsymbol{E}$.

Comments.

(1) In continuum mechanics, the stress tensor has a real physical meaning by giving the shearing forces, *etc.*, inside the continuum. The stress tensor there is concentrated to the body of the continuum, it is zero where there is no material. In electrostatics $\boldsymbol{L}$ is not zero where ρ is zero, thus, *it is unjustified* to consider $\boldsymbol{L}$ the static electric stress tensor *of the charge distribution ρ.*

(2) Shearing forces, *etc.*, cannot be observed in the electric field where ρ is zero, thus, *it is more unjustified* to consider $\boldsymbol{L}$ the stress tensor *of the static electric field* $\boldsymbol{E}$; this mis-switching is due to the fact that $\boldsymbol{L}$ refers to the electric field only (does not refer to any charges).

(3) Only the negative divergence of $\boldsymbol{L}$ has a real physical meaning: the self-force density of the charge distribution.

(4) For an electric field $\boldsymbol{\mathcal{E}}$, the products $\boldsymbol{\mathcal{E}}\otimes\boldsymbol{\mathcal{E}}$ and $|\boldsymbol{\mathcal{E}}|^2=\boldsymbol{\mathcal{E}}\cdot\boldsymbol{\mathcal{E}}$, as the products of two Distributions, make no sense, in general. For a field function $\boldsymbol{E}$ the following cases can occur:
   A. $\boldsymbol{L}$ and $\nabla\cdot\boldsymbol{L}$ are locally integrable; then $\boldsymbol{L}$ defines a regular Distribution, nevertheless, it cannot be considered a stress tensor, and *it has the only physical meaning* that its negative divergence is the self-force density.
   B. $\boldsymbol{L}$ is locally integrable but $\nabla\cdot\boldsymbol{L}$ is not; then $\boldsymbol{L}$ defines a regular Distribution but *it is questionable* whether it has some physical meaning.
   C. $\boldsymbol{L}$ is not locally integrable; then it does not define a Distribution without further ado.

3.4. Stress tensor, point charge. According to the first step, *it can be stated unmistakably that the self-force of a point charge is zero.* The same result can be derived in a correct way by taming the pole of the fictitious "stress tensor"

$$\boldsymbol{L}(q)=\frac{e^2}{16\pi^2}\left(-\frac{(q-q_o)\otimes(q-q_o)}{|q-q_o|^6}+\frac{\mathbf{1}}{2|q-q_o|^4}\right)\qquad(q\neq q_o)\tag{45}$$

of a point charge e resting in the space point q_o which has a pole at q_o and is not locally integrable. Introducing $\boldsymbol{q}:=q-q_o$ and $\boldsymbol{n}(\boldsymbol{q}):=\frac{\boldsymbol{q}}{|\boldsymbol{q}|}$,

$$\boldsymbol{L}(q_o+\boldsymbol{q})=\frac{e^2}{16\pi^2}\left(\frac{-\boldsymbol{n}(\boldsymbol{q})\otimes\boldsymbol{n}(\boldsymbol{q})}{|\boldsymbol{q}|^4}+\frac{\mathbf{1}}{2|\boldsymbol{q}|^4}\right)=\tag{46}$$

$$=\frac{e^2}{16\pi^2}\frac{\mathbf{1}-2\boldsymbol{n}(\boldsymbol{q})\otimes\boldsymbol{n}(\boldsymbol{q})}{2|\boldsymbol{q}|^4}\qquad(\boldsymbol{q}\neq 0).\tag{47}$$

The numerator is an even tensor power of $\boldsymbol{n}$, therefore the pole can be tamed and

$$(\mathrm{tm}\boldsymbol{L}\mid\psi)=\int_{\mathbf{S}_0}\boldsymbol{L}(q_o+\boldsymbol{q})\big(\psi(\boldsymbol{q})-\psi(\mathbf{0})\big)\ d\boldsymbol{q}\ !!\tag{48}$$

holds for all test functions ψ; by the equality (see (**??**))

$$\int_{S_1(\mathbf{0})}(\mathbf{1}-2\boldsymbol{n}\otimes\boldsymbol{n})\ d\boldsymbol{n}=\frac{4\pi}{3}\mathbf{1},\tag{49}$$

(48) can be rewritten in the form

$$(\mathrm{tm}\boldsymbol{L}\mid\psi)=\int_{\mathbf{S}_0}\left(\boldsymbol{L}(q_o+\boldsymbol{q})\psi(\boldsymbol{q})-\frac{e^2}{16\pi^2}\frac{\psi(\mathbf{0})}{6|\boldsymbol{q}|^4}\mathbf{1}\right)\ d\boldsymbol{q}.\ !!\tag{50}$$

The quantity $\mathrm{tm}\boldsymbol{L}$ is a mathematical object; it is not a measure, and even if it were, it could not be considered the stress distribution. Its only physical meaning could be that its negative divergence is zero, the self-force of the point charge.

Proposition 3.1.

$$-\nabla\cdot\mathrm{tm}\boldsymbol{L}=0.\tag{51}$$

Proof. The divergence of the function $\boldsymbol{L}$ (not defined at q_o) is zero: there is no force outside the charge, but this does not mean that the divergence of tm$\boldsymbol{L}$ is zero as well. According to the definition of the derivatives of Distributions,

$$(-\nabla\cdot\mathrm{tm}\boldsymbol{L}\mid\psi)=(\mathrm{tm}\boldsymbol{L}\mid\cdot\nabla\psi)= \tag{52}$$

$$=\int_{\mathbf{S}_0}\left(\boldsymbol{L}(q_e+\boldsymbol{q})\cdot\nabla\psi(\boldsymbol{q})-\frac{e^2}{16\pi^2}\frac{\nabla\psi(\mathbf{0})\mathbf{1}}{6|\boldsymbol{q}|^4}\right)\,d\boldsymbol{q}.\ !! \tag{53}$$

The first term in the integral is $\nabla\cdot(\boldsymbol{L}\psi)$ because $\nabla\cdot\boldsymbol{L}=0$ (except q_o). The second term, with the notation $r(\boldsymbol{q}):=|\boldsymbol{q}|$ and

$$\nabla\cdot\frac{\boldsymbol{n}}{r^3}=-\frac{1}{r^4}, \tag{54}$$

can be written in the form $\nabla\cdot\left(\frac{e^2}{16\pi^2}\frac{(\nabla\psi(\mathbf{0}))\otimes\boldsymbol{n}}{6r^3}\right)$. Then, taking radial parametrization (see Subsection A.**3**) and $\int_{\mathbf{S}_0}=\lim_{R\to 0}\int_R^\infty\int_{S_1(0)}$ and applying Gauss' theorem, we obtain

$$\begin{aligned}(-\nabla\cdot\mathrm{tm}\boldsymbol{L}\mid\psi)&=\\ &=\frac{e^2}{16\pi^2}\lim_{R\to 0}\int_{S_1(\mathbf{0})}\left(\frac{\mathbf{1}-2\boldsymbol{n}\otimes\boldsymbol{n}}{2R^2}\psi(R\boldsymbol{n})+\frac{(\nabla\psi(\mathbf{0}))\otimes\boldsymbol{n}}{6R}\right)\cdot(-\boldsymbol{n})\ d\boldsymbol{n}=\\ &=\frac{e^2}{16\pi^2}\lim_{R\to 0}\int_{S_1(\mathbf{0})}\left(\frac{\boldsymbol{n}\psi(R\boldsymbol{n})}{2R^2}-\frac{\nabla\psi(\mathbf{0}))}{6R}\right)\ d\boldsymbol{n}.\end{aligned}$$

