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SHOCK AND RAREFACTION WAVES IN A HYPERBOLIC MODEL OF INCOMPRESSIBLE MATERIALS

ANDREA MENTRELLI [a] AND TOMMASO RUGGERI [a*]

ABSTRACT. The aim of the present paper is to investigate shock and rarefaction waves in a hyperbolic model of incompressible materials. To this aim, we use the so-called *extended-quasi-thermal-incompressible* (EQTI) model, recently proposed by Gouin & Ruggeri [H. Gouin, T. Ruggeri, *Internat. J. Non-Linear Mech.* **47** 688–693 (2012)]. In particular, we use as constitutive equation a variant of the well-known Bousinnesq approximation in which the specific volume depends not only on the temperature but also on the pressure. The limit case of ideal incompressibility, namely when the thermal expansion coefficient and the compressibility factor vanish, is also considered.

Dedicated to Professor Giuseppe Grioli
on his 100th birthday.

1. Introduction

Mathematical modelling of incompressible materials has been given considerable attention during the past decades. This is due to the fact that, even thought fully incompressible materials do not exist in nature, incompressibility is a useful idealization when materials which exhibit extreme resistance to volume change are studied.

In the case of purely mechanical problems, i.e. when no change in temperature comes into play, the definition of incompressibility is clearly understood: a material is considered incompressible if its specific volume (or density) can be assumed to be constant. In this case, a broad literature is available concerning qualitative analysis as well as numerical methods for building the solutions of incompressible model equations as limits of the solutions of compressible ones, as the Mach number vanishes [1, 2, 3, 4].

In contrast to the purely mechanical case, the non isothermal case is not even well defined and several definitions of incompressibility, leading to different models – the most relevant of which are briefly reviewed in Section **3** – have been proposed over the years.

In order to unify the treatment of compressible and incompressible materials, the first step to be taken is to choose the pressure, instead of the density, as unknown field variable. In the case of a perfect fluid, i.e. when viscosity and heat conductivity may be neglected, and in the absence of external body forces, the system of conservation laws of mass, momentum and energy (Euler equations) is the following $(i, j = 1, 2, 3)$:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0, \\
&\frac{\partial \rho v_j}{\partial t} + \frac{\partial}{\partial x_i} \left(\rho v_i v_j + p \right) = 0, \\
&\frac{\partial \left(\rho \varepsilon + \frac{1}{2} \rho v^2 \right)}{\partial t} + \frac{\partial}{\partial x_i} \left(\left(\rho \varepsilon + \frac{1}{2} \rho v^2 + p \right) v_i \right) = 0,
\end{aligned} \tag{1}$$

where ρ, $\mathbf{v} \equiv (v_i)$, p and ε are, respectively, the density, the velocity, the pressure and the internal energy density (t represents the time and $\mathbf{x} \equiv (x_i)$ represents the space variable).

The system given in Eq. (1), together with the constitutive equations

$$\rho \equiv \rho \left(p, T \right), \qquad \varepsilon = \varepsilon \left(p, T \right), \tag{2}$$

is a closed system of equations in which the unknowns are the pressure p, the temperature T, and the velocity $\mathbf{v}$, i.e. the physical quantities commonly assumed as field variables.

The aim of this paper is to investigate shock and rarefaction waves in a hyperbolic model of incompressible materials. The so-called *extended-quasi-thermal-incompressible* (EQTI) model, recently proposed by Gouin & Ruggeri [5], is used to this aim. In this context, we use as constitutive equation a variant of the well-known Bousinnesq approximation, in which the specific volume depends not only on the temperature but also on the pressure. The sudy of shock and rarefaction waves is performed in the limit case of ideal perfect incompressibility, namely when the thermal expansion coefficient and the compressibility factor vanish. The case of practical interest in which the model parameters are chosen as to model the behaviour of water is studied as well.

Shock and rarefaction waves are crucial ingredients in the construction of the solution of the well-known Riemann problem, and the understanding of their features in EQTI materials will be utilized in the study of the Riemann problem which will be the subject of a forthcoming paper.

The paper is organized as follows. In Section **2** the restrictions that should be satisfied by the constitutive equations of a material in order to be thermodynamically consistent (i.e. the entropy principle and the requirement of thermodynamic stability) are briefly recalled.

In Section **3** the model of *perfectly incompressible* material proposed by I. Müller is reviewed, along with two models which have recently been proposed with the aim of overcoming the main drawbacks of the Müller's model, namely the *quasi-thermal-incompressible* (QTI) model [6] and the *extended-quasi-thermal-incompressible* (EQTI) model [5].

In Section **4** a simple equation of state of a EQTI material, first proposed in [5], is presented and discussed.

In Section **4.1**, the Rankine-Hugoniot conditions governing the propagation of a shock wave in the hyperbolic system of Euler equations are exploited for a material modelled by the proposed equation of state and the analysis of the behaviour of shock waves is performed in the case in which the model parameters as chosen in agreement with the experimental data of water as well as in the *incompressible limit*, i.e. when the thermal expansion coefficient and the compressibility factor of the material vanish.

In Section **4.2**, making use of the theory of hyperbolic system, the integral curves of the genuinely nonlinear fields of the system are analytically calculated and, as done for the case of shock waves, the propagation of rarefaction waves is analysed both in the case of water and in the incompressible limit.

Finally, in Section **5** the conclusions concerning the study of wave propagation in a EQTI material are drawn.

2. Thermodynamic restrictions

Assuming that the constitutive equations of a material are given by Eq. (2), the entropy principle and the requirement of thermodynamic stability lead to thermodynamic restrictions that narrow the arbitrariness of the constitutive equations [5].

