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ON THE SYMMETRIC ALGEBRA OF SYZYGY MODULES OF MONOMIAL IDEALS

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ABSTRACT. We consider the symmetric algebra of the first syzygy module of a monomial ideal generated by an s-sequence. We introduce on that algebra an admissible term order which allows us to compute its algebraic invariants.

Introduction

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring in n indeterminates over a field K and let M be a finitely generated R-module. The symmetric algebra $Sym_R(M)$ is a very important algebra from many points of view. In particular, if M is a linear type ideal I in the sense of Valla (1979), then $Sym_R(I) = Rees(I)$ and it is evident its importance in algebraic geometry. The main goal of this paper is to compute the invariants of the symmetric algebra of the first syzygy module of M, $Syz_1(M)$, where M is a monomial ideal I of R. We look to the computation of invariants of $Sym(Syz_1(I))$ via the theory of s-sequences, that has been introduced by Herzog, Restuccia, and Tang (2001) and having a useful role in this direction (La Barbiera and Restuccia 2011; Restuccia 2006; Restuccia, Utano, and Tang 2014). If the R-module $Syz_1(I)$ is generated by an s-sequence, we give formulas for dimension, depth (with respect to the maximal irrelevant ideal of R), multiplicity and Castelnuovo-Mumford regularity of $Sym_R(Syz_1(I))$ only in terms of the same invariants of ideals of R. More precisely, in Section 1 we give two sufficient conditions for $Syz_1(I)$ to be generated by a s-sequence, I a monomial ideal of R. In Section 2, we investigate the module $Syz_1(\mathfrak{m})$, $\mathfrak{m} = (x_1, \ldots, x_n)$ the maximal irrelevant ideal of R. We prove that $Syz_1(\mathfrak{m})$ is generated by a s-sequence if and only if $n = 3$. The complete study of the invariants of $Sym_R(Syz_1(\mathfrak{m}))$, for $n > 3$ is given by Restuccia, Utano, and Tang (2014). Also $Sym_R(Syz_{n-1}(\mathfrak{m}))$ and $Sym_R(Syz_n(\mathfrak{m}))$ are studied, finding that $Syz_{n-1}(\mathfrak{m})$ and $Syz_n(\mathfrak{m})$ are generated by an s-sequence, hence we proceed to calculate their invariants.

1. Preliminaries

Let R be a Noetherian graded ring and let M a finitely generated R-module. We give a short introduction of the notion of $s-$sequence generating a finitely generated R-module M and we will conclude with the main theorem concerning the computation of the invariants of $Sym(M)$: the Krull dimension of M, $dim(M)$, the multiplicity of M, $e(M)$, the Castelnovo-Mumford regularity of M, $reg(M)$, and the depth of M with respect to the maximal irrelevant ideal of R, $depth(M)$.

Let M be a finitely generated module on a Noetherian ring R, with generators $f_1, f_2, \ldots, f_n$. We denote by $(a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ the relation matrix, by $Sym_i(M)$ the ith symmetric power, and by $Sym_R(M) = \bigoplus_{i\geq 0} Sym(M)_i$ the symmetric algebra of M. Note that

$$Sym_R(M) = R[Y_1, \ldots, Y_n]/J,$$

where

$$J = (g_1, \ldots, g_m), \text{ and } g_i = \sum_{j=1}^{n} a_{ij} Y_j.$$

We consider $S = R[Y_1, \ldots, Y_n]$ a graded ring by assigning to each variable Y_i the degree 1 and to the elements of R the degree 0. Then J is a graded ideal and the natural epimorphism $S \to Sym_R(M)$ is a homomorphism of graded $R-$algebras.

Let $<$ be monomial order on the monomials in $Y_1, \ldots, Y_n$ with the order $Y_1 < Y_2 < \ldots < Y_n$. We call such an order admissible. For any polynomial $f \in R[Y_1, \ldots, Y_n], f = \sum_\alpha a_\alpha Y^\alpha$, we put $in_<(f) = a_\alpha Y^\alpha$ where Y^α is the largest monomial in f with $a_\alpha \neq 0$, and we set

$$in_<(J) = (in_<(f) : f \in J).$$

For $i = 1, \ldots, n$ we set $M_i = \sum_{j=1}^{1} Rf_j$, and let I_i be the colon ideal $M_{i-1} : \langle f_i \rangle$. In other words, I_i is the annihilator of the cyclic module M_i/M_{i-1} and so $M_i/M_{i-1} \cong R/I_i$. For convenience we also set $I_0 = (0)$.