The integral of the second term is $-\frac{4\pi}{6R}\nabla\psi(\mathbf{0})$. As concerns the first term, we take the expansion

$$\psi(R\boldsymbol{n})=\psi(\mathbf{0})+\nabla\psi(\mathbf{0})\cdot(R\boldsymbol{n})+\frac{1}{2}(R\boldsymbol{n})\cdot\nabla^2\psi(\mathbf{0})\cdot(R\boldsymbol{n})+\mathrm{ordo}(R^2)T(\boldsymbol{n}) \tag{55}$$

where $T(\boldsymbol{n})$ is a linear combination of tensor powers of $\boldsymbol{n}$. Then, because of (**??**),

$$\int_{S_1(\mathbf{0})}\frac{\big(\nabla\psi(\mathbf{0})\cdot(R\boldsymbol{n})\big)\boldsymbol{n}}{2R^2}\ d\boldsymbol{n}=\frac{4\pi}{6R}\nabla\psi(\mathbf{0}), \tag{56}$$

and

- the integrals of $\frac{\psi(\mathbf{0})\boldsymbol{n}}{2R^2}$ and $\frac{\left(\big(R\boldsymbol{n}\big)\cdot\nabla^2\psi(\mathbf{0})\cdot(R\boldsymbol{n})\right)\boldsymbol{n}}{4R^2}$ are zero,
- the integral of $\frac{\mathrm{ordo}(R^2)T(\boldsymbol{n})}{2R^2}$ tends to zero as R tends to zero, ■

As a consequence, not without any reservations, tm$\boldsymbol{L}$ can be considered the *Distribution of self-stress* (which is not a self-stress distribution!) of a point charge, keeping in mind that only its negative divergence has a physical meaning: the self-force is zero.

3.5. Self-forces, charges and dipoles. An inertial point dipole does not act on itself, its self-force is zero. The self-force distribution of the dipole distribution (29), with the earlier manipulation with zero terms,

$$\sum_{k=1}^{n} \boldsymbol{p}_k \cdot \nabla \boldsymbol{E} \delta_{q_k} \tag{57}$$

where $\boldsymbol{E}$ is the electric field produced by all the dipoles. Then

$$\boldsymbol{P} \cdot \nabla \boldsymbol{E} = (\nabla \boldsymbol{E}) \cdot \boldsymbol{P} \tag{58}$$

can be accepted as the electric self-force density of the **given** dipole density $\boldsymbol{P}$ in its own electric field $\boldsymbol{E}$, provided that it is locally integrable. Note that for dipoles, too, the extraneous force density and the self-force density is given by the same formula.

Let us now consider a system consisting of point charges and point dipoles having the distribution (33). The force acting on the charge e_i in the field $\boldsymbol{E}_j$ of the dipole $\boldsymbol{p}_j$ is $e_i \boldsymbol{E}_j(q_i)$; the force acting on the dipole $\boldsymbol{p}_j$ in the electric field $\boldsymbol{E}_i$ of the charge e_i is $\boldsymbol{p}_j \cdot (\nabla \boldsymbol{E}_i)(q_j)$ and we have $e_i \boldsymbol{E}_j(q_i) = -\boldsymbol{p}_j \cdot (\nabla \boldsymbol{E}_i)(q_j)$. Then by an equality similar to (34), *it can be accepted that*

$$\rho \boldsymbol{E} + \boldsymbol{P} \cdot \nabla \boldsymbol{E} \tag{59}$$

is the self-force density of the **given** *charge density* ρ *and the* **given** *dipole density* $\boldsymbol{P}$ *in their own fields*, provided that it is locally integrable. Now the fictitious "stress tensor"

$$\boldsymbol{L} := -\boldsymbol{E} \otimes \boldsymbol{E} + \frac{1}{2}|\boldsymbol{E}|^2 \mathbf{1} - \boldsymbol{E} \otimes \boldsymbol{P} = -\boldsymbol{E} \otimes \boldsymbol{D} + \frac{1}{2}|\boldsymbol{E}|^2 \mathbf{1} \tag{60}$$

has the property that its negative divergence is the formal self-force density

$$-\nabla \cdot \boldsymbol{L} = \rho \boldsymbol{E} + \boldsymbol{P} \cdot \nabla \boldsymbol{E}. \tag{61}$$

Then we can repeat the problems listed at the end of Subsection **3.3**, crowned by the fact that $\boldsymbol{L}$ is not given by field quantities only (contains $\boldsymbol{P}$ explicitly or hidden in $\boldsymbol{D}$). Moreover, for a point dipole it is questionable even how the fictitious "stress tensor" can be defined.

4. Energies and forces in electrostatics, a summary.

Let us survey the ways we followed treating self-energies and self-forces in statics.

(1) The starting points were zero self-energy and zero self-force of a point charge.
(2) Formulae (20) and (40) of self-energy and self-force of a system of point charges suggested formulae (21) and (42) of self-energy and self-force density of a continuous charge distribution.
(3a) The formula of self-energy density is usually transformed by (25) into the false form $\frac{1}{2}|\boldsymbol{E}|^2$ which can have the only real physical meaning that its integral – if it is integrable – is the total self-energy of the continuous charge distribution.
(3b) The usual interpretation of (43) as a stress tensor is not right because it has not the properties of stress tensors in continuum theory; it can have

the only real physical meaning that its negative divergence is the self-force density of the continuous charge distribution.

(4a) The fictitious "self-energy density" and "stress tensor" of a point charge are not locally integrable; we attached to them Distributions by pole-taming.

(4b) Those Distributions have the correct physical meaning: the self-energy and the self-force of a point charge are zero.

We have made a tour from a point charge to charge densities and back, by giving sense to originally doubtful or unjustified formulae. This going round and arriving back at the starting point will be useful for later considerations. For dipoles, steps (1) – (3) are similar but step (4) is questionable as concerns the stress tensor.

5. Beyond statics with respect to a standard inertial frame.

5.1. Basic notions. For an **arbitrary** standard inertial frame $\boldsymbol{u}$ the electric and magnetic quantities are Distributions on $\mathtt{I}_{\boldsymbol{u}} \times \mathtt{S}_{\boldsymbol{u}}$ ($\boldsymbol{u}$-time and $\boldsymbol{u}$-space):

$\boldsymbol{\mathcal{E}}_{\boldsymbol{u}}$: $\boldsymbol{u}$-electric field, $\boldsymbol{\mathcal{B}}_{\boldsymbol{u}}$: $\boldsymbol{u}$-magnetic field,

$\mathfrak{e}_{\boldsymbol{u}}$: $\boldsymbol{u}$-charge distribution, $\mathfrak{i}_{\boldsymbol{u}}$: $\boldsymbol{u}$-current distribution,

$\boldsymbol{\mathcal{P}}_{\boldsymbol{u}}$: $\boldsymbol{u}$-electric dipole distribution, $\boldsymbol{\mathcal{M}}_{\boldsymbol{u}}$: $\boldsymbol{u}$-magnetic moment distribution;

for the sake of simplicity, the subscript $\boldsymbol{u}$ will be omitted in the sequel. The Maxwell equations are

$$\nabla \cdot \boldsymbol{\mathcal{E}} = \mathfrak{e} - \nabla \cdot \boldsymbol{\mathcal{P}}, \qquad -\partial_t \boldsymbol{\mathcal{E}} + \nabla \times \boldsymbol{\mathcal{B}} = \mathfrak{i} + \partial_t \boldsymbol{\mathcal{P}} + \nabla \times \boldsymbol{\mathcal{M}}, \tag{62}$$

$$\nabla \cdot \boldsymbol{\mathcal{B}} = 0, \qquad \partial_t \boldsymbol{\mathcal{B}} + \nabla \times \boldsymbol{\mathcal{E}} = 0. \tag{63}$$