Entropy principle. In local thermodynamic equilibrium, the entropy principle requires that the Gibbs equation holds:

$$TdS = d\varepsilon + pdV \tag{3}$$

where S is the entropy density and V is the specific volume ($V = 1/\rho$). As the most convenient independent variables to adopt are the pressure p and the temperature T, the natural thermodynamic potential to be introduced is the chemical potential μ:

$$\mu = \varepsilon + pV - TS. \tag{4}$$

Combining Eq. (3) and Eq. (4), it is easily seen that

$$d\mu = Vdp - SdT$$

and

$$V = \mu_p, \qquad S = -\mu_T, \qquad \varepsilon = \mu - p\mu_p - T\mu_T. \tag{5}$$

Above and in the following the subscript denotes partial differentiation, i.e.

$$f_p = \left(\frac{\partial f}{\partial p}\right)_T, \qquad f_T = \left(\frac{\partial f}{\partial T}\right)_p,$$

and a superscript $'$ will be used to denote ordinary derivatives of functions that depend only on a single variable: $g'(T) = dg/dT$.

Once the constitutive equation $(2)_1$ is given, in order for the entropy principle to be fulfilled, the chemical potential μ, the entropy density S and the internal energy density ε must satisfy the following relations:

$$\begin{aligned} \mu &= \int V(p,T)\,dp + \tilde{\mu}(T), \\ S &= -\int V_T(p,T) - \tilde{\mu}'(T), \\ \varepsilon &= \int Vdp - pV - T\int V_T dp + e(T), \end{aligned} \tag{6}$$

where $e \equiv e(T)$ is a constitutive function which, together with $V \equiv V(p,T)$, fully characterize the material.

Upon comparison of Eq. (5) and Eq. (6), it is seen that:

$$e(T) = \tilde{\mu}(T) - T\tilde{\mu}'(T), \quad \text{or} \quad \tilde{\mu}(T) = -T\int \frac{e(T)}{T^2}dT. \tag{7}$$

Thermodynamic stability. The condition that guarantees the thermodynamic stability is the concavity of the chemical potential μ. Introducing the specific heat at constant pressure, c_p, defined as the partial derivative with respect to the temperature, at constant pressure, of the enthalpy h (the latter being defined as $h \equiv \varepsilon + pV$) i.e.

$$c_p = e'(T) - T \int V_{TT} dp,$$

the concavity of the chemical potential μ requires

$$\begin{aligned} &\mu_{pp} = V_p < 0, \\ &\mu_{TT} = -c_p/T < 0, \\ &J \equiv \mu_{TT}\mu_{pp} - \mu_{Tp}^2 = -c_p V_p/T - V_T^2 > 0 \end{aligned} \tag{8}$$

which are equivalent to

$$c_p > 0, \qquad V_p < -\frac{TV_T^2}{c_p}. \tag{9}$$

Recalling that the adiabatic sound velocity, c, is given by

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_S} = \sqrt{-V^2 \left(\frac{\partial p}{\partial V}\right)_S}$$

and that, from Eq. (5),

$$dV = \mu_{Tp} dT + \mu_{pp} dp, \qquad dS = -\mu_{TT} dT - \mu_{Tp} dp,$$

the adiabatic sound velocity c may be written as

$$c = \sqrt{-\frac{\mu_p^2 \mu_{TT}}{J}}, \tag{10}$$

which clearly shows that the concavity of the chemical potential μ automatically ensures the positivity of the adiabatic sound velocity c, thus guaranteeing the hyperbolicity of the Euler system of equations given in Eq. (1).

When focusing on the compressibility features of a material, it is generally useful to introduce the *thermal expansion coefficient*, α, and the *compressibility factor*, β, as follows:

$$\alpha = \frac{V_T}{V}, \qquad \beta = -\frac{V_p}{V}.$$

The condition given in Eq. $(9)_2$ may thus be rewritten as

$$\beta > \beta_{cr}, \qquad \beta_{cr} = \frac{\alpha^2 TV}{c_p} > 0, \tag{11}$$

and the adiabatic sound velocity may be written as:

$$c = \sqrt{\frac{V}{\beta - \beta_{cr}}}. \tag{12}$$

From the above discussion it is seen that, in order for a material to satisfy the entropy principle and the requirement of thermodynamic stability, the following conditions must hold:

(1) the constitutive functions $V \equiv V(p,T)$ and $e \equiv e(T)$ – or, equivalently, $V \equiv V(p,T)$ and $\varepsilon \equiv \varepsilon(p,T)$ – must satisfy the relations given in Eq. (6)-(7);
(2) the compressibility factor β must be greater than a threshold value as stated in Eq. (11).

It is noticeable how the compressibility of a thermodynamically consistent material may be possibly *very small* but not zero, thus confirming the experimental evidence according to which fully incompressible materials do not exist in nature.

3. Models of incompressibility

With the aim of providing useful models for incompressible materials in the non-isothermal case, several definitions of incompressibility have been proposed over the years. Each of the proposed models is based on some simplifying assumptions concerning the constitutive equations (2). Unfortunately, the proposed model are not always compatible with the thermodynamic restrictions discussed in Section **2**.

Perfectly incompressible material. A simple model, well justified by experimental evidence, was proposed by I. Müller [7]. According to the Müller's model, a material is incompressible when its constitutive equations do not depend on the pressure. Eq. (2) may thus be written as:

$$\rho \equiv \rho(T), \quad \varepsilon \equiv \varepsilon(T).$$

This model, although being very attractive for its simplicity and its adherence to the most intuitive idea of incompressibility, is affected by a serious inconvenience: the only constitutive function $\rho(T)$ which does not violate the entropy principle is the constant function $\rho = \rho_0$.

