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ON A CLASS OF VECTOR OPTIMIZATION PROBLEMS WITH A VARIATIONAL APPROACH

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ABSTRACT. A class of vector optimization problems is considered. In particular, we reformulate this problem by means of a suitable vector variational inequality, involving the normal cone operator to the adjusted sublevel sets. Finally, this variational approach allows us to obtain the existence of solutions to our vector optimization problem.

1. Introduction

The aim of this paper is to study a class of vector optimization problem by means of a suitable vector variational inequality (for short, VVI). Optimization may be regarded as the cornerstone of many areas of applied mathematics, computer science, engineering, and a number of other scientific disciplines. Among other things, optimization plays a key role in finding feasible solutions to real-life problems, from mathematical programming to operations research, economics, management science, business, medicine, life science, artificial intelligence, etc..

However, in most real-life problems, optimization problems concern the simultaneous minimization of several objective functions. These problems fit in the vector optimization theory. This branch of optimization began with the pioneering paper of Kuhn and Tucker (1951). Subsequently, Edgeworth (1881) and Pareto (1971) gave the definition of standard optimality concepts in the multiobjective optimization.

A fundamental aspect of optimization theory is represented by variational inequality theory (see, for example, Barbagallo *et al.* 2014; Benedetti, Donato, and Milasi 2013; Donato and Milasi 2011; Maugeri and Vitanza 2008; Milasi 2014). The variational inequality theory was introduced by Fichera and Stampacchia in the early 1960s, in connection with several equilibrium problems originating from mathematical physics. These early studies were set in the context of the calculus of variation and optimal control theory, and in connection with the boundary value problems posed in the form of differential equations. In 1980, the classical Stampacchia variational inequality was extended for vector-valued functions by Giannessi (1980) and is called *vector Stampacchia variational inequality*. Since then, Stampacchia vector variational inequalities and their generalizations represented one of the most important tools to study vector optimization problems.

The paper deals with a class of vector optimization problems. More precisely, we consider a problem in which an additional condition is simultaneously requested for the solution of a vector minimization problem (see Section **2.2**). In order to obtain a wide applicability of the theoretical problem, we suppose that the objective function is quasiconvex. In fact, even if the convexity of functions plays a central role in many branches of applied mathematics, not all real-life problems can be described by a convex mathematical model; as for instance in economics, engineering and various applied sciences. This led to the introduction of several generalizations of the classical concept of convex function and, by 1949, a new research field about the generalized convexity began. For this reason, we require quasiconvexity and continuity assumptions on the objective function, without requiring any differentiability assumption. Under these assumptions we connect the vector optimization problem with the solution to a suitable generalized vector quasi-variational inequality. By relaxing assumptions of convexity and differentiability on the objective function, a new operator, introduced by Aussel and Hadjisavvas (2005), must be considered: the normal operator to the adjusted sublevel sets. In order to obtain a suitable variational formulation of the problem, tools of generalized convex analysis are used. Furthermore, thanks to this variational formulation, we are able to give an existence result.

To conclude, we mention the paper by Donato, Milasi, and Vitanza (2014), in which an economic equilibrium problem is studied by means a suitable generalized quasi-variational inequality. This equilibrium problem represents an interesting application to the theoretical results presented in the present paper. In fact it can be set and studied as a vector optimization problem (see also Aussel and Dutta (2008) for further applications).

The paper is organized as follows. In Section 2, firstly we recall some basic definitions and results and subsequently we introduce our class of constrained vector optimization problems. In Section 3 we reformulate these problems as suitable vector variational inequalities involving the normal cone operator to the adjusted sublevel sets. Finally, in Section 4 we use this variational approach in order to investigate on the existence of solutions to our vector optimization problem.

2. The problem

In this section, firstly we recall some basic concepts of the generalized convex analysis, which will play an important role in the following. Subsequently, we present a class of vector optimization problems, which will be studied in the next sections.

2.1. Some basic notations and definitions. In this subsection we give some fundamental definitions and characterizations of generalized convex scalar and vector functions. For further details we refer to Aussel and Hadjisavvas (2005), Aussel and Ye (2006), Luc (1989), and references contained therein.

Let X be a convex and nonempty subset of $\mathbb{R}^l$ and let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function. The domain of f is denoted by

$$domf = \{x \in X : f(x) < +\infty\}$$

which is always assumed nonempty.