A special case is when the charge, *etc.*, distributions have continuously differentiable densities ρ, $\boldsymbol{i}$, $\boldsymbol{P}$ and $\boldsymbol{M}$, respectively, and the fields are regular Distributions corresponding to continuously differentiable functions $\boldsymbol{E}$ and $\boldsymbol{B}$; then the Maxwell equations are

$$\nabla \cdot \boldsymbol{E} = \rho - \nabla \cdot \boldsymbol{P}, \qquad -\partial_t \boldsymbol{E} + \nabla \times \boldsymbol{B} = \boldsymbol{i} + \partial_t \boldsymbol{P} + \nabla \times \boldsymbol{M}, \tag{64}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0. \tag{65}$$

Further, with the quantities

$$\boldsymbol{D} := \boldsymbol{E} + \boldsymbol{P}, \qquad \boldsymbol{H} := \boldsymbol{B} - \boldsymbol{M}, \tag{66}$$

(64) gets its most frequently used form

$$\nabla \cdot \boldsymbol{D} = \rho, \qquad -\partial_t \boldsymbol{D} + \nabla \times \boldsymbol{H} = \boldsymbol{i}. \tag{67}$$

This simpler form, however, is misleading because $\boldsymbol{D}$ and $\boldsymbol{H}$ are usually considered proper field quantities though they are not because of including dipoles and magnetic moments hidden. (And even, from the start of the theory of electromagnetism, for a long time, the roles of $\boldsymbol{H}$ and $\boldsymbol{B}$ have been confused.)

Two charge, *etc.*, distributions are **extraneous** to each other if their supports are disjoint. A charge, *etc.*, distribution and an electromagnetic field are called **extraneous** to each other if the charge, *etc.*, distribution is extraneous to the charge, *etc.*, distribution producing the electromagnetic field. The **self-field** of a charge, *etc.*, distribution is the electromagnetic field produced by them. In the following we assume that convenient conditions are fulfilled in order that the formulae make sense as Distributions, so "provided that ..." will be omitted.

5.2. Extraneous forces, charges. Let us consider a point charge e moving under the action of an **extraneous** electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively, its **given motion** being described by the function $t \mapsto \boldsymbol{r}(t)$. Then the **extraneous force** acting on the particle is the well-known Lorentz force

$$e\big(\boldsymbol{E}(t,\boldsymbol{r}(t)) + \boldsymbol{r}'(t) \times \boldsymbol{B}(t,\boldsymbol{r}(t))\big), \tag{68}$$

$\boldsymbol{r}'$ being the velocity of the point charge. Further, the **extraneous** mechanical power is

$$e\boldsymbol{E}(t,\boldsymbol{r}(t)) \cdot \boldsymbol{r}'(t). \tag{69}$$

The corresponding charge and current distribution are $\mathfrak{e} := e\lambda_{\mathrm{Ran}(\boldsymbol{r})}$ and $\boldsymbol{i} := e\boldsymbol{r}'\lambda_{\mathrm{Ran}(\boldsymbol{r})}$, respectively, where $\lambda_{\mathrm{Ran}(\boldsymbol{r})}$ is the Lebesgue measure of the curve $\{(t,\boldsymbol{r}(t) \mid \in \mathbb{T}\}$; as a Distribution, it acts on test functions ψ by integration,

$$(\lambda_{\mathrm{Ran}(\boldsymbol{r})} \mid \psi) := \int_{\mathbb{T}} \psi(t,\boldsymbol{r}(t))\; dt. \tag{70}$$

Then (68) and (69) says that

$$\mathfrak{e}(\boldsymbol{E} + \boldsymbol{r}' \times \boldsymbol{B}) \qquad \text{and} \qquad \mathfrak{e}\boldsymbol{E} \cdot \boldsymbol{r}' \tag{71}$$

are the force distribution and power distribution. From that it is quite obvious that

$$\rho(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) = \rho\boldsymbol{E} + \boldsymbol{i} \times \boldsymbol{B} \qquad \text{and} \qquad \rho\boldsymbol{E} \cdot \mathbf{v} = \boldsymbol{E} \cdot \boldsymbol{i} \tag{72}$$

can be accepted as the electromagnetic force density and mechanical power density, respectively, acting on a **given** charge density ρ, with **given** velocity field $\mathbf{v}$ (and $\boldsymbol{i} := \rho\mathbf{v}$), in an **extraneous** electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$.

5.3. Self-forces, charges. Let us try to follow the route suggested by statics.

First step. A non-inertial point charge acts on itself depending on how it moves but its self-force, denoted by $\boldsymbol{f}_s$, is not known at present.

Second step. Let us consider a system of point charges $e_1, \ldots, e_n$ with given motions $\boldsymbol{r}_1, \ldots \boldsymbol{r}_n$, respectively; then the self-force acting in this system is

$$\sum_{k=1}^{n} e_k \left(\sum_{k \neq i = 1}^{n} \Big(\boldsymbol{E}_i + \boldsymbol{r}_k' \times \boldsymbol{B}_i \Big) \lambda_{\mathrm{Ran}(\boldsymbol{r}_k)} \right) + \sum_{k=1}^{n} \boldsymbol{f}_{s,k} \tag{73}$$

and the self-power in the system is

$$\sum_{k=1}^{n} e_k \left(\sum_{k \neq i = 1}^{n} \boldsymbol{E}_i \lambda_{\mathrm{Ran}(\boldsymbol{r}_k)} \right) \cdot \boldsymbol{r}_k' + \sum_{k=1}^{n} \boldsymbol{f}_{s,k} \cdot \boldsymbol{r}_k'. \tag{74}$$

Contrary to statics, where the self-force of a point charge is known to be zero, here the self-force is not zero and is not known. There, defining an originally senseless expression to be zero and adding zero to the formula in question, is a harmless manipulation. Here, on the contrary, it is doubtful that the originally senseless $e_k(\boldsymbol{E}_k + \mathbf{r}'_k \times \boldsymbol{B}_k)\lambda_{\mathrm{Ran}(\boldsymbol{r}_k)}$ can be defined to be equal to the unknown $\boldsymbol{f}_{s,k}$. Nevertheless, we do so, keeping in mind that this is a questionable guess. We get

$$\sum_{k=1}^{n} e_k\Big(\boldsymbol{E} + \mathbf{r}'_k \times \boldsymbol{B}\Big)\lambda_{\mathrm{Ran}(\boldsymbol{r}_k)} \quad (?) \tag{75}$$

and

$$\sum_{k=1}^{n} e_k \boldsymbol{E} \cdot \mathbf{r}'_k \lambda_{\mathrm{Ran}(\boldsymbol{r}_k)} \quad (?) \tag{76}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field produced by all the charges; the question-marks remind us of our questionable guess.