This may be seen differentiating Eq. $(5)_3$ with respect to p and taking into account that $\varepsilon_p = V_p = 0$:

$$\varepsilon_p = -p\mu_{pp} - T\mu_{Tp} = -pV_p - TV_T \quad \Rightarrow \quad V_T = 0.$$

This result, which is clearly in contrast with the experimental evidence as well as with a successful model as the Boussinesq approximation [8, 9], is known as *Müller paradox* and was first pointed out by Müller himself [7, 6].

Quasi-thermal-incompressible material. A second, less restrictive, model of incompressibility requires that the only constitutive function not depending on the pressure be the density. Under this assumption, Eq. (2) is written as:

$$\rho \equiv \rho(T), \quad \varepsilon \equiv \varepsilon(p,T).$$

This model, which has been named *quasi-thermal-incompressible* (QTI) model [6] is not affected by the stark shortcoming of the Müller's model and has the additional advantage that the latter can be obtained as a limit case from the quasi-thermal-incompressible model.

In fact, in this case Eq. $(6)_1$ reads

$$\mu(p,T) = V(T)\,p + \tilde{\mu}(T) \tag{13}$$

which, combined with Eq. $(5)_3$, gives

$$\varepsilon(p,T) = -TV'(T)\,p + e(T). \tag{14}$$

For *sufficiently small* values of the pressure p, more specifically when

$$p \ll \frac{e\,(T)}{\|V'\|T},$$

the quasi-thermal-incompressible model approximates the perfectly incompressible model by Müller [6].

Nonetheless, the quasi-thermal-incompressible model is still not completely satisfying. In fact, it is easily seen from Eq. $(9)_1$ that a constitutive function $\rho \equiv \rho\,(T)$ leads to a non-concave chemical potential μ which, in turn, leads to imaginary sound velocity: the system of the Euler equations becomes of elliptic type and instabilities in wave propagation occur [10].

Extended-quasi-thermal-incompressible material. In order to overcome also the limitations of the quasi-thermal-incompressible model, a new model of incompressibility has recently been proposed by Gouin & Ruggeri [5].

According to this model, a material is said to be *extended-quasi-thermal-incompressible* (EQTI) if its constitutive equations satisfy the thermodynamic restriction outlined in Section **2** and if there exist two functions $\hat{V} \equiv \hat{V}\,(T)$ and $\hat{\varepsilon} \equiv \hat{\varepsilon}\,(T)$ such that:

$$\begin{aligned} &V\,(p,T) = \hat{V}\,(T) + \mathcal{O}\left(\delta^2\right), \qquad \text{with} \quad \hat{V}'\,(T) = \mathcal{O}\,(\delta)\,, \\ &\varepsilon\,(p,T) = \hat{\varepsilon}\,(T) + \mathcal{O}\left(\delta^2\right) \end{aligned} \tag{15}$$

where δ is a dimensionless parameter such that $\delta \ll 1$. Moreover, given the reference state (V_0, p_0, T_0), it is assumed that the thermal expansion coefficient and the compressibility factor at the reference state – respectively, α_0 and β_0 – are such that

$$\alpha_0 T_0 = \delta, \qquad \beta_0 p_0 = \mathcal{O}\left(\delta^2\right). \tag{16}$$

The latter assumptions are in agreement with experimental results, according to which materials usually considered as incompressible exhibit a *small* thermal expansion coefficient and a *very small* compressibility factor.

After some calculations (see [5]), it is seen that for a EQTI material, the specific volume V and the specific internal energy ε are represented as

$$\begin{aligned} V\,(p,T) &= V_0 + \delta W\,(T) - \delta^2 U\,(p,T)\,, \\ \varepsilon\,(p,T) &= e\,(T) - \delta p T W'\,(T) + \mathcal{O}\left(\delta^2\right), \end{aligned}$$

where $W\,(T)$ and $U\,(p,T)$ are two constitutive functions chosen in agreement with the thermodynamic restrictions outlined in Section **2**. Under these assumptions, a an EQTI material is a thermodynamically consistent compressible material which approximates a perfectly incompressible material to the order δ^2, provided that the pressure p does not exceed the critical value p_{cr}:

$$p \ll p_{cr}, \qquad p_{cr} = \frac{e\,(T)}{\delta T W'\,(T)}. \tag{17}$$

4. An example of EQTI material

A first example of constitutive equations for an EQTI material is obtained performing a linear expansion of the specific volume V near the reference state (V_0, p_0, T_0) and assuming for the constitutive function e a linear dependence on the temperature:

$$\begin{aligned} &V\,(p,T) = V_0\,(1 + \alpha_0\,(T - T_0) - \beta_0\,(p - p_0))\,, \\ &e\,(T) = c_p\,T, \end{aligned} \tag{18}$$

where α_0, β_0 and c_p are positive constants. It is seen that α_0 and β_0 represent, respectively, the thermal expansion coefficient and the compressibility factor at the reference state (V_0, p_0, T_0), and c_p represents the specific heat at constant pressure.

It is worth noticing that Eq. $(18)_1$ coincides, for $\beta_0 = 0$, with the well-known Bousinnesq approximation. As seen in Section **2**, the necessity of assuming, in order to have a thermodinamically stable material, $\beta_0 > \beta_{cr}$ (with $\beta_{cr} \geq 0$), suggests that Eq. $(18)_1$ may be regarded as the natural correction of the Bousinnesq equation.