Definition 1.1. *The colon ideals I_i are called the annihilator ideals of the sequence $f_1, \ldots, f_n$.*

Notice that $(I_1Y_1, I_2Y_2, \ldots, I_nY_n) \subseteq in_<(J)$, and that the two ideals coincide in degree 1.

Definition 1.2. *The generators $f_1, \ldots, f_n$ of M are called $s-$sequence (with respect to an admissible order $<$), if*

$$in_<(J) = (I_1Y_1, I_2Y_2, \ldots, I_nY_n)$$

If in addition $I_1 \subset I_2 \subset \ldots \subset I_n$, then $f_1, \ldots, f_n$ is called a strong $s-$sequence.

The invariants of the symmetric algebra of a module which is generated by an s-sequence are computed by the corresponding invariants of R. We have:

Proposition 1.1. *Let M be generated by an s-sequence $f_1,\dots,f_n$ with annihilator ideals $I_1,\dots,I_n$. Then*

$$d := dim(Sym_R(M)) = max_{\substack{0\le r\le n,\\ 1\le i_1<\dots<i_r\le n}} \{dim(R/(I_{i_1}+\dots+I_{i_r}))+r\};$$

$$e(Sym_R(M)) = \sum_{\substack{0\le r\le n, 1\le i_1<\dots<i_r\le n\\ dim(R/(I_{i_1}+\dots+I_{i_r}))=d-r}} e(R/(I_{i_1}+\dots+I_{i_r})).$$

In particular, if $f_1,\dots,f_n$ is a strong $s-$sequence, then

$$d = max_{0\le r\le n}\{dim(R/I_r)+r\};$$

$$e(Sym_R(M)) = \sum_{\substack{r\\ dim(R/I_r)=d-r}} e(R/I_r).$$

Proof. (see Herzog, Restuccia, and Tang 2001, Proposition 2.4) □

Proposition 1.2. *Let $R = K[x_1,\dots,x_m]$ be a polynomial ring, and let M be a graded R-module. If M is generated by a strong s-sequence and $I_1 \subseteq \cdots \subseteq I_n$ are the annihilator ideals of this sequence, then*

$$reg(Sym_R(M)) \le max\{reg(I_i) : i = 1,\dots,n\};$$

and

$$depht(Sym_R(M)) \ge min\{depth(R/I_i)+i : i = 0,1,\dots,n\}.$$

Proof. (see Herzog, Restuccia, and Tang 2001, Proposition 2.5) □

2. First syzygy module of a monomial ideal

Let I be a monomial ideal in $R = K[x_1,\dots,x_n]$, generated by monomials $m_1,\dots,m_s$, $I = (m_1,\dots,m_s)$, and consider the beginning of the Taylor resolution (Eisenbud 1995; Taylor 1966):

$$\bigwedge^2 R^s \xrightarrow{\varphi_2} R^s \xrightarrow{\varphi_1} R \to R/I \to 0$$

We denote by $e_1,\dots,e_s$ the canonical basis of R^s. Then $\varphi_1(e_i) = m_i$ for all i and

$$\varphi_2(e_1\wedge e_j) = \frac{lcm(m_i,m_j)}{m_i}e_i - \frac{lcm(m_i,m_j)}{m_j}e_j \quad \text{for all } i<j$$

The Taylor resolution is multigraded with $deg(e_i) = deg(m_i)$,
$deg(e_i\wedge e_j) = deg(lcm(m_i,m_j))$.
We choose $i_1,\dots,i_l \in \{1,\dots,m\}$ and set

$$s(i_1,\dots,i_l) = \sum_{k=1}^{l} \frac{lcm(m_{i_1},\dots,m_{i_l})}{lcm(m_{i_k},m_{i_{k+1}})} e_{i_k}\wedge e_{i_{k+1}} \quad (i_{l+1} = i_1)$$

Then $\varphi_2(s(i_1,\dots,i_l)) = 0$. We call the elements $s(i_1,\dots,i_l)$ cyclic syzygies.