Third step. Putting together more and more particles with smaller and smaller charges, we obtain the self-force density

$$\rho(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}) \ (?) = \rho\boldsymbol{E} + \boldsymbol{i} \times \boldsymbol{B}) \ (?) \tag{77}$$

and the self-power density

$$\rho\boldsymbol{E} \cdot \mathbf{v} \ (?) = \boldsymbol{E} \cdot \boldsymbol{i} \ (?) \tag{78}$$

of a charge density ρ with a velocity field $\boldsymbol{v}$.

5.4. Balance equations? (1). Let the electric and magnetic field $\boldsymbol{E}$ and $\boldsymbol{B}$ be produced by the charge density ρ and current density $\boldsymbol{i}$ (electric dipoles and magnetic moments are not present). A usual argumentation (see, *e.g.*, de Groot and Suttorp 1972) runs as follows.

(1a) Subtracting the scalar product of the second equation in (65) by $\boldsymbol{B}$ from the scalar product of the second equation in (64) by $\boldsymbol{E}$ one gets

$$-\partial_t \frac{1}{2}(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) - \nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = \boldsymbol{E} \cdot \boldsymbol{i} \tag{79}$$

which is considered an energy balance equation, with

- the energy density
$$\frac{1}{2}(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2). \tag{80}$$
- the energy flow density (Poynting vector)
$$\boldsymbol{E} \times \boldsymbol{B}, \tag{81}$$
- the power density
$$\boldsymbol{E} \cdot \boldsymbol{i}, \tag{82}$$

There are some serious doubts, however.

(1) According to Subsection **2.2**, it is unjustified to take (80) either for self-energy or energy density. By the way, which is not mentioned usually, the presumed energy density is unique only up to a time-independent term.

(2) The doubts regarding the energy flow density are well-known (Feynman, Leighton, and Sands 1963): first, it is unique only up to a divergence-free term; second, it is not zero for a static $\boldsymbol{E}$ and a static $\boldsymbol{B}$ when nothing changes in time with respect to the reference frame in question; does the energy really flow in that reference frame?

(3) Since the fields in the Maxwell equations are just the ones produced by the charges, *etc.*, $\boldsymbol{E}\cdot\boldsymbol{i}$ should be the self-power density which is questionable according to (78).

(1b) Adding the vectorial product of the second equation in (64) by $\boldsymbol{B}$ to the vectorial product of the second equation in (65) by $\boldsymbol{E}$, using the other two equations, too, one gets

$$-\partial_t(\boldsymbol{E}\times\boldsymbol{B})-\nabla\cdot\left(-\boldsymbol{E}\otimes\boldsymbol{E}-\boldsymbol{B}\otimes\boldsymbol{B}+\frac{1}{2}(|\boldsymbol{E}|^2+|\boldsymbol{B}|^2)\ \mathbf{1}\right)=\rho\boldsymbol{E}+\boldsymbol{i}\times\boldsymbol{B} \tag{83}$$

which is considered a momentum balance equation, with

- the momentum density
$$\boldsymbol{E}\times\boldsymbol{B}, \tag{84}$$
- the momentum flow density
$$-\boldsymbol{E}\otimes\boldsymbol{E}-\boldsymbol{B}\otimes\boldsymbol{B}+\frac{1}{2}(|\boldsymbol{E}|^2+|\boldsymbol{B}|^2)\ \mathbf{1}, \tag{85}$$
- the force density
$$\rho\boldsymbol{E}+\boldsymbol{i}\times\boldsymbol{B}, \tag{86}$$

There are some serious doubts, however.

(1) The Poynting vector can hardly be a momentum density for static fields which "do not move" (consider a static electric field $\boldsymbol{E}$ generated by a "standing" charge distribution and a static magnetic field $\boldsymbol{B}$ generated by a "standing" magnetic moment distribution; is $\boldsymbol{E}\times\boldsymbol{B}$ a momentum density?)
(2) According to Subsection **3.3**, it is unjustified to take (85) for the momentum flow density (stress tensor).
(3) Since the fields in the Maxwell equations are just the ones produced by the charges, *etc.*, $\rho\boldsymbol{E}+\boldsymbol{i}\times\boldsymbol{B}$ should be the self-force density which is questionable according to (77).

Remark. In spite of the listed doubts, it is not excluded that *beyond statics, in some circumstances,* (80) is the energy density and (81) is the energy flow density, *etc. The problem is how those "some circumstances" can be precisely formulated; it is sure only that they are connected with radiation because it is an everyday experience that energy flows with light.*

5.5. Balance equations? (2).

(2a) If there are an electric dipole density $\boldsymbol{P}$ and magnetic moment density $\boldsymbol{M}$, too, then instead of (79) one gets

$$-\partial_t\frac{1}{2}(|\boldsymbol{E}|^2+|\boldsymbol{B}|^2)-\nabla\cdot(\boldsymbol{E}\times\boldsymbol{B})=\boldsymbol{E}\cdot\boldsymbol{i}+\boldsymbol{E}\cdot\partial_t\boldsymbol{P}-\boldsymbol{B}\cdot\partial_t\boldsymbol{M}+\nabla\cdot(\boldsymbol{E}\times\boldsymbol{M}) \tag{87}$$

which has not the form of a balance equation for energy. It becomes simpler with the quantities (66), without becoming a balance equation:

$$-\big(\boldsymbol{E}\cdot\partial_t\boldsymbol{D}+\boldsymbol{H}\cdot\partial_t\boldsymbol{B}\big)-\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})=\boldsymbol{E}\cdot\boldsymbol{i} \tag{88}$$

To obtain a balance-like equation at all costs, one considers (Jackson 1998, p. 189) the special case when there are no independent dipoles and magnets, the whole space of a given inertial frame is filled with a homogeneous material which is polarized and magnetized, *i.e.* $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and $\mu\boldsymbol{H} = \boldsymbol{B}$ holds; then

$$-\partial_t \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H}) - \nabla\cdot(\boldsymbol{E}\times\boldsymbol{H}) = \boldsymbol{E}\cdot\boldsymbol{i}. \tag{89}$$

This is a balance-like equation, but in vain, more serious doubts arise as previously.

(2b) The situation is even worse for the general momentum balance: then the formula obtained instead of (83) cannot be converted into a balance-like equation for momentum even in the special case above.

Remarks.

(1) There are some other forms of energy and momentum balance (see Jackson 1998 and a good review in Jiménez, Campos, and López-Mariño 2013); all of them are unsatisfactory in some aspect, and all of them have the common fault that $\frac{1}{2}(|\boldsymbol{E}|^2+|\boldsymbol{B}|^2)$ or $\frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D}+\boldsymbol{B}\cdot\boldsymbol{H})$ or similar formulae constructed from $\boldsymbol{E},\boldsymbol{D},\boldsymbol{B}$ and $\boldsymbol{H}$ are accepted as energy density.

(2) The not too convincing method in statics to obtain the energy density $\frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{D}$ (see the Remark at the end of Subsection **2.4**) is probably invented for an acceptable explanation of the balance equation (89).

(3) The problems connected with static dipole distributions, outlined in Subsections **2.4** and **3.5**, are precursors of the more significant problems that emerged in balance equations.