It may be seen that, after introducing the parameter σ such that $\sigma = \beta_0 p_0/\delta^2$ and taking into account that $\alpha_0 = \delta/T_0$ (see Eq. $(16)_1$), the constitutive equation $(18)_1$ may be written as:

$$V = V_0\left(1 + \delta\,\frac{T - T_0}{T_0} - \sigma\,\delta^2\,\frac{p - p_0}{p_0}\right). \tag{19}$$

The thermal expansion coefficient and the compressibility factor for compressible materials described by the equations of state (18) are given by:

$$\alpha = \alpha_0 V_0/V, \qquad \beta = \beta_0 V_0/V,$$

or, in terms of the parameters δ and σ,

$$\alpha = \frac{\delta}{V}\,\frac{V_0}{T_0}, \qquad \beta = \frac{\sigma\,\delta^2}{V}\,\frac{V_0}{p_0}.$$

As shown in [5], the compressible material modelled by the constitutive equations given in Eqs. $(18)_2$-(19) is an EQTI material for which the constitutive functions W and U introduced in Section **3** are given by:

$$W\,(T) = V_0\frac{T - T_0}{T_0}, \qquad U\,(p,T) = \sigma\,V_0\frac{p - p_0}{p_0}.$$

Following Eq. $(17)_2$, the critical pressure p_{cr} becomes

$$p_{cr} = \frac{1}{\delta}\frac{c_p T_0}{V_0}$$

and the condition (11), which guarantees the thermodynamic stability of the material, is written in terms of the parameter σ as follows:

$$\sigma > \sigma_{cr}, \qquad \sigma_{cr} = \frac{V_0 p_0}{c_p T_0}. \tag{20}$$

Provided that Eq. (20) is satisfied, the Euler system of equations (1) is hyperbolic and the adiabatic sound velocity c reads:

$$c = \frac{1}{\delta}\sqrt{\frac{p_0 T_0 V^2}{V_0\,(T_0\sigma - T\sigma_{cr})}}. \tag{21}$$

From the Gibbs equation (see Eq. (3)), it is obtained:

$$dS = \left(\frac{c_p}{T} - \frac{p_0}{\sigma \rho_0 T_0^2}\right) dT - \frac{p_0}{\delta \sigma T_0 \rho^2} d\rho \tag{22}$$

and the entropy density S reads

$$S = c_p \log T - \frac{p_0}{\sigma \rho_0 T_0^2} T + \frac{p_0}{\delta \sigma T_0} \frac{1}{\rho} + S_0 \tag{23}$$

where S_0 is a constant.

4.1. Rankine-Hugoniot conditions and shock waves. The system of conservation laws of mass, momentum and total energy for a perfect fluid (Euler equations), already recalled in Section **1**, may be written in the one–dimensional case as follows:

$$\partial_t \mathbf{u} + \partial_x \mathbf{F}(\mathbf{u}) = 0, \tag{24a}$$

$$\mathbf{u} \equiv \begin{pmatrix} \rho \\ \rho v \\ \rho\varepsilon + \rho v^2/2 \end{pmatrix}, \quad \mathbf{F} \equiv \begin{pmatrix} \rho v \\ \rho v^2 + p \\ \left(\rho\varepsilon + \rho v^2/2 + p\right) v \end{pmatrix}. \tag{24b}$$

The characteristic velocities of the above system, i.e. the eigenvalues of the matrix $\nabla \mathbf{F}$ ($\nabla \equiv \partial/\partial \mathbf{u}$), are the following:

$$\lambda^{(1)} = v - c, \qquad \lambda^{(2)} = v, \qquad \lambda^{(3)} = v + c, \tag{25}$$

and the corresponding eigenvectors are:

$$\mathbf{r}^{(1)} = \begin{pmatrix} 1 \\ v - c \\ \varepsilon + \dfrac{v^2}{2} + \dfrac{p}{\rho} - c\,v \end{pmatrix}, \quad \mathbf{r}^{(2)} = \begin{pmatrix} 1 \\ v \\ \dfrac{v^2}{2} \end{pmatrix}, \quad \mathbf{r}^{(3)} = \begin{pmatrix} 1 \\ v + c \\ \varepsilon + \dfrac{v^2}{2} + \dfrac{p}{\rho} + c\,v \end{pmatrix}.$$

A *shock wave* is a weak solution of the system (24) characterized by a discontinuity propagating with velocity s. Denoting as *unperturbed* and *perturbed* states (respectively, $\mathbf{u}_0$ and $\mathbf{u}_1$), the states before and after the discontinuity *(shock front)*, it is well-known that a shock wave must satisfy the Rankine-Hugoniot conditions [11]:

$$-\, s[\![\mathbf{u}]\!] + [\![\mathbf{F}(\mathbf{u})]\!] = 0, \tag{26}$$

where $[\![\varphi(\mathbf{u})]\!] = \varphi(\mathbf{u}_1) - \varphi(\mathbf{u}_0)$ represents the discontinuity (jump) of the generic quantity φ across the shock front.

In the case of a material described by Eq. (24), the Rankine-Hugoniot conditions (26) are written as follows:

$$-s[\![\rho]\!] + [\![\rho v]\!] = 0, \tag{27a}$$

$$-s[\![\rho v]\!] + [\![\rho v^2 + p]\!] = 0, \tag{27b}$$

$$-s[\![\rho\varepsilon + \rho v^2/2]\!] + [\![\left(\rho\varepsilon + \rho v^2/2 + p\right) v]\!] = 0. \tag{27c}$$

The above system of equations admits a one-parameter family of solutions; denoting with η the parameter, the perturbed states $\mathbf{u}_1$ that can be connected to a given unperturbed state $\mathbf{u}_0$ (which in the following will be assumed to be coincident with the reference state introduced in Section **4**) through a shock wave, and the velocity of propagation of the shock front s, may be written as

$$\mathbf{u}_1 \equiv \mathbf{u}_1 (\mathbf{u}_0, \eta), \qquad s \equiv s(\mathbf{u}_0, \eta). \tag{28}$$