Definition 1. *The function f is said to be*

- *quasiconvex if, for any $x, y \in dom f$ and $\lambda \in [0,1]$ one has*
$$f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\};$$
- *semistrictly quasiconvex if it is quasiconvex and for any $x, y \in dom f$ such that $f(y) \neq f(x)$, one has*
$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \quad \forall \lambda \in (0,1);$$
- *strictly quasiconvex if for any $x, y \in dom f$, with $x \neq y$, one has*
$$f(\lambda x + (1-\lambda)y) < \max\{f(x), f(y)\}, \quad \forall \lambda \in (0,1).$$

An equivalent characterization of quasiconvexity is the following:

Proposition 2.1. *(see Aussel and Ye 2006) A function f is quasiconvex if and only if, for any $\lambda \in f(X)$, the sublevel set $S_\lambda = \{y \in X : f(y) \leq \lambda\}$ is nonempty convex.*

Aussel and Hadjisavvas (2005) introduced a new concept of adjusted sublevel set of a function which will adapt very well to all variants of quasiconvex functions.

Definition 2. *Let f be any function. To any element $x \in dom f$ is associated the adjusted sublevel set $S_f^a(x)$ defined by*

$$S_f^a(x) = \begin{cases} S_{f(x)} \cap \overline{B}\big(S^<_{f(x)}, \rho_x\big), & if\, x \notin argmin_X f; \\ \\ S_{f(x)}, & otherwise \end{cases}$$

where: $S^<_\lambda = \{y \in X : f(y) < \lambda\}$ is the strict sublevel set of f, $\rho_x = dist(x, S^<_{f(x)}) = \inf_{y \in S^<_{f(x)}} ||x-y||$, $\forall x \in dom f \backslash argmin_X f$ and $\overline{B}\big(S^<_{f(x)}, \rho_x\big) = \{z \in X : dist(z, S^<_{f(x)}) \leq \rho_x\}$.

The characterization of the quasiconvexity of functions still holds true for the adjusted sublevel sets.

Proposition 2.2. *(see Aussel and Hadjisavvas 2005) f is quasiconvex if and only if $S_f^a(x)$ is convex, $\forall x \in dom f$.*

In order to relate the minimization of quasiconvex and lower semicontinuous (lsc) functions to a variational inequality problem, Aussel and Hadjisavvas (2005) introduced the following multimap, based on adjusted sublevel sets.

Definition 3. *To any function f is associated the multimap:*

$$N^a : dom\, f \rightrightarrows X^*$$

defined, for any $x \in dom\, f$, as

$$N^a(x) = \{h \in X^* : \langle h, y-x\rangle \leq 0 \quad \forall y \in S_f^a(x)\}.$$

If the function f is quasiconvex, since $S_f^a(x)$ is a convex set (from Proposition **2.2**), the multimap N^a coincides with the normal cone to the adjusted sublevel set $S_f^a(x)$ at x, i.e., for all $x \in dom\, f$ one has $N^a(x) = N_{S_f^a}(x)$ is a convex cone.

Remark 1. *The adjusted sublevel set lies between the strict sublevel set and the sublevel set:*

$$S^{<}_{f(x)} \subseteq S^a_f(x) \subseteq S_{f(x)}\,, \quad \forall x \in dom\, f\,.$$

This fact implies that

$$N(x) \subseteq N^a(x) \subseteq N^{<}(x)\,, \quad \forall x \in dom\, f\,,$$

where $\quad N(x) = \{h \in X^* :\ \langle h, y-x\rangle \le 0 \quad for\ all\ y \in S_{f(x)}\},$

$$N^{<}(x) = \{h \in X^* :\ \langle h, y-x\rangle \le 0 \quad for\ all\ y \in S^{<}_{f(x)}\}\,.$$

Moreover, if f *is semistrictly quasiconvex,* $\forall x \in dom f \setminus argmin_X f$*, one has* $S^a_f(x) = S_{f(x)} = \overline{S}^{<}_{f(x)}$*, and thus* $N^a(x) = N(x) = N^{<}(x)$*.*

One has the following

Proposition 2.3. *(see Aussel and Hadjisavvas 2005) Let* f *be a continuous and semistrictly quasiconvex function. Then the multimap* $N^a : dom f \rightrightarrows X^*$ *is closed for all* $x \in dom f \setminus argmin_X f$*.*

Finally, we need to recall the quasiconvexity concept of maps in the vector context. Let $X \subseteq \mathbb{R}^l$ be convex, $C \subseteq \mathbb{R}^n$ a convex cone and $\mathscr{F} : X \to \mathbb{R}^n$ a vector function.