6. Spacetime formulation

6.1. Basic notions. As it is said in the Introduction, coordinate-free spacetime notations (Lorentz product, differentiation, *etc.*) will be used according to Matolcsi (2020, 2023). Electric and magnetic fields, $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$, according to a standard inertial frame, are split components of an electromagnetic field $\boldsymbol{\mathcal{F}}$ which is an antisymmetric spacetime tensor Distribution. Precisely speaking, up to now, the magnetic field has been considered, as usual, a three-dimensional vector Distribution, but the spacelike component of $\boldsymbol{\mathcal{F}}$ is a three-dimensional antisymmetric tensor which can be identified with a three dimensional vector. The charge distribution and current distribution, $\mathfrak{e}$ and $\mathfrak{i}$, according to a standard inertial frame, are split components of an absolute current $\mathfrak{j}$ which is a four dimensional vector Distribution. The electric dipole distribution $\boldsymbol{\mathcal{P}}$ and magnetic moment distribution $\boldsymbol{\mathcal{M}}$, according to a standard inertial frame, are split components of an absolute dipole-moment $\boldsymbol{\mathcal{N}}$ which is an antisymmetric spacetime Distribution. More explicitely: for a standard inertial frame corresponding to the absolute velocity $\boldsymbol{u}$,

$$\boldsymbol{\mathcal{E}}_{\boldsymbol{u}} = \boldsymbol{\mathcal{F}}\boldsymbol{u}, \qquad \boldsymbol{\mathcal{B}}_{\boldsymbol{u}} = \boldsymbol{\mathcal{F}} + \boldsymbol{u}\otimes(\boldsymbol{u}\boldsymbol{\mathcal{F}}) + (\boldsymbol{\mathcal{F}}\boldsymbol{u})\otimes\boldsymbol{u} = \boldsymbol{\mathcal{F}} - \boldsymbol{u}\wedge(\boldsymbol{\mathcal{F}}\boldsymbol{u}),$$

$$\boldsymbol{\mathcal{P}}_{\boldsymbol{u}} = -\boldsymbol{\mathcal{N}}\boldsymbol{u}, \qquad \boldsymbol{\mathcal{M}}_{\boldsymbol{u}} = \boldsymbol{\mathcal{N}} - \boldsymbol{u}\wedge(\boldsymbol{\mathcal{N}}\boldsymbol{u}),$$

$$\mathfrak{e}_{\boldsymbol{u}} = -\boldsymbol{u}\cdot\mathfrak{j}, \qquad \mathfrak{i}_{\boldsymbol{u}} = \mathfrak{j} + \boldsymbol{u}(\boldsymbol{u}\cdot\mathfrak{j}).$$ [2]

[2] In coordinates: $\boldsymbol{\mathcal{E}}_k = \boldsymbol{\mathcal{F}}_{0k},\ \boldsymbol{\mathcal{B}}_{ik} = \boldsymbol{\mathcal{F}}_{ik},\ \mathfrak{e} = \mathfrak{j}_0,\ \mathfrak{i}_k = \mathfrak{j}_k \quad (i,k = 1,2,3)$

If D denotes spacetime differentiation (then $\mathrm{D}_{\boldsymbol{u}} := \boldsymbol{u}\cdot\mathrm{D}$ is the $\boldsymbol{u}$-timelike differentiation and $\nabla_{\boldsymbol{u}} := \mathrm{D} + \boldsymbol{u}(\boldsymbol{u}\cdot\mathrm{D})$ is the $\boldsymbol{u}$-spacelike differentiation), the Maxwell equations are

$$\mathrm{D}\cdot\boldsymbol{\mathcal{F}} = \mathrm{j} + \mathrm{D}\cdot\boldsymbol{\mathcal{N}}, \qquad \mathrm{D}\wedge\boldsymbol{\mathcal{F}} = 0. \tag{90}$$

With $\boldsymbol{\mathcal{G}} := \boldsymbol{\mathcal{F}} - \boldsymbol{\mathcal{N}}$ (whose split components according to $\boldsymbol{u}$ are $\boldsymbol{\mathcal{D}}_{\boldsymbol{u}} = \boldsymbol{\mathcal{G}}\boldsymbol{u}$ and $\boldsymbol{\mathcal{H}}_{\boldsymbol{u}} = \boldsymbol{\mathcal{G}} - \boldsymbol{u}\wedge(\boldsymbol{\mathcal{N}}\boldsymbol{u})$, the first equation becomes

$$\mathrm{D}\cdot\boldsymbol{\mathcal{G}} = \mathrm{j}; \tag{91}$$

it is advisable to keep away from it because $\boldsymbol{\mathcal{G}}$ is not a proper field quantity.

A special case is when the absolute current has a continuous density $\boldsymbol{j}$, the absolute dipole-moment has a continuously differentiable density $\boldsymbol{N}$, and the field is the regular Distribution corresponding to a continuously differentiable function $\boldsymbol{F}$; then the Maxwell equations are

$$\mathrm{D}\cdot\boldsymbol{F} = \boldsymbol{j} + \mathrm{D}\cdot\boldsymbol{N}, \qquad \mathrm{D}\wedge\boldsymbol{F} = 0. \tag{92}$$

(77) and (78) are the split components of

$$\boldsymbol{F}\cdot\boldsymbol{j} \tag{93}$$

which is the absolute force density acting on a **given** absolute current density $\boldsymbol{j}$ in an **extraneous** electromagnetic field $\boldsymbol{F}$. On the other hand, it is questionable that $\boldsymbol{F}\cdot\boldsymbol{j}$ is the absolute self-force density, too, acting on $\boldsymbol{j}$ in its own electromagnetic field $\boldsymbol{F}$: (77) and (78) are the split components of

$$\boldsymbol{F}\cdot\boldsymbol{j}\ (?). \tag{94}$$

6.2. Energy-momentum tensor. For lack of electromagnetic dipole-moment, the "energy density" (80), the "energy flow" (81) and the "stress tensor" (85) are split components, according to a standard inertial frame, of the "energy-momentum tensor"

$$\boldsymbol{T} := -\boldsymbol{F}\cdot\boldsymbol{F} - \frac{1}{4}\mathrm{Tr}(\boldsymbol{F}\cdot\boldsymbol{F})\mathbf{1}; \tag{95}$$

here $\mathbf{1}$ is the identity map of spacetime vectors. Using the Maxwell equation $\mathrm{D}\cdot\boldsymbol{F} = \boldsymbol{j}$ we find that

$$-\mathrm{D}\cdot\boldsymbol{T} = \boldsymbol{F}\cdot\boldsymbol{j}, \tag{96}$$

whose split components are the "balance equations" (79) and (83).

According to the doubts emerged in Subsection **5.4**, indicated by the question-mark, *it is questionable*, in general, that (96) is really an energy-momentum balance. Correspondingly, even if $\boldsymbol{F}\cdot\boldsymbol{j}$ is the absolute self-force density (which is questionable), *it is not right* to consider (95), *except in some circumstances*, a proper energy-momentum tensor. In particular, in statics, it is definitely not right. *The problem is, how those "some circumstances" can be precisely formulated; it is sure only that they are connected with radiation because it is an everyday experience that light heats material objects by transporting energy.* If electric dipole and magnetic moment are present, (87) can be transformed into a balance-like equation (89) only in exceptional cases; nevertheless, it is not a real balance equation of energy. Moreover, no manipulation results in a balance-like equation for momentum. This means that it is questionable whether a general "energy-momentum tensor" exists whose negative spacetime divergence gives a reasonable energy-momentum balance. A number of proposals followed the first ones given by Abraham and Minkowski, as it is pointed out in the Introduction. The proposed different energy-momentum balances correspond to different physical situations (different kinds of materials and their thermodynamical properties), and only experiments could verify or refute them.