Differentiating Eq. (26) with respect to η ($' = d/d\eta$) and then setting $\eta = 0$, gives:

$$\nabla \mathbf{F}(\mathbf{u}_0)\, \mathbf{u}_1' (\mathbf{u}_0, 0) = s(\mathbf{u}_0, 0)\, \mathbf{u}_1' (\mathbf{u}_0, 0),$$

which shows that $\mathbf{u}_1' (\mathbf{u}_0, 0)$ is an eigenvector of the matrix $\nabla \mathbf{F}$ and $s(\mathbf{u}_0, 0)$ is the corresponding eigenvalue. The curve $\mathbf{u}_1' (\mathbf{u}_0, \eta)$ is thus tangent to an eigenvector of $\nabla \mathbf{F}$: if it is tangent to the k^{th} eigenvector, denoted as $\mathbf{r}^{(k)}$, the states lying on this curve are said to form, together with $\mathbf{u}_0$, a *k–shock*, and the curve $\mathbf{u}_1 (\mathbf{u}_0, \eta)$ is called the *Hugoniot locus* of the k–family of shocks and is denoted with $\mathcal{S}^{(k)} (\mathbf{u}_0)$.

In the wide literature concerning shock waves in fluids, the shock front velocity s is often replaced by the *unperturbed Mach number* $M_0 = (s - v_0)/c_0$, where v_0 and c_0 are respectively the fluid velocity and the adiabatic sound velocity – see Eq. (21) – in the unperturbed state, and M_0 is usually chosen as parameter η. Nonetheless, in the present situation it turns out that it is more convenient to choose as parameter the pressure of the perturbed state, p_1.

Assuming, without loss of generality due to Galilean invariance, $v_0 = 0$, and considering only the 3–shock wave (propagating in the positive x-direction), after the introduction of the dimensionless variables $\hat{p}$, $\hat{V}$, $\hat{T}$ and $\hat{v}$ as follows:

$$\hat{p} = p/p_0, \quad \hat{V} = V/V_0, \quad \hat{T} = T/T_0, \quad \hat{v} = v/v_* \tag{29}$$

(being $v_* = \sqrt{p_0 V_0}$ a suitable parameter introduced in order to conveniently nondimensionalize the velocities), the Rankine-Hugoniot conditions provide the dimensionless specific volume $\hat{V}_1$, temperature $\hat{T}_1$, and velocity $\hat{v}_1$ as functions of the dimensionless pressure $\hat{p}_1$ (the subscripts '0' and '1' denote quantities evaluated, respectively, in the unperturbed and perturbed states) which, assuming that δ is *small*, take the form:

$$\begin{aligned} \hat{V}_1 &= 1 - (\sigma - \sigma_{cr})(\hat{p}_1 - 1)\,\delta^2 + \mathcal{O}\left(\delta^3\right), \\ \hat{T}_1 &= 1 + \sigma_{cr}(\hat{p}_1 - 1)\,\delta + \mathcal{O}\left(\delta^2\right), \\ \hat{v}_1 &= (\sigma - \sigma_{cr})^{1/2}(\hat{p}_1 - 1)\,\delta + \mathcal{O}\left(\delta^2\right). \end{aligned} \tag{30}$$

which define the Hugoniot locus $\mathcal{S}^{(3)}$.

The Mach number in the unperturbed and perturbed state (respectively, M_0 and M_1), and the dimensionless velocity of the shock front $\hat{s}$ ($\hat{s} = s/v_*$), are the following:

$$M_0 = 1 + \frac{\sigma_{cr}^2 (\hat{p}_1 - 1)}{4(\sigma - \sigma_{cr})}\,\delta + \mathcal{O}\left(\delta^2\right), \tag{31a}$$

$$M_1 = 1 - \frac{\sigma_{cr}^2 (\hat{p}_1 - 1)}{4(\sigma - \sigma_{cr})}\,\delta + \mathcal{O}\left(\delta^2\right), \tag{31b}$$

and

$$\hat{s} = \left(\frac{\sigma_{cr}(\hat{p}_1 - 1)}{\sigma - \sigma_{cr}}\right)^{1/2} \frac{1}{\delta^{1/2}} + \mathcal{O}\left(\delta^{1/2}\right). \tag{32}$$

Shock waves in water. In order to carry out a quantitative analysis of the features of a shock wave propagating in a EQTI material satisfying Eq. (18), the following values are assumed for the parameters included in the model [12]:

$$V_0 = 10^{-3}\,\mathrm{m}^3\,\mathrm{Kg}^{-1}, \quad p_0 = 10^5\ \mathrm{Pa}, \quad T_0 = 293\ \mathrm{K},$$
$$c_p = 4.2 \times 10^3 \mathrm{J}\,\mathrm{Kg}^{-1}\,\mathrm{K}^{-1}, \quad \delta = 0.061, \quad \sigma = 0.014,$$

which correspond to the experimental values obtained for water at the room reference state.

In this case, the solution of the Rankine-Hugoniot conditions put into evidence that the specific volume $\hat{V}_1$, the temperature $\hat{T}_1$ and the velocity $\hat{v}_1$ of the perturbed state all show almost a linear variation with the pressure $\hat{p}_1$.

It may be appreciated how an increase in pressure induced by the shock wave up to 10^3 times the reference (atmospheric) pressure causes only a *small* increase in density and temperature. Interestingly enough, the variations of the Mach number (both perturbed and unperturbed) from the unity is almost negligible.

It is also noted that the value of the critical pressure, given the above-listed values, is $p_{cr} \approx 2.02 \times 10^{10}$, well above the phisically meaniungful values of the pressure.