Definition 4. *The function* $\mathscr{F}$ *is C-quasiconvex on X if for every* $y \in \mathbb{R}^n$*,* $x_1, x_2 \in X$*,* $t \in [0,1]$

$$\mathscr{F}(x_1), \mathscr{F}(x_2) \in y - C \text{ implies } \mathscr{F}(tx_1 + (1-t)x_2) \in y - C$$

One has the following

Proposition 2.4. *(see Luc 1989) A function* $\mathscr{F}$ *is C-quasiconvex if and only if for every* $y \in \mathbb{R}^n$ *the level set* $\{x \in X : \mathscr{F}(x) \in y - C\}$ *is convex.*

Remark 2. *We remind that when* $C = \mathbb{R}^n_+$*, the function* $\mathscr{F} = (f_1, ..., f_n)$ *is* $\mathbb{R}^n_+$*-quasiconvex if and only if every component function,* $f_i : X \to \mathbb{R}$*, is quasiconvex.*

Moreover, we recall the following definition that will be useful in the sequel:

Definition 5. *(see Aubin and Frankowska 1990) A multimap* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$*, with* $Dom F = \{x \in \mathbb{R}^n :\ F(x) \neq \emptyset\} \neq \emptyset$ *is said to be:*

(a) upper semicontinuous (usc) *at* $x \in \mathbb{R}^n$ *if for each open set* $V \subset \mathbb{R}^m$*, where* $F(x) \subset V$*, there exists a neighborhood* $U \subset \mathbb{R}^n$ *of* x *such that for all* $x' \in U : F(x') \subset V$*;*
(b) lower semicontinuous (lsc) *at* $x \in \mathbb{R}^n$ *if for any sequence of elements* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$*,* $x_n \to x$*, and for any* $y \in F(x)$*, there exists a sequence of elements* $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$*, with* $y_n \in F(x_n)\ \forall n$ *and* $y_n \to y$*;*
(c) closed *if for any sequences* $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^n$*,* $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^m$*, if* $x_n \to x$ *and* $y_n \in F(x_n)$*,* $y_n \to y$ *then* $y \in F(x)$*.*

2.2. Vector optimization problem. In this paper we will consider the following class of constrained vector optimization problems.

Let Z_i, for $i = 1,...,n$, and Y, be $n+1$ compact sets of $\mathbb{R}^l$; we pose $Z = \prod_{i=1}^n Z_i \subseteq \mathbb{R}^{n\times l}$ and $X = \{x_i\}_{i=1}^n = \begin{pmatrix} x_1 \\ .... \\ x_n \end{pmatrix} \in \mathbb{R}^{n\times l}$, where $x_i \in \mathbb{R}^l$ represents the i^{th} row of X. Let C_l, C_n and C_s be, respectively, the ordering cones of $\mathbb{R}^l$, $\mathbb{R}^n$ and $\mathbb{R}^s$. The following vector-valued functions are given:

$$\mathscr{F} : Z \to \mathbb{R}^n, \quad with \quad \mathscr{F} := (f_1,...,f_n), \; f_i : Z_i \to \mathbb{R}, \; \forall i = 1,...,n$$
$$and \quad h_i : Z_i \times Y \to \mathbb{R}^s, \quad with \quad h_i := (h_i^1,...,h_i^s) \quad for\,each \; i = 1,...,n\,.$$

Fixed $y \in Y$, we consider the following parametric vector optimization problem (VOP):

$$\text{“}Find \;\; \overline{X} \in K(y) \;\; such\ that \;\; \mathscr{F}(\overline{X}) = \min_{X\in K(y)} \mathscr{F}(X),\text{”}$$

where:

$$K(y) = \{X = \{x_i\}_{i=1}^n \in Z : \, h_i(x_i,y) \in -C_s, \; i = 1,...,n\} \neq \emptyset\,.$$

Now, let $g : Z \times Y \to \mathbb{R}^l$ be a given vector-valued function, with $g := (g^1,...,g^l)$, and let $\mu, \lambda \in \mathbb{R}^l$ be such that, for each $j = 1,...,l$, one has $\lambda^j \le y^j \le \mu^j$ for all $y = (y^1,...,y^l) \in Y$. We consider the problem:

$$\text{“}Find \;\; \overline{y} \in Y \;\; such\ that\ for\ all \;\; j = 1,...,l$$
$$g^j(\overline{X},\overline{y}) \begin{cases} \ge 0 & if \;\; \overline{y}^j = \lambda^j \\ = 0 & if \;\; \lambda^j < \overline{y}^j < \mu^j \\ \le 0 & if \;\; \overline{y}^j = \mu^j \end{cases} \text{”}$$

Hence, we are concerned with the following class of vector optimization problems:

Problem 2.1. *Find* $\overline{X} \in K(\overline{y}) \subseteq \mathbb{R}^{n\times l}$ *and* $\overline{y} \in Y \subseteq \mathbb{R}^l$ *such that*

$$\min_{X\in K(\overline{y})} \mathscr{F}(X) = \mathscr{F}(\overline{X}), \tag{1}$$

$$g^j(\overline{X},\overline{y}) \begin{cases} \ge 0 & if \;\; \overline{y}^j = \lambda^j \\ = 0 & if \;\; \lambda^j < \overline{y}^j < \mu^j \\ \le 0 & if \;\; \overline{y}^j = \mu^j \end{cases} \tag{2}$$

We remind that $\overline{X} \in K(\overline{y})$ is said to be an efficient solution (or minimal solution) for the VOP (1), if:

$$\mathscr{F}(X) \notin \mathscr{F}(\overline{X}) - C_n \backslash \{0\} \qquad \forall X \in K(\overline{y}). \tag{3}$$

Remark 3. *Fixed* $y \in Y$, *let* $\overline{X} = \{\overline{x}_i\}_{i=1}^n$ *be such that, for all* $i = 1,...,n$ *one has* $\overline{x}_i \in K_i(y)$ *and*

$$f_i(\overline{x}_i) = \min_{x_i\in K_i(y)} f_i(x_i), \tag{4}$$

where $K_i(y) := \{x_i \in Z_i : \; h_i(x_i,y) \in -C_s\}$.
Then, $\overline{X}$ *is a solution to vector optimization problem (1).*

Hence, we can consider the following scalar problem:

Problem 2.2. *Find $\overline{X} \in K(\overline{y}) \subseteq \mathbb{R}^{n\times l}$ and $\overline{y} \in Y \subseteq \mathbb{R}^l$ such that*

$$for\ all\ i = 1,...,n \qquad \min_{x_i \in K_i(\overline{y})} f_i(x_i) = f_i(\overline{x}_i), \tag{5}$$

$$g^j(\overline{X},\overline{y}) \begin{cases} \geq 0 & if\ \ \overline{y}^j = \lambda^j \\ = 0 & if\ \ \lambda^j < \overline{y}^j < \mu^j \\ \leq 0 & if\ \ \overline{y}^j = \mu^j \end{cases} \tag{6}$$

Then, thanks to Remark 3, if $(\overline{X},\overline{y})$ is a solution to Problem **2.2**, then it is a solution to Problem **2.1**.

3. Variational approach

Our purpose is to study the Problem **2.1** by means of a suitable vector variational inequality (for short, VVI). In order to relate the Problem **2.1** with a suitable VVI, we introduce the multivalued map $G_i : Z \rightrightarrows \mathbb{R}^l$, such that for all $X = \{x_i\}_{i=1}^n \in Z$:

$$G_i(X) = \begin{cases} \overline{B}(0,1) & if\ \ x_i \in argmin_{Z_i}(f_i) \\ \\ conv\Big(N^a(x_i) \cap S(0,1)\Big) & if\ \ x_i \in Z_i \setminus argmin_{Z_i}(f_i) \end{cases}$$

where $\overline{B}(0,1) = \{y \in \mathbb{R}^l : \|y\| \leq 1\}$ and $S(0,1) = \{y \in \mathbb{R}^l : \|y\| = 1\}$ are, respectively, the closed unit ball and the unit sphere of $\mathbb{R}^l$.
We define $n+1$ vector-valued functions

$$\Phi_1(A) = \begin{pmatrix} G_1(X) \\ 0 \\ .... \\ 0 \end{pmatrix}, ..., \Phi_n(A) = \begin{pmatrix} 0 \\ .... \\ G_n(X) \\ 0 \end{pmatrix}, \Phi_{n+1}(A) = \begin{pmatrix} 0 \\ .... \\ 0 \\ g(X,y) \end{pmatrix}$$

for all $A = \begin{pmatrix} X \\ y \end{pmatrix}$ with $X \in Z$ and $y \in Y$. We set

$$\Phi := (\Phi_1,...,\Phi_n,\Phi_{n+1}) \text{ and } \Phi(A)(B) := (\langle \Phi_1(A),B\rangle, ..., \langle \Phi_{n+1}(A),B\rangle)$$

for every A and $B = \begin{pmatrix} V \\ t \end{pmatrix} \in \mathbb{R}^{(n+1)\times l}$, with $V = \{v_i\}_{i=1}^n \in \mathbb{R}^{n\times l}$ and $t = (t^1,...,t^l) \in \mathbb{R}^l$, where $\langle \Phi_i(A),B\rangle = \sum_{j=1}^l G_i^j(X)v_i^j$ and $\langle \Phi_{n+1}(A),B\rangle = \sum_{j=1}^l g^j(X,y)t^j$.