6.3. Point charge, self-force. Let us consider a point charge e with a **given** world line function r. For a spacetime point x, $\boldsymbol{s}_r(x)$ denotes the retarded proper time of r. Then with the notations (Matolcsi 2023)

$$\boldsymbol{R}_r(x) := x - r(\boldsymbol{s}_r(x)), \qquad \boldsymbol{u}_r(x) := \dot{r}(\boldsymbol{s}_r(x)), \qquad \boldsymbol{a}_r(x) := \ddot{r}(\boldsymbol{s}_r(x)), \tag{97}$$

$$\boldsymbol{L}_r := \frac{\boldsymbol{R}_r}{-\boldsymbol{u}_r \cdot \boldsymbol{R}_r} = -\mathrm{D}\boldsymbol{s}_r, \qquad \boldsymbol{d}_r := \boldsymbol{a}_r + (\boldsymbol{a}_r \cdot \boldsymbol{L}_r)\boldsymbol{u}_r, \tag{98}$$

the electromagnetic field produced by the point charge (*i.e.*, by the absolute current distribution $e\lambda : \mathrm{Ran}(r)$) is the regular Distribution defined by the locally integrable function

$$\boldsymbol{F} := \frac{e}{4\pi} \frac{\boldsymbol{u}_r \wedge \boldsymbol{L}_r}{(-\boldsymbol{u}_r \cdot \boldsymbol{R}_r)^2} + \frac{e}{4\pi} \frac{\boldsymbol{d}_r \wedge \boldsymbol{L}_r}{-\boldsymbol{u}_r \cdot \boldsymbol{R}_r}; \tag{99}$$

the first term is the *tied field* $\boldsymbol{F}^{td}$ which is never zero, the second term is the *radiated field* $\boldsymbol{F}^{rd}$ which is zero if and only if the charge is not accelerated. Then the fictitious "energy-momentum tensor" (95) is of the form

$$\boldsymbol{T} = \boldsymbol{T}^{td} + \boldsymbol{T}^{rtd} + \boldsymbol{T}^{rd} \tag{100}$$

where

$$\boldsymbol{T}^{td} := -\boldsymbol{F}^{td} \cdot \boldsymbol{F}^{td} - \frac{1}{4}\mathrm{Tr}(\boldsymbol{F}^{td} \cdot \boldsymbol{F}^{td})\mathbf{1}, \tag{101}$$

$$\boldsymbol{T}^{rtd} := -(\boldsymbol{F}^{td} \cdot \boldsymbol{F}^{rd} + \boldsymbol{F}^{rd} \cdot \boldsymbol{F}^{td}) - \frac{1}{4}\mathrm{Tr}(\boldsymbol{F}^{td} \cdot \boldsymbol{F}^{rd} + \boldsymbol{F}^{rd} \cdot \boldsymbol{F}^{td})\mathbf{1}, \tag{102}$$

$$\boldsymbol{T}^{rd} := -\boldsymbol{F}^{rd} \cdot \boldsymbol{F}^{rd} - \frac{1}{4}\mathrm{Tr}(\boldsymbol{F}^{rd} \cdot \boldsymbol{F}^{rd})\mathbf{1}. \tag{103}$$

Only the last term, the purely radiated part is locally integrable in spacetime. It is proved in the paper Matolcsi 2023 that a Distribution can be attached to the fictitious "energy-momentum tensor" (100) by pole-taming and the negative spacetime divergence of that Distribution is just the self-force, the radiation reaction force:

$$-\mathrm{D} \cdot \mathrm{tm}\boldsymbol{T} = \boldsymbol{f}_s \ \left(= \frac{1}{4\pi} \frac{2e^2}{3} (\dot{r} \wedge \dddot{r}) \cdot \dot{r}) \lambda_{\mathrm{Ran}(r)} \right). \tag{104}$$

The fictitious "self-energy density" and "stress tensor" which appeared in statics now are included in the fictitious "energy-momentum tensor". For a static point charge, those fictitious quantities are not locally integrable in space; Distributions can be attached to them by pole-taming and those Distributions give back the starting points, the zero self-energy and the zero self-force. Now, the fictitious "energy-momentum tensor" for a point charge gives the originally not known self-force.

7. A general summary

Let us compare our present considerations with those in statics, see Section **4**.

(1) Starting point: a non-inertial point charge acts on itself, the self-force is unknown.
(2) Formulae (73) and (74) of self-force and self-power of a system of point charges suggested formulae (77) and (78) of self-force and self-power density of a continuous charge distribution.
(3) The false "energy density" (80), the questionable "energy flow" (81) and unjustified "stress tensor" (85) make questionable the balance equations (79) and (83).
   (3a) In spacetime formulation, the absolute self-force density is (94).
   (3b) It is not right to consider (95) a proper energy-moment tensor, in general; moreover, it is questionable that its negative spacetime divergence (96) is the absolute self-force density.
(4a) A Distribution can be attached to the fictitious "energy-momentum tensor" of a point charge by pole-taming.
(4b) The negative spacetime divergence of that Distribution is just the self-force due to radiation reaction.

Thus, at the end we obtained what was unknown in the first step. The negative divergence of the pole-tamed energy-momentum tensor of a point charge is the self-force which suggests us to accept that (96) gives the self-force in general, *i.e.*, (?) in (77), (78), and (94) can be omitted.

8. Radiation

8.1. Plane waves. Elementary experimental fact is that light has wave properties and transports energy and momentum. Light far from its source can be modelled approximately by plane wave solutions of the Maxwell equations. The assumption that in a standard inertial frame $\boldsymbol{E}\times\boldsymbol{B}$ is the energy flow density, *etc.*, of light agrees with experimental data, *i.e.* (79) and (83) (in a spacetime setting (96)) can be considered real balance equations for radiated electromagnetic fields.

8.2. Point charge. The Distribution

$$\mathrm{tm}\boldsymbol{T} := \boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}}^{td} + \boldsymbol{\mathcal{T}}^{rtd} + \boldsymbol{\mathcal{T}}^{rd} \tag{105}$$

is attached to the fictitious "energy-momentum tensor" (100) of a point charge on a **given** world line. Its negative divergence is the radiation self-force. The formulae given by Matolcsi (2023) show that both $\boldsymbol{\mathcal{T}}^{td}$ and $\boldsymbol{\mathcal{T}}^{rd}$ have zero divergence. Then it is seen clearly that the self-force is due to the interaction between the tied field and the radiated field.

It is known (Jackson 1998, p. 269) that the electric field and the magnetic field in $\boldsymbol{F}^{rd}$ have properties, similar to the properties of the fields in a plane wave. Then it seems natural the **conjecture** that (103) is a real energy-momentum tensor in the sense that its split components (80), (81) and (85) have the usual physical meaning.