Shock waves in the incompressible limit. Since in order to have a thermodynamically stable material it must be $\sigma > \sigma_{cr}$, it is clearly seen that the solution of the Rankine-Hugoniot conditions admits the following *incompressible limit*, obtained as $\delta \to 0$, i.e. as the thermal expansion coefficient and the compressibility factor vanish:

$$\lim_{\delta\to 0} V_1 = V_0, \qquad \lim_{\delta\to 0} T_1 = T_0, \qquad \lim_{\delta\to 0} v_1 = v_0 = 0,$$

and

$$\lim_{\delta\to 0} M_0 = \lim_{\delta\to 0} M_1 = 1, \qquad \lim_{\delta\to 0} s = +\infty.$$

In order to analyse in some detail how the shock wave behaves in the *incompressible limit*, it may be helpful to consider the solution of the Rankine-Hugoniot conditions for decreasing values of δ, namely $\delta = 10^{-4}$, $\delta = 0.5 \times 10^{-4}$ and $\delta = 10^{-5}$, given $\sigma = 0.014$ as in the case of water.

The behaviours of the dimensionless specific volume $\hat{V}_1$, temperature $\hat{T}_1$, and velocity $\hat{v}_1$ of the perturbed state, shown in Fig. 1 together with the dimensionless velocity of the shock front $\hat{s}$, put into evidence that along with the expected reduction of the variation of the specific volume as δ decreases, the variations in temperature and in velocity of the perturbed state become also less and less relevant as the fluid approaches the incompressible limit (i.e. when $\delta \to 0$). Interestingly, the velocity of the shock front, which is only slightly affected by the value of the pressure jump across the shock, becomes larger and larger as the incompressible limit is attained, thus confirming the expectations according to which the shock velocity becomes infinite in the purely theoretical case of a perfectly incompressible fluid ($\delta = 0$). Nonetheless, the variations of the Mach number across the shock wave, shown in Fig. 2, become smaller and smaller as $\delta \to 0$.

It should also be mentioned that the behaviours of the quantities given in Eq. (30)-(32) are plotted in Fig. 1 and Fig. 2 for a broad range of values of the perturbed state pressure, $1 < \hat{p}_1 < 3 \times 10^8$, which may seem to be not physically meaningful. This is done in order to put into evidence the behaviour of the physical quantities over the whole range of validity of the model, i.e. up to the critical pressure p_{cr} (the curves are plotted in

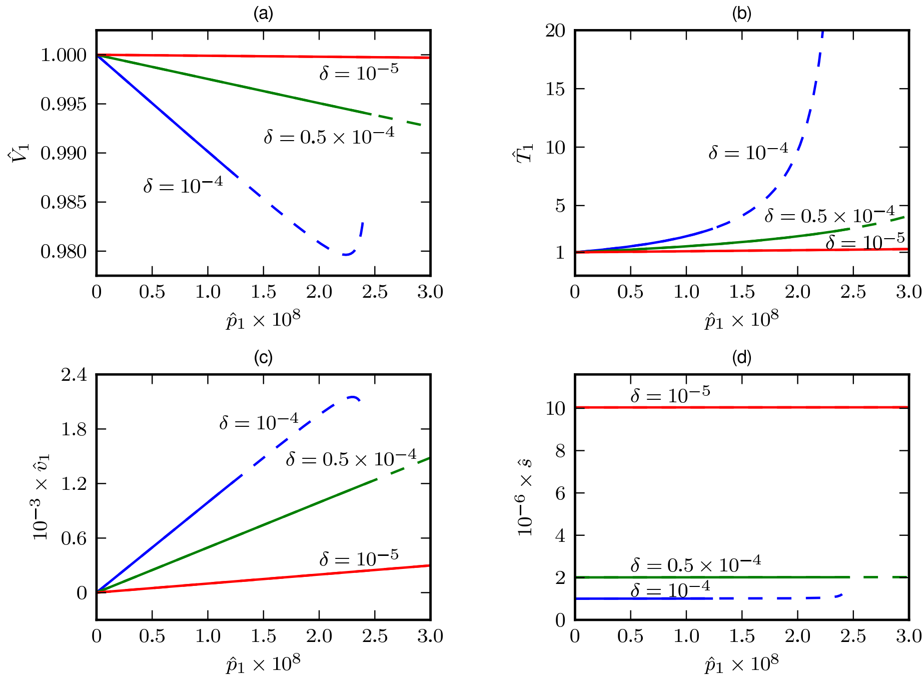

Figure 1. From top to bottom and from left to right: dimensionless specific volume ($\hat{V}_1$), temperature ($\hat{T}_1$), velocity ($\hat{v}_1$) of the perturbed state of the 3–shock and velocity of the shock front ($\hat{s}$) as functions of the dimensionless pressure of the perturbed state ($\hat{p}_1$) for three different degrees of compressibility ($\delta = 10^{-4}$, $\delta = 0.5 \times 10^{-4}$ and $\delta = 10^{-5}$).

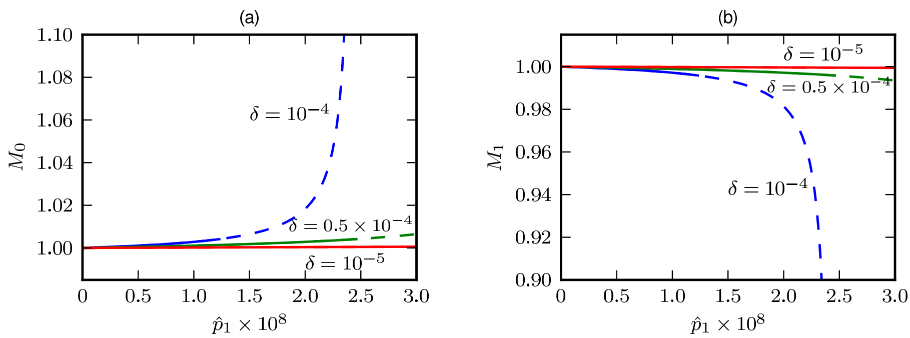

Figure 2. From left to right: Mach number of the unperturbed state (M_0) and of the perturbed state (M_1) for the 3–shock as functions of the dimensionless pressure of the perturbed state ($\hat{p}_1$) for three different degrees of compressibility ($\delta = 10^{-4}$, $\delta = 0.5 \times 10^{-4}$ and $\delta = 10^{-5}$).

dashed lines above the critical pressure, i.e. when the fluid may not be considered EQTI). Physically meaningful values of the pressure are well below the critical value, and for these values the variations in the specific volume are so small as to be considered negligible, as it is expected in an incompressible fluid.