Theorem 2.1. *Let $I = (m_1,\dots,m_s) \subset R = K[x_1,\dots,x_n]$ be a monomial ideal, let U be a subset of $\{(i,j) : 1 \le i < j \le s\}$ and set $F = \sum_{(i,j)\in U} Se_i \wedge e_j$. Then $Ker(\varphi_2|F)$ is generated by cyclic syzygies.*

Proof. (see Bruns and Herzog 1995, Proposition 5.1) □

Example 2.1. *Let $I = (x,y,z,t) \subset K[x,y,z,t]$ be a monomial ideal. We consider the subsets of cardinality 3 of $\{1,2,3,4\}$. We have the following cyclic syzygies*

$$s(1,2,3) = ze_1 \wedge e_2 + xe_2 \wedge e_3 - ye_1 \wedge e_3$$

$$s(1,2,4) = te_1 \wedge e_2 + xe_2 \wedge e_4 - ye_1 \wedge e_4$$

$$s(1,3,4) = te_1 \wedge e_3 + xe_3 \wedge e_4 - ze_1 \wedge e_4$$

$$s(2,3,4) = te_2 \wedge e_3 + ye_3 \wedge e_4 - ze_2 \wedge e_4$$

If we consider the subset of cardinality 4, we obtain

$$s(1,2,3,4) = zte_1 \wedge e_2 + xte_2 \wedge e_3 + xye_3 \wedge e_4 - yze_1 \wedge e_4 = zs(1,2,3) + xs(2,3,4)$$

In fact, for this ideal, any cyclic syzygy has length 3. Same result for $I = (x_1,x_2,\dots,x_n) \subset K[x_1,x_2,\dots,x_n]$

Theorem 2.2. *Let $I = (m_1,\dots,m_t)$ be a monomial ideal of $K[x_1,x_2,\dots,x_n]$. Let $1 \le i_1 < i_2 < i_3 \le t$ and put:*

$$m_{i_1,i_2i_3} = \frac{lcm(m_{i_1},m_{i_2},m_{i_3})}{lcm(m_{i_2},m_{i_3})}.$$

Suppose that:

(a) *the second syzygy module of I is generated by trinomials;*
(b) *for $i_2 \ne j_2$ or $i_3 \ne j_3$, we have: $gcd\left(\frac{lcm(m_{i_1,i_2i_3})}{lcm(m_{i_2},m_{i_3})}, \frac{lcm(m_{j_1,j_2j_3})}{lcm(m_{j_2},m_{j_3})}\right) = 1$, $1 \le i_1 < i_2 < i_3 \le t$, $1 \le j_1 < j_2 < j_3 \le t$*
(c) *for $i_2 = j_2$, $i_3 = j_3$ and $i_1 \ne j_1$, we have:*
  (1) $m_{i_2,i_1i_3} = m_{i_2,j_1i_3} = m_{i_2,i_1j_1}$
  $m_{i_3,i_1i_2} = m_{i_3,j_1i_2} = m_{i_3,i_1j_1}$,
  $m_{i_1,j_1i_3} = m_{i_1,i_2i_3} = m_{i_1,j_1i_2}$
  (2) *if $q = gcd(m_{i_1,i_2i_3}, m_{j_1,j_2j_3})$, then $q/m_{i_2,i_1i_3}$ and $q/m_{i_3,i_1i_2}$.*

Then $Syz_1(I)$ is generated by an s-sequence.

Proof. The relation ideal Z of $Syz_1(I)$ is generated by

$$s_{i_1i_2i_3} = m_{i_1,i_2i_3}T_{i_2i_3} - m_{i_2,i_1i_3}T_{i_1i_3} + m_{i_3,i_1i_2}T_{i_1i_2}$$

with $T_{i_1i_2} < T_{i_1i_3} < T_{i_2i_3}$. We consider two trinomials $s_{i_1i_2i_3}$ and $s_{j_1j_2j_3}$, then $in_<(s_{i_1i_2i_3}) = m_{i_1,i_2i_3}T_{i_2i_3}$ and $in_<(s_{j_1j_2j_3}) = m_{j_1,j_2j_3}T_{j_2j_3}$. The $S-$polynomial is

$$S(s_{i_1i_2i_3}, s_{j_1j_2j_3}) = \frac{m_{j_1,j_2j_3}T_{j_2j_3}}{gcd(m_{i_1,i_2i_3}T_{i_2i_3}, m_{j_1,j_2j_3}T_{j_2j_3})}s_{i_1i_2i_3} - \frac{m_{i_1,i_2i_3}T_{i_2i_3}}{gcd(m_{i_1,i_2i_3}T_{i_2i_3}, m_{j_1,j_2j_3}T_{j_2j_3})}s_{j_1j_2j_3}$$