8.3. Integral of point charges. The conjecture in the previous subsection seems satisfactory for a point charge; but what about the radiated energy-momentum tensor of an arbitrary absolute current? This question, connected with the problem how to define a center of mass, emerged (Gralla, Harte, and Wald 2009, p. 9). We give a particular answer as follows. The absolute current of a point charge e having the world line function r is $e\dot{r}\lambda_{\mathrm{Ran}(r)}$, where $\lambda_{\mathrm{Ran}(r)}$ is the Lebesgue measure of the world line $\mathrm{Ran}(r)$ (given by the usual integration along the world line). The absolute current consisting of point charges $e_1, \dots, e_n$ with world line functions $r_1, \dots, r_n$ is

$$\mathsf{j} := \sum_{k=1}^{n} e_k \dot{r}_k \lambda_{\mathrm{Ran}(r_k)}. \tag{106}$$

Then the radiated field is the sum of the radiated fields of the point charges,

$$\boldsymbol{F}^{rd} := \sum_{k=1}^{n} \boldsymbol{F}_k^{rd} \tag{107}$$

and the radiated energy-momentum tensor is

$$\begin{aligned}\boldsymbol{T}^{rd} :=& -\boldsymbol{F}^{rd} \cdot \boldsymbol{F}^{rd} - \frac{1}{4}\mathrm{Tr}\left(\boldsymbol{F}^{rd} \cdot \boldsymbol{F}^{rd}\right)\mathbf{1} = \\ =& \sum_i \sum_k \left(-\boldsymbol{F}_k^{rd} \cdot \boldsymbol{F}_k^{rd} - \frac{1}{4}\mathrm{Tr}(\boldsymbol{F}_i^{rd} \cdot \boldsymbol{F}_k^{rd})\mathbf{1}\right),\end{aligned}$$

We conceive that any current consists of point charges flowing in spacetime, as it occurs in our everyday electricity. This can be described by a simple generalization of the above formulae in such a way that there is a measure α on a σ-algebra of subsets of a set A and for each element a of A there is given a point charge e_a with world line function r_a; then

$$\mathsf{j} := \int_A e_a \dot{r}_a \lambda_{\mathrm{Ran}(r_a)} \, d\alpha(a) \tag{108}$$

which is defined in such a way that for an arbitrary test function ϕ

$$(\mathsf{j} \mid \phi) := \int_A \left(\int_{\mathbb{T}} (e_a \dot{r}_a(\boldsymbol{s}) \phi(r_a(\boldsymbol{s})) \, d\boldsymbol{s} \right) d\alpha(a). \tag{109}$$

Then the radiated field $\boldsymbol{F}^{rd}$ of the current is the integral of the radiated fields of the point charges, defined by

$$(\boldsymbol{F}^{rd} \mid \phi) := \int_A (\boldsymbol{F}_a^{rd} \mid \phi) \, d\alpha(a), \tag{110}$$

provided, of course, that the integral exists. The radiated energy-momentum tensor of the current is

$$(\boldsymbol{T}^{rd} \mid \phi) := \int_A \int_A \left(-\boldsymbol{F}_a^{rd} \cdot \boldsymbol{F}_b^{rd} - \frac{1}{4}(\boldsymbol{F}_a^{rd} \cdot \boldsymbol{F}_b^{rd})\mathbf{1} \,\Big|\, \phi \right) \, d\alpha(a) \, d\alpha(b), \tag{111}$$

It is a question, however, whether it suffices to consider absolute currents of this type only. The following answer could be encouraging. A continuous absolute current is a function $x \mapsto \rho(x)\boldsymbol{U}(x)$ defined in spacetime, where ρ is the absolute charge density and $\boldsymbol{U}$ is an absolute velocity field.

Proposition 8.1. *If $\boldsymbol{U}$ is smooth then the current $\rho\boldsymbol{U}$ is the integral of point charges if ρ is constant along every world line function r for which $\dot{r}(\boldsymbol{s}) = \boldsymbol{U}(r(\boldsymbol{s}))$ holds.*

Proof. Let us choose a spacelike hyperplane H in spacetime. For $q \in H$ let r_q be the world line function of $\boldsymbol{U}$ for which $r_q(0) = q$ holds. Then the map

$$F : H \times \mathbb{T} \to \mathrm{M}, \qquad (q, \boldsymbol{s}) \mapsto r_q(\boldsymbol{s}) \tag{112}$$

is injective because the solutions of a differential equation are uniquely determined by their initial values. Moreover, F is smooth:

$$\begin{aligned} r_{q'}(\boldsymbol{s}') - r_q(\boldsymbol{s}) &= \big(q' + \boldsymbol{U}(q')\boldsymbol{s}' + \mathrm{ordo}(\boldsymbol{s}')\big) - \big(q + \boldsymbol{U}(q)\boldsymbol{s} + \mathrm{ordo}(\boldsymbol{s})\big) = \\ &= \boldsymbol{q}' - \boldsymbol{q} + \boldsymbol{U}(q')\boldsymbol{s}' - \boldsymbol{U}(q)\boldsymbol{s} + \mathrm{ordo}(\boldsymbol{s}'\boldsymbol{s}) \end{aligned} \tag{113}$$

and

$$\begin{aligned} \boldsymbol{U}(q')\boldsymbol{s}' - \boldsymbol{U}(q)\boldsymbol{s} &= \big(\boldsymbol{U}(q) + (q' - q)\cdot \mathrm{D}\boldsymbol{U}(q) + \mathrm{ordo}(q' - q)\big)\boldsymbol{s}' - \boldsymbol{U}(q)\boldsymbol{s} = \\ &= \boldsymbol{U}(q)(\boldsymbol{s}' - \boldsymbol{s}) + (q' - q)\cdot \mathrm{D}\boldsymbol{U}(q) + \mathrm{ordo}(q' - q)\big)\boldsymbol{s}'. \end{aligned} \tag{114}$$

Then for a test function ψ

$$\int_{\mathrm{M}} \rho(x)\boldsymbol{U}(x)\psi(x)\,dx = \int_H \left(\int_{\mathbb{T}} \rho(r_q(\boldsymbol{s}))\boldsymbol{U}(r_q(\boldsymbol{s}))\psi(r_q(\boldsymbol{s}))Z(r_q(\boldsymbol{s}))\ d\boldsymbol{s} \right) dq \tag{115}$$

where $Z := |\det \mathrm{D}F| \circ F^{-1}$. ρ is constant on every world line, $\rho(r_q(\boldsymbol{s})) = \rho(r_q(0)) =: e_q$ and $\phi := \psi Z$ is a test function, thus we have got (109). ■

Remarks.

(1) The requirement that ρ be constant along every world line function of $\boldsymbol{U}$ is equivalent to that the velocity field be divergence-free. Indeed, if ρ is constant on the integral curves of $\boldsymbol{U}$, then $\boldsymbol{U}\cdot \mathrm{D}\rho = 0$, and the charge conservation requires $0 = \mathrm{D}\cdot \rho\boldsymbol{U} = \boldsymbol{U}\cdot \mathrm{D}\rho + \rho \mathrm{D}\cdot \boldsymbol{U}$.

(2) It suffices that $\boldsymbol{U}$ be continuously differentiable because (109) is meaningful for continuous test functions, too.

Acknowledgements

I am grateful to A. László and T. Fülöp for useful comments and to P. Ván for helping me to compose this article.

Statements and declarations

There are no financial interests related to the work.

Appendix A.

A.**1. Measures.** The Borel sets in spacetime or in the space of an observer are the elements of the σ-algebra generated by the open subsets. Elementary notions of (vector) measures and integration by them can be found in the article by Dinculeanu (1967). In this paper a (vector) measure is defined (at least) on the bounded Borel sets. The Dirac measure at a point x is defined by $\delta_x(H)=1$ if $x \in H$ and $\delta_x(H)=0$ is $x \notin H$, H being a Borel set.

A.**2. Distributions in spacetime.** Spacetime test functions are smooth functions in M or in **M** with compact support.

A (scalar, vector, tensor) Distribution is a (scalar, vector, tensor) valued continuous linear map defined on test functions. The action of a Distribution $\mathcal{K}$ on a test function ϕ is denoted by $(\mathcal{K} \mid \phi)$.