4.2. Shock admissibility and rarefaction waves. According to the theory of hyperbolic systems, not every solution of the Rankine-Hugoniot conditions represents a shock wave acceptable from a physical point of view: A selection rule to determine which of the states $\mathbf{u}_1 \in \mathcal{S}^{(k)}(\mathbf{u}_0)$ are perturbed states that, together with $\mathbf{u}_0$, form *admissible* k–shocks is needed.

For genuinely non-linear waves, i.e. for the waves such that $\nabla\lambda \cdot \mathbf{r} \neq 0 \ \forall \mathbf{u}$, the selection rule is given by the *Lax condition*, according to which a shock wave is admissible if there exists a characteristic velocity λ such that [11]:

$$\lambda_0 < s < \lambda_1$$

where $\lambda_0 \equiv \lambda(\mathbf{u}_0)$ and $\lambda_1 \equiv \lambda(\mathbf{u}_1)$. If λ is the k^{th} eigenvalue of the system, namely if $\lambda \equiv \lambda^{(k)}$, assuming that $\lambda^{(1)} < \lambda^{(2)} < \ldots < \lambda^{(N)}$ (where N is the order of the system and assuming that all the eigenvalues have multiplicity equal to one), such a shock wave is indeed the *k–shock* [13].

On the other hand, for linearly degenerate wave, i.e. for the waves such that $\nabla\lambda \cdot \mathbf{r} \equiv 0 \ \forall \mathbf{u}$, admissible *k–shocks* are called *characteristic shocks* and they propagate with velocity $s = \lambda_0 = \lambda_1$.

Since in the case of the equation of states discussed here the case of locally linearly degerate waves (waves such that $\nabla\lambda \cdot \mathbf{r} = 0$ for some $\mathbf{u}$) is not relevant, the selection rule to be adopted in this circumstance (which has already been thoroughly discussed elsewhere [14, 15, 16, 17]) shall not be discussed here.

When, according to the selection rule, the shock wave connecting two states $\mathbf{u}_0$ and $\mathbf{u}_1$ is not admissible ($\mathbf{u}_1 \notin \mathcal{S}^{(k)}(\mathbf{u}_0)$ for any k), the initial discontinuity cannot propagate: instead, it breaks in general in the combination of a shock wave, a discontinuity wave and a rarefaction wave, leading to the so-called *Riemann problem*.

A rarefaction wave is a similarity solution of the system (24), namely it is a solution constant along all the rays of the form $x = \xi t$. A rarefaction wave is thus a solution of the form:

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_1 & x \leq \xi_1 t \\ \tilde{\mathbf{u}}(x/t) & \xi_1 t < x < \xi_0 t \\ \mathbf{u}_0 & x \geq \xi_0 t \end{cases},$$

where $\tilde{\mathbf{u}}$ is a smooth function such that $\tilde{\mathbf{u}}(\xi_0) = \mathbf{u}_0$ and $\tilde{\mathbf{u}}(\xi_1) = \mathbf{u}_1$. As seen for the case of shock waves, given a state $\mathbf{u}_0$ (which, in the following, shall be assumed to be coincident with the reference state previously introduced in Section **4**) it is possible to obtain the locus of the states $\mathbf{u}_1$ which can be connected to $\mathbf{u}_0$ by a rarefaction wave. This locus is called *integral curve* and is denoted by $\mathcal{R}(\mathbf{u}_0)$.

As known from the theory (see, for example, [18]), except for the linearly degenerate field ($k = 2$), the integral curves are obtained by solving the following system of ordinary differential equations:

$$
\begin{aligned}
\frac{d\rho}{d\xi} &= \frac{\rho}{\rho c_\rho + c}, \\
\frac{dv}{d\xi} &= (k-2)\,\frac{c}{\rho c_\rho + c}, \qquad (k=1,3)\,, \\
\frac{dS}{d\xi} &= 0.
\end{aligned} \tag{33}
$$

Combining Eq. $(33)_1$ and Eq. $(33)_2$, one obtains

$$
\frac{dv}{d\rho} = (k-2)\;\frac{c}{\rho}
$$

and, taking into account Eq. $(33)_3$ and Eq. (22), it is easily seen that:

$$
\frac{dv}{dT} = (k-2)\,\frac{\sigma c_p \rho_0 T_0^2 - p_0 T}{T\sqrt{\rho_0 p_0 T_0\left(\sigma T_0 - \sigma_{cr} T\right)}}. \tag{34}
$$