Now, we consider the following generalized vector quasi-variational inequality problem:

Problem 3.1. *Find $\overline{X} = \{\overline{x}_i\}_{i=1}^n \in K(\overline{y})$ and $\overline{y} \in Y$ such that there exists*

$$t := (\varphi_1,...,\varphi_n, g(\overline{X},\overline{y})) \in \Phi \begin{pmatrix} \overline{X} \\ \overline{y} \end{pmatrix}$$

such that

$$t \begin{pmatrix} X - \overline{X} \\ y - \overline{y} \end{pmatrix} \notin C_l \setminus \{\theta_Y\} \qquad \forall \begin{pmatrix} X \\ y \end{pmatrix} : X \in K(\overline{y}),\ y \in Y\,.$$

The above variational problem can be rewritten in the following form:

Problem 3.2. *Find* $\overline{X} = \{\bar{x}_i\}_{i=1}^n \in K(\bar{y})$ *and* $\bar{y} \in Y$ *such that, for all* $i = 1,...,n$ *there exist* $\varphi_i \in G_i(\overline{X})$ *such that:*

$$\begin{aligned} &\langle \varphi_i, x_i - \bar{x}_i \rangle \geq 0 \qquad \forall x_i \in K_i(\bar{y}), \\ &\langle g(\overline{X},\bar{y}), y - \bar{y} \rangle \geq 0 \qquad \forall y \in Y\,. \end{aligned} \tag{7}$$

Clearly, if $(\overline{X},\bar{y})$ is a solution to Problem **3.2**, then it is a solution to Problem **3.1**.

Theorem 3.1. *Let* $\mathscr{F} : Z \to \mathbb{R}^n$ *be continuous and* $\mathbb{R}^n_+$*-quasiconvex. If, for all* $i = 1,...,n$ *and* $y \in Y$ *one has* $(K_i(y))^\perp = \{0\}$*, then any solution* $\bar{y} \in Y$ *and* $\overline{X} = \{\bar{x}_i\}_{i=1}^n \in K(\bar{y})$ *to Problem* ***3.2*** *is solution to Problems* ***2.2*** *and* ***2.1****.*

Proof. Let $\overline{X},\bar{y}$ be solutions to Problem **3.2**. For all $i = 1,...,n$, $\bar{x}_i \in K_i(\bar{y})$ is such that:

$$\exists\ \varphi_i \in G_i(\overline{X}) :\ \langle \varphi_i, x_i - \bar{x}_i \rangle \geq 0 \qquad \forall x_i \in K_i(\bar{y}) \tag{8}$$

and

$$\langle g(\overline{X},\bar{y}), y - \bar{y} \rangle \geq 0 \qquad \forall y \in Y\,. \tag{9}$$

Firstly, we prove that $\bar{x}_i \in K_i(\bar{y})$ *is a solution to optimization problem*

$$\min_{x_i \in K_i(\bar{y})} f_i(x_i). \tag{10}$$

If $\bar{x}_i \in argmin_{Z_i} f_i$, $\bar{x}_i$ is a solution to (10).
If $\bar{x}_i \in Z_i \backslash argmin_{Z_i} f_i$, one has $0 \notin G_i(\bar{x}_i) \subseteq N^a(\bar{x}_i)$, hence $\varphi_i \in N^a(\bar{x}_i) \setminus \{0\}$. Furthermore taking into account that $(K_i(\bar{y}))^\perp = \{0\}$ one has $\varphi_i \notin (K_i(\bar{y}))^\perp : t_i \in K_i(\bar{y})$ such that $\langle \varphi_i, t_i \rangle \neq \langle \varphi_i, \bar{x}_i \rangle$. Moreover, for all $x_i \in K_i(\bar{y})$ and $\lambda \in (0,1)$, $x_\lambda = \lambda t_i + (1-\lambda) x_i \in conv(K_i(\bar{y}))$. Then, we have:

$$\langle \varphi_i, x_\lambda - \bar{x}_i \rangle = \langle \varphi_i, \lambda t_i + (1-\lambda) x_i - \bar{x}_i \rangle = \lambda \langle \varphi_i, t_i - \bar{x}_i \rangle + (1-\lambda) \langle \varphi_i, x_i - \bar{x}_i \rangle > 0.$$

Being, by assumption, f_i quasiconvex and $\varphi_i \in N^a(\bar{x}_i) \subseteq N^<(\bar{x}_i)$ we have:

$$\langle \varphi_i, x_\lambda - \bar{x}_i \rangle > 0 \quad \Rightarrow \quad f_i(x_\lambda) \geq f_i(\bar{x}_i) \quad \forall \lambda \in (0,1)\,.$$

Since, f_i is continuous, we get $f_i(x_i) \geq f_i(\bar{x}_i)$, namely $\bar{x}_i$ is a solution to optimization problem (10). It remains to prove condition (6). We fix $j^* \in \{1,...,l\}$ and we consider the following cases.