A (vector) measure is considered a Distribution by integrating test functions. Originally a Distribution acts on test functions but in some cases the action of a Distribution $\mathcal{K}$ on a locally integrable function f can be defined as follows. A sequence $n \mapsto \omega_n$ of test functions is said to converge to f in the Distribution sense if

$$\lim_{n\to\infty} \int \omega_n(x)\phi(x)\ dx = \int f(x)\phi(x)\ dx \tag{116}$$

for all test functions ψ. If $f=\lim_n \omega_n$ in the Distribution sense, then

$$(\mathcal{K} \mid f) := \lim_{n\to\infty} (\mathcal{K} \mid \omega_n) \tag{117}$$

if the limit exists. This definition is consistent: if f is a test function itself then the limit exists and equals the action of $\mathcal{K}$ on f.

Similar notions hold, according to the sense, for test functions and Distributions in the affine space S or vector space **S** of an inertial observer. In particular, for all $0<a<b$ there is a test function $\omega_{a,b}$ in **S** such that

$$\omega_{a,b}(\boldsymbol{q}) \begin{cases} =1 & \text{if} \quad |\boldsymbol{q}|<a, \\ =0 & \text{if} \quad |\boldsymbol{q}|>b, \\ \text{is between 0 and 1 if} & a \le |\boldsymbol{q}| \le b. \end{cases}, \tag{118}$$

and $1=\lim_{n\to\infty} \omega_{na,nb}$ in the Distribution sense.

A.**3. Three-dimensional integration and pole taming.** These topics can be found in the Appendices of the paper by Matolcsi (2023).

References

Barnett, S. M. (2010). “Resolution of the Abraham-Minkowski Dilemma”. *Physical Review Letters* **104**(7), 070401. doi: 10.1103/PhysRevLett.104.070401.

Bergmann, P. G. (1962). *Basic Theories of Physics : Mechanics and Electrodynamics*. New York: Dover Publications.

Bild, C., Deckert, D.-A., and Ruhl, H. (2019). “Radiation reaction in classical electrodynamics”. *Physical Review D* **99**(9), 096001. doi: 10.1103/PhysRevD.99.096001.

de Groot, S. R. and Suttorp, L. G. (1972). *Foundations of Electrodynamics*. Amsterdam: North-Holland.

Demidov, A. S. (2001). *Generalized Functions in Mathematical Physics*. Huntington: Nova Sciences.

Dinculeanu, N. (1967). *Vector Measures*. Vol. 95. International Series of Monographs on Pure and Applied Mathematics. Oxford: Pergamon Press. DOI: 10.1016/C2013-0-07847-4.

Feynman, R. P., Leighton, R. B., and Sands, M. (1963). *The Feynman Lectures on Physics*. Vol. 2. Reading, MA: Addison-Wesley.

Gel'fand, I. M. and Shilov, G. E. (1964). *Generalized Functions. Properties and Operations*. Vol. 1. New York: Academic Press. URL: https://archive.org/details/gelfand-shilov-generalized-functions-vol-1-properties-and-operations.

Gralla, S. E., Harte, A. I., and Wald, R. M. (2009). "Rigorous derivation of electromagnetic self-force". *Physical Review D* **80**(2), 024031. DOI: 10.1103/PhysRevD.80.024031.

Gsponer, A. (2008). "A concise introduction to Colombeau generalized functions and their applications in classical electrodynamics". *European Journal of Physics* **30**(1), 109. DOI: 10.1088/0143-0807/30/1/011.

Horváth, J. (2012). *Topological Vector Spaces and Distributions*. Mineola: Dover Publications.

Jackson, J. D. (1998). *Classical Electrodynamics*. 3rd. New York: Wiley.

Jiménez, J. L., Campos, I., and López-Mariño, M. A. (2011). "A new perspective of the Abraham-Minkowski controversy". *The European Physical Journal Plus* **126**(5), 50. DOI: 10.1140/epjp/i2011-11050-8.

Jiménez, J. L., Campos, I., and López-Mariño, M. A. (2013). "Maxwell's equations in material media, momentum balance equations and force densities associated with them". *The European Physical Journal Plus* **128**(4), 46. DOI: 10.1140/epjp/i2013-13046-8.

Leonhardt, U. (2006). "Momentum in an uncertain light". *Nature* **444**(7121), 823–824. DOI: 10.1038/444823a.

Lombardi, G. G. (1983). "Feynman's disk paradox". *American Journal of Physics* **51**(3), 213–214. DOI: 10.1119/1.13272.

Matolcsi, T. (2020). *Spacetime Without Reference Frames*. Montreal, Quebec, Canada: Minkowski Institute Press.

Matolcsi, T. (2021). *Klasszikus Elektromágnesség Téridőszemléletben*. Budapest, Hungary: Egyesület a Tudomány és Technológia Egységéért.

Matolcsi, T. (2023). "On the radiation reaction force". *Atti della Accademia Peloritana dei Pericolanti. Classe di Scienze Fisiche, Matematiche e Naturali* **101**(1), A1 [23 pages]. DOI: 10.1478/AAPP.1011A1.

Matolcsi, T., Fülöp, T., and Weiner, M. (2017). "Second-order equation of motion for electromagnetic radiation back-reaction". *Modern Physics Letters A* **32**(27), 1750147. DOI: 10.1142/S0217732317501474.

Maugin, G. A. (1988). "Continuum Mechanics of Electromagnetic Solids". In: *North-Holland Series in Applied Mathematics and Mechanics*. Ed. by J. D. Achenbach, B. Budiansky, W. T. Koiter, H. A. Lauwerier, and L. van Wijngaarden. 1st ed. Vol. 33. Amsterdam, The Netherlands: North-Holland.

Raju, S. and Raju, C. K. (2011). "Radiative damping and functional differential equations". *Modern Physics Letters A* **26**(35), 2627–2638. DOI: 10.1142/S021773231103698X.

Rohrlich, F. (2007). *Classical Charged Particles*. 3rd ed. Singapore: World Scientific.

Rowe, E. G. P. (1978). "Structure of the energy tensor in the classical electrodynamics of point particles". *Physical Review D* **18**(10), 3639–3654. DOI: 10.1103/PhysRevD.18.3639.

Spohn, H. (2000). "The critical manifold of the Lorentz–Dirac equation". *Europhysics Letters* **50**(3), 287–292. doi: 10.1209/epl/i2000-00268-x.

Taylor, J. G. (1958). "Classical electrodynamics as a distribution theory. II". *Mathematical Proceedings of the Cambridge Philosophical Society* **54**(2), 258–264. doi: 10.1017/S0305004100033429.

Ván, P. (1998). "Thermodynamical stabiliy of dia- and paramagnetic materials". *Periodica Polytechnica Chemical Engineering* **42**(2), 97–102. url: https://pp.bme.hu/ch/article/view/2551.

van Dijk, G. (2013). *Distribution Theory. Convolution, Fourier Transform, and Laplace Transform*. De Gruyter Textbook. Berlin: De Gruyter. doi: 10.1515/9783110298512.

Wang, Z. L. (2017). "On Maxwell's displacement current for energy and sensors: the origin of nanogenerators". *Materials Today* **20**(2), 74–82. doi: 10.1016/j.mattod.2016.12.001.

---

* Eötvös Loránd University,
Department of Applied Analysis and Computational Mathematics,
Budapest, Hungary

Email: matolcsi.tamas@gmail.com

Communicated 19 May 2023; manuscript received 24 June 2023; published online 26 January 2024