Eq. (34) provides:

$$
v = v_c + \frac{2\;(k-2)}{\sigma_{cr}\sqrt{\rho_0 p_0}}\left(p_0\sqrt{\sigma - \frac{\sigma_{cr}T}{T_0}} - \sigma_{cr}\rho_0 T_0 c_p\sqrt{\sigma}\,\mathrm{arctanh}\sqrt{1-\frac{\sigma_{cr}T}{\sigma T_0}}\right), \tag{35}
$$

where $k=1,3$ and

$$
v_c = v_0 - \frac{2\;(k-2)}{\sigma_{cr}\sqrt{\rho_0 p_0}}\left(p_0\sqrt{\sigma-\sigma_{cr}} - \sigma_{cr}\rho_0 T_0 c_p\sqrt{\sigma}\,\mathrm{arctanh}\sqrt{1-\sigma_{cr}/\sigma}\right),
$$

being v_0 the velocity in the state $\mathbf{u}_0$ (which can be assumed without loss of generality, as remarked in Section **4.1**, to be $v_0=0$). Eq. (35), together with Eq. $(33)_3$ and Eq. (23), allows to completely determine the states $\mathbf{u}$ lying on the integral curves $\mathcal{R}^{(k)}\left(\mathbf{u}_0\right)$ $(k=1,3)$. These states, in terms of the dimensionless variables and using the pressure $\hat{p}$ as parameter, are defined by (when δ is *small*):

$$
\begin{aligned}
\hat{V} &= 1 - (\sigma-\sigma_{cr})\,(\hat{p}-1)\,\delta^2 + \mathcal{O}\left(\delta^3\right), \\
\hat{T} &= 1 + \sigma_{cr}\,(\hat{p}-1)\,\delta + \mathcal{O}\left(\delta^2\right), \\
\hat{v} &= \hat{v}_0 + 2\,(k-2)\,(\sigma-\sigma_{cr})^{1/2}\,(\hat{p}-1)\,\delta + \mathcal{O}\left(\delta^2\right).
\end{aligned} \tag{36}
$$

where $\hat{v}_c = v_c/v_*$. Being $\lambda^{(3)} = v + c$ and making use of Eq. (21), it is moreover possible to see that

$$
\hat{\lambda}^{(3)} - \hat{\lambda}_0^{(3)} = \frac{1}{2}\sigma_{cr}^2\,(\sigma-\sigma_{cr})^{-3/2}\,(\hat{p}-1) + \mathcal{O}\left(\delta\right).
$$

Rarefaction waves in water. The analysis of the behaviours of the (dimensionless) specific volume $\hat{V}$, temperature $\hat{T}$ and velocity $\hat{v}$ of the states $\mathbf{u}$ such that $\mathbf{u}\in\mathcal{R}^{(3)}\left(\mathbf{u}_0\right)$ as functions of the pressure $\hat{p}$ shows that, for the case of water, the variations in specific volume, temperature and fluid velocity on the rarefaction wave are always very *small*. Moreover, the variations of the (dimensionless) wave velocity on the rarefaction wave, i.e. the difference of the wave velocity in the state $\mathbf{u}$, $\hat{\lambda}^{(3)} = \lambda^{(3)}/v_*$, and the wave velocity of the right state $\mathbf{u}_0$, $\hat{\lambda}_0^{(3)} = \lambda_0^{(3)}/v_*$, is so small that the development of the rarefaction profile is expected to be very *slow*.

The quantity $\hat{\lambda}^{(3)} - \hat{\lambda}_0^{(3)}$ represents the velocity of development of the rarefaction wave profile, and the fact that the the states $\mathbf{u}_0$ and $\mathbf{u}_1$ between which the rarefaction wave develops are very *close* one to the other, together with the fact that the velocity of development of the wave profile is very small, possibly make the rarefaction wave difficult to detect, according to the EQTI model.

Rarefaction waves in the incompressible limit. The analysis of Eq. (36), clearly show that in the incompressible limit ($\delta \to 0$):

$$\lim_{\delta\to 0} V = V_0, \qquad \lim_{\delta\to 0} T = T_0, \qquad \lim_{\delta\to 0} v_1 = v_0,$$

and

$$\lim_{\delta\to 0}\left(\lambda^{(3)} - \lambda_0^{(3)}\right) = \frac{1}{2}\sigma_{cr}^2\left(\sigma - \sigma_{cr}\right)^{-3/2}\left(p_0 V_0\right)^{1/2}\frac{p - p_0}{p_0}.$$

The latter shows that the wave velocity along the rarefaction wave has a finite, different from zero, limit as $\delta \to 0$. These results point out that, as the incompressible limit is attained, the rarefaction waves which can develop in the EQTI material becomes more and more difficult to detect as $\delta \to 0$, and their velocity of development is very small but finite.

5. Conclusions

The properties of a constitutive equation of an extended-quasi-thermal-incompressible material from the viewpoint of the propagation of shock and rarefaction waves has been discussed, both for the case in which the model parameters are set as to study wave propagation in water and in the incompressible limit, i.e. when the thermal expansion coefficient and the compressibility factor vanish.

Making use of the theory of hyperbolic systems, the Rankine-Hugoniot conditions have been exploited in order to study the features of shock propagation and the integral curve representing the locus of rarefaction waves have been analytically determined. The results show that the propagation of shock waves in an EQTI material is always characterized by *small* jump in specific volume and temperature, even when the jump in pressure in relevant and, as expected, the velocity of propagation of the shock front becomes larger as the degree of compressibility of the material decreases, becoming infinite in the incompressible limit, i.e. when the thermal expansion coefficient and the compressibility factor vanish.

The analysis of the propagation of rarefaction waves in EQTI materials shows that an initial discontinuity which is not an admissible shock wave develops in a rarefaction wave *very slowly* and is characterized by a very steep profile continuously connecting two states that become closer as the incompressible limit is attained.

The knowledge of the the Hugoniot loci and of the integral curves, namely of the loci of the states that can be connected to a given state by a shock wave or a rarefaction wave, will allow to completely solve the Riemann problem for any initial profile characterized by two constant states and a single discontinuity.

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[a] University of Bologna
Department of Mathematics & Research Center of Applied Mathematics (CIRAM)
Via Saragozza 8
40123 Bologna, Italy

[*] Email: tommaso.ruggeri@unibo.it

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