- If $\bar{y}^{j^*} = \lambda^{j^*}$, we pose:

$$\tilde{y} = \begin{cases} \bar{y}^s & \forall s \neq j^* \\ \tilde{y}^{j^*} & with\ \ \lambda^{j^*} < \tilde{y}^{j^*} < \mu^{j^*} \end{cases} \qquad \Rightarrow\ \ \tilde{y} \in Y\,.$$

Replacing $\tilde{y}$ in (9), it results:

$$\langle g(\overline{X},\bar{y}), \tilde{y} - \bar{y} \rangle = g^{j^*}(\overline{X},\bar{y})(\tilde{y}^{j^*} - \bar{y}^{j^*}) \geq 0 \ \ \Rightarrow\ \ g^{j^*}(\overline{X},\bar{y}) \geq 0.$$

- If $\bar{y}^{j^*} = \mu^{j^*}$, we pose:

$$\tilde{y} = \begin{cases} \bar{y}^s & \forall s \neq j^* \\ \tilde{y}^{j^*} & with\ \ \lambda^{j^*} < \tilde{y}^{j^*} < \mu^{j^*} \end{cases} \qquad \Rightarrow\ \ \tilde{y} \in Y\,.$$

Replacing $\widetilde{y}$ in (9), it results:

$$\langle g(\overline{X},\overline{y}),\hat{y}-\overline{y}\rangle = g^{j^*}(\overline{X},\overline{y})(\widetilde{y}^{j^*}-\overline{y}^{j^*}) \geq 0 \;\Rightarrow\; g^{j^*}(\overline{X},\overline{y}) \leq 0.$$

- If $\lambda^{j^*} < \overline{y}^{j^*} < \mu^{j^*}$, we pose, with $0 < \varepsilon < \min\{\overline{y}^{j^*}-\lambda^{j^*}, \mu^{j^*}-\overline{y}^{j^*}\}$:

$$\widetilde{y}_1 = \begin{cases} \overline{y}^s & \forall s \neq j^* \\ \\ \overline{y}^{j^*}+\varepsilon & otherwise \end{cases} \quad and \quad \widetilde{y}_2 = \begin{cases} \overline{y}^s & \forall s \neq j^* \\ \\ \overline{y}^{j^*}-\varepsilon & otherwise \end{cases} \;\Rightarrow\; \widetilde{y}_1, \widetilde{y}_2 \in Y\,.$$

Replacing $\widetilde{y}_1$ and $\widetilde{y}_2$ in (9), one has

$$\langle g(\overline{X},\overline{y}),\widetilde{y}_1-\overline{y}\rangle = \varepsilon g^{j^*}(\overline{X},\overline{y}) \geq 0,\; \langle g(\overline{X},\overline{y}),\widetilde{y}_2-\overline{y}\rangle = \varepsilon g^{j^*}(\overline{X},\overline{y}) \leq 0$$
$$\Rightarrow\; g^{j^*}(\overline{X},\overline{y}) = 0.$$

□

4. Existence

Now, our aim is to give an existence result to Problem **2.1** by means of variational Problem **3.2**.

Proposition 4.1. *Let, for all $i = 1,....,n$, f_i be continuous and semistrictly quasiconvex. Then, the multimap $G_i : Z \rightrightarrows \mathbb{R}^l$ is usc, with convex and compact values.*

Proof. For all $X \in Z$, from the definition, one has that $G_i(X)$ is a convex and compact set. Moreover, since for all $X \in Z$, $G_i(X) \subseteq \overline{B}(0,1)$ it results that $\overline{G_i(Z_i)}$ is compact, that is G_i is compact. Now, we prove that G_i is closed, namely:

Let $\{X_n\}_{n\in\mathbb{N}} \subset Z$ and $\{y_{i,n}\}_{n\in\mathbb{N}} \subset \mathbb{R}^l$ be sequences such that $X_n \to X$ and $y_{i,n} \to y_i$ with $y_{i,n} \in G_i(X_n)$; we have to prove that $y_i \in G_i(X)$.
If $x_i \in argmin_{Z_i}(f_i)$ we have $G_i(X) = \overline{B}(0,1)$. Since $y_{i,n} \in G_i(X_n) \subseteq \overline{B}(0,1)$, one has $y_i \in \overline{B}(0,1) = G_i(X)$.
If $x_i \notin argmin_{Z_i}(f_i)$, $G_i(X) = conv(N^a(x_i) \cap S(0,1))$. Since f_i is continuous there exists $\nu \in \mathbb{N}$ such that $x_{i,n} \notin argmin_{Z_i}(f_i)$, $\forall n > \nu$, then $y_{i,n} \in conv(N^a(x_{i,n}) \cap S(0,1))$:

$$y_{i,n} = \sum_{k=1}^{l+1} \lambda_n^k v_{i,n}^k\,, \qquad \lambda_n^k \geq 0,\ \forall k = 1,...,l+1,\ \sum_{k=1}^{l+1} \lambda_n^k = 1\,.$$

with $v_{i,n}^k \in (N^a(x_{i,n}) \cap S(0,1))$ for all $k = 1,...,l+1$. Furthermore, $\forall n > \nu$ and $\forall k = 1,...,l+1$, from $v_{i,n}^k \in S(0,1)$, we can suppose that $v_{i,n}^k \to v_i^k \in S(0,1)$; moreover, because $v_{i,n}^k \in N^a(x_{i,n})$ and N^a is closed it follows $v_i^k \in N^a(x_i)$. Then $v_i^k \in (N^a(x_i) \cap S(0,1))$.
Hence: $y_{i,n} = \sum_{k=1}^{l+1} \lambda_n^k v_{i,n}^k \to y_i = \sum_{k=1}^{l+1} \lambda^k v_i^k$, with $v_i^k \in (N^a(x_i) \cap S(0,1))$, $\lambda^k \geq 0$ and $\sum_{k=1}^{l+1} \lambda^k = 1$; that is $y_i \in conv(N^a(x_i) \cap S(0,1)) = G_i(X)$.
We can conclude that G_i is usc. □

Theorem 4.1. *Let f_i be semistrictly quasiconvex and continuous for all $i = 1,...,n$. Then there exists a solution to Problem **3.2**.*

Proof. For all $i = 1,...,n$, we fix $y \in Y$ and we consider the generalized variational inequality:

$$\langle \varphi_i, x_i - \overline{x}_i \rangle \geq 0 \qquad \forall x_i \in K_i(y)\,. \tag{11}$$

This inequality admits solution, since, from Proposition **4.1**, the multimap G_i is usc with convex and compact values. Then, we can define the multimap:

$$\Theta_i : Y \rightrightarrows Z_i$$

such that for all $y \in Y$, $\Theta_i(y) = \{\overline{x}_i : \overline{x}_i \text{ is a solution to } (11)\} \neq \emptyset$.
The map Θ_i satisfies the following properties.

Θ_i *is closed:*

Let $\{y_n\} \subseteq Y$ and $\{\overline{x}_{i,n}\} \subseteq Z_i$ be two sequences with $\overline{x}_{i,n} \in \Theta_i(y_n)$ and such that $y_n \to y$ and $\overline{x}_{i,n} \to \widetilde{x}_i$. We have to prove that $\widetilde{x}_i \in \Theta_i(y)$. Firstly, we observe that $\widetilde{x}_i \in K_i(y)$, being X_i closed and h_i continuous. Moreover, $\overline{x}_{i,n}$ is a local minimum of f_i in $K_i(y_n)$, then, for the semistrict quasiconvexity of f_i, it is a global minimum: $f_i(\overline{x}_{i,n}) \leq f_i(x_i), \forall x_i \in Z_i$.

Passing to the limit, it follows that $\widetilde{x}_i$ is a global minimum of f_i and $\widetilde{x}_i \in K_i(y)$. Then $\widetilde{x}_i \in \Theta_i(y)$, hence Θ_i is closed.

Θ_i *is with compact values*, since $K_i(y)$ is compact for all $y \in Y$.

Θ_i *is usc*:

Firstly, we observe that, since for all $y \in Y$, $\Theta_i(y) \subseteq Z_i$. Hence $\overline{\Theta_i(Y)}$ is compact, that is Θ_i is compact. Namely, Θ_i is usc.

Θ *is with convex values.*

For all $y \in Y$, let $\overline{x}_i$, $\overline{x}'_i \in \Theta_i(y)$. Since f_i is semistrictly quasiconvex, it follows that $\overline{x}_i$ and $\overline{x}'_i$ are global minima of f_i:

$$S^a_{(f_i)}(\overline{x}_i) = S_{f_i(\overline{x}_i)}\,, \qquad S^a_{(f_i)}(\overline{x}'_i) = S_{f_i(\overline{x}'_i)}\,,$$

then $S^a_{(f_i)}(\overline{x}_i) = S^a_{(f_i)}(\overline{x}'_i) = S_\mu$, where $\mu = \min_{Z_i} f_i(x_i)$.
For all $\lambda \in [0,1]$, $z = \lambda\overline{x}_i + (1-\lambda)\overline{x}'_i \in K_i(y)$. Moreover, being f_i is quasiconvex, the set S_μ is convex, then $z \in S_\mu$. Hence $z \in K_i(y)$ is a global minimum of f_i: $z \in \Theta_i(y)$.