(b):

(i) If $j_2 \neq i_2$ or $j_3 \neq i_3$ and $j_1 \neq i_1$ then

$$gcd(m_{i_1,i_2i_3}T_{i_2i_3}, m_{j_1,j_2j_3}T_{j_2j_3}) = gcd(m_{i_1,i_2i_3}, m_{j_1,j_2j_3}) = 1$$

by hypothesis. Then the $S-$polynomial $S(s_{i_1i_2i_3}, s_{j_1j_2j_3})$ reduces to zero.

(ii) If $j_1 = i_1$, $j_2 \neq i_2$ or $j_3 \neq i_3$, then $s_{j_1j_2j_3} = s_{i_1j_2j_3}$ and

$$gcd(m_{i_1,i_2i_3}T_{i_2i_3}, m_{i_1j_2j_3}T_{j_2j_3}) = gcd(m_{i_1,i_2i_3}, m_{i_1,j_2j_3}) = 1$$

by hypothesis. Then the $S-$polynomial $S(s_{i_1i_2i_3}, s_{i_1j_2j_3})$ reduces to zero.

(c):

If $j_2 = i_2$ and $j_3 = i_3$, $i_1 \neq j_1$, we can suppose $i_1 < j_1$. Consider the set $\{i_1, j_1, i_2, i_3\}$, where $i_1 < j_1 < i_2 < i_3$, then we have:

$$gcd(m_{i_1,i_2i_3}T_{i_2i_3}, m_{j_1,i_2i_3}T_{i_2i_3}) = gcd(m_{i_1,i_2i_3}, m_{j_1,i_2i_3})T_{i_2}T_{i_3}$$

Put $gcd(m_{i_1,i_2i_3}, m_{j_1,i_2i_3}) = q$, then the $S-$polynomial is

$$S(s_{i_1i_2i_3}, s_{j_1i_2i_3}) = \frac{m_{j_1,i_2i_3}}{q}s_{i_1i_2i_3} - \frac{m_{i_1,i_2i_3}}{q}s_{j_1i_2i_3} =$$

$$= \frac{m_{j_1,i_2i_3}}{q}\left(-m_{i_2,i_1i_3}T_{i_1i_3} + m_{i_3,i_1i_2}T_{i_1i_2}\right) - \frac{m_{i_1,i_2i_3}}{q}\left(-m_{i_2,j_1i_3}T_{j_1i_3} + m_{i_3,j_1i_2}T_{j_1i_2}\right).$$

In the following $gcd(m_{i_1}, m_{i_2}) = [m_{i_1}, m_{i_2}]$, for $i_1 \neq i_2$.
By $m_{i_2,i_1i_3} = m_{i_2,j_1i_3}$ and by $m_{i_3,i_1i_2} = m_{i_3,j_1i_2}$, we can write:

$$S(s_{i_1i_2i_3}, s_{j_1i_2i_3}) = m_{i_2,j_1i_3}(-\frac{m_{j_1,i_2i_3}}{q}T_{i_1i_3} + \frac{m_{i_1,i_2i_3}}{q}T_{j_1i_3}) + m_{i_3,i_1i_2}(\frac{m_{j_1,i_2i_3}}{q}T_{i_1i_2} - \frac{m_{i_1,i_2i_3}}{q}T_{j_1i_2}) =$$

$$= m_{i_2,i_1i_3}(\frac{m_{i_3,i_1j_1}}{q}T_{i_1j_1} - \frac{m_{j_1,i_2i_3}}{q}T_{i_1i_3} + \frac{m_{i_1,i_2i_3}}{q}T_{j_1i_3}) - m_{i_2,i_1i_3}\frac{m_{i_3,i_1j_1}}{q}T_{i_1j_1} +$$

$$+ m_{i_3,i_1i_2}(\frac{m_{j_1,i_2i_3}}{q}T_{i_1i_2} - \frac{m_{i_1}, i_2i_3}{q}T_{j_1i_2}) =$$

$$= m_{i_2,i_1i_3}(\frac{m_{i_1,i_2i_3}}{q}T_{j_1i_3} - \frac{m_{j_1,i_2i_3}}{q}T_{i_1i_3} + \frac{m_{i_3,i_1j_1}}{q}T_{i_1j_1}) +$$

$$- \frac{m_{i_3,i_1i_2}}{q}(-m_{i_2,i_1i_3}T_{i_1j_1} + m_{j_1,i_2i_3}T_{i_1i_2} - m_{i_1,i_2i_3}T_{j_1i_2}).$$