Now, we can consider the following generalized variational inequality:

Find $\overline{y} \in Y$ such that there exists $\gamma_i \in \Theta_i(\overline{y})$, with $i = 1,...,n$:

$$\langle \sum_{i \in I} g(\gamma_i, \overline{y}), y - \overline{y} \rangle \geq 0 \qquad \forall y \in Y\,. \tag{12}$$

The operator of this variational problem, from properties of Θ_i, results to be usc with compact and convex values. Being Y a compact set, then there exist $\overline{y} \in Y$ and $\gamma_i \in \Theta_i(\overline{y})$ solutions to (12). Then, the pair $(\overline{X}, \overline{y})$ with $\overline{y} \in Y$ and $\overline{X} = \{\overline{x}_i\}_{i=1}^n \in \prod_{i=1}^n K_i(y)$ is a solution to Problem **3.2**. □

Hence we can conclude with the following

Theorem 4.2. *Let $\mathscr{F}$ be such that every f_i, $i = 1,...,n$, is semistrictly quasiconvex and continuous function. Then there exists a solution to Problem **2.1**.*

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References

Aubin, J. P. and Frankowska, H. (1990). *Set Valued Analysis*. Boston: Birkhäuser.

Aussel, D. and Dutta, J. (2008). "Generalized Nash equilibrium problem, variational inequality and quasiconvexity". *Operation Research Letters* **36**(4), 461–464. DOI: 10.1016/j.orl.2008.01.002.

Aussel, D. and Hadjisavvas, N. (2005). "Adjusted sublevel sets, normal operator, and quasiconvex programming". *SIAM Journal on Optimization* **16**(2), 358–367. DOI: 10.1137/040606958.

Aussel, D. and Ye, J. J. (2006). "Quasiconvex programming with locally starshaped constraint region and applications to quasiconvex MPEC". *Optimization. A Journal of Mathematical Programming and Operations Research* **55**(5-6), 433–457. DOI: 10.1080/02331930600808830.

Barbagallo, A., Daniele, P., Giuffrè, S., and Maugeri, A. (2014). "Variational approach for a general financial equilibrium problem: The Deficit Formula, the Balance Law and the Liability Formula. A path to the econpmy recovery." *European Journal of Operational Research* **237**(1), 231–244. DOI: 10.1016/j.ejor.2014.01.033.

Benedetti, I., Donato, M. B., and Milasi, M. (2013). "Existence for competitive equilibrium by means of generalized quasivariational inequalities". *Abstract and Applied Analysis*, 648986, 8 pages. DOI: 10.1155/2013/648986.

Donato, M. B. and Milasi, M. (2011). "Lagrangean variables in infinite dimensional spaces for a dynamic economic equilibrium problem". *Nonlinear Analysis: Theory, Methods & Applications* **74**(15), 5048–5056. DOI: 10.1016/j.na.2011.04.070.

Donato, M. B., Milasi, M., and Vitanza, C. (2014). "Variational problem, generalized convexity, and application to a competitive equilibrium problem". *Numerical Functional Analysis and Optimization* **35**(7-9), 962–983. DOI: 10.1080/01630563.2014.895758.

Edgeworth, F. Y. (1881). *Mathematical Psychics. An essay on the application of mathematics to the moral sciences*. London: C. Kegan Paul & Co.

Giannessi, F. (1980). "Theorems of the Alternative, Quadratic Programs and Complementarity Problems". In: *Variational Inequalities and Complementarity Problems*. Ed. by R. W. Cottle, F. Giannessi, and J. L. Lions. Kluwer Academic, pp. 151–186.

Kuhn, H. W. and Tucker, A. W. (1951). "Nonlinear programming". In: *Proceedings of the second Berkeley symposium on mathematical statistics and probability*. Ed. by J. Neyman. Berkeley: University of California Press, pp. 481–492.

Luc, T. D. (1989). *Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems*. Berlin: Springer.

Maugeri, A. and Vitanza, C. (2008). "Time-Dependent Equilibrium Problems". In: *Pareto Optimality, Game Theory and Equilibria*. Ed. by A. Chinchuluun, P. M. Pardalos, A. Migdalas, and L. Pitsoulis. Optimization and its applications. Springer, pp. 249–266.

Milasi, M. (2014). "Existence theorem for a class of generalized quasi-variational inequalities". *Journal of Global Optimization* **60**(4), 679–688. DOI: 10.1007/s10898-013-0114-6.

Pareto, V. (1971). *Manuale di economia politica*. Milano: Società Editrice Libraria.

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