By $m_{i_3,i_1j_1} = m_{i_3,i_1i_2}$, $m_{i_1,j_1i_3} = m_{i_1,i_2i_3}$, $m_{i_1,j_1i_2} = m_{i_1,i_2i_3}$,
$m_{j_1,i_2i_3} = m_{j_1,i_1i_2}$, we can write:

$$S(s_{i_1i_2i_3}, s_{j_1i_2i_3}) = \frac{m_{i_2,i_1i_3}}{q}(m_{i_1,j_1i_3}T_{j_1i_3} - m_{j_1,i_1i_3}T_{i_1i_3} + m_{i_3,i_1j_1}T_{i_1j_1}) +$$

$$-\frac{m_{i_3,i_1i_2}}{q}(m_{i_1,j_1i_2}T_{j_1i_2} - m_{j_1,i_1i_2}T_{i_1i_2} + m_{i_2,i_1j_1}T_{i_1j_1}) =$$

$$= \frac{m_{i_2,i_1i_3}}{q}s_{i_1j_1i_3} - \frac{m_{i_3,i_1i_2}}{q}s_{i_1j_1i_2}$$

and $S(s_{i_1i_2i_3}, s_{j_1i_2i_3})$ reduces to zero by $\{s_{i_1j_1i_3}, s_{i_1j_1i_2}\}$ □

Example 2.2. *Let I_k be the square-free monomial ideal of $K[x_1,\ldots,x_n]$ generated by all square-free monomials of degree k. For $k = n-1$, I_k satisfies all conditions of Theorem* **2.2**, *then $Syz_1(I_{n-1})$ is generated by an s-sequence.*

The following lemmas contain some equalities between monomials of $K[x_1,\ldots,x_n]$, in view of giving other sufficient conditions in order that the S-polynomials of the generators of the relation ideal of the symmetric algebra of the first syzygy module of I reduce to zero.

Lemma 2.1. *Let m_1, m_2, m_3 monomials of $K[x_1,\ldots,x_n]$. Then*

$$\frac{lcm(m_{i_1}, m_{i_2}, m_{i_3})}{lcm(m_{i_1}, m_{i_3})} = \frac{m_{i_1}}{lcm(gcd(m_{i_1}, m_{i_2}), gcd(m_{i_1}, m_{i_3}))}$$

Proof. Put $m_1 = x_1^{i_1}x_2^{i_2}\ldots x_n^{i_n}$, $m_2 = x_1^{j_1}x_2^{j_2}\ldots x_n^{j_n}$, $m_3 = x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$

$$\frac{lcm(m_1,m_2,m_3)}{lcm(m_2,m_3)} = \frac{x_1^{max(i_1,j_1,k_1)}\ldots x_n^{max(i_n,j_n,k_n)}}{x_1^{max(j_1,k_1)}\ldots x_n^{max(j_n,k_n)}} =$$

$$= x_1^{max(i_1,j_1,k_1)-max(j_1,k_1)}\ldots x_n^{max(i_n,j_n,k_n)-max(j_n,k_n)}$$

Moreover

$$\frac{m_1}{lcm(gcd(m_1,m_2), gcd(m_1,m_3))} =$$

$$= \frac{x_1^{i_1}\ldots x_n^{i_n}}{lcm(gcd(x_1^{i_1},x_1^{j_1}),\ldots,gcd(x_n^{i_n},x_n^{j_n}),gcd(x_1^{i_1},x_1^{k_1}),\ldots,gcd(x_n^{i_n},x_n^{k,n}))} =$$

$$x_1^{i_1-max(min(i_1,j_1),min(i_1,k_1))}\ldots x_n^{i_n-max(min(i_n,j_n),min(i_n,k_n))}$$

Suppose $i_t < j_t < k_t, \forall t$, $1 \le t \le n$, it results:

$$max(i_t, j_t, k_t) - max(j_t, k_t) = k_t - k_t = 0$$

and

$$i_t - max(min(i_t, j_t), min(i_t, k_t)) = i_t - max(i_t, k_t) = i_t - i_t = 0.$$

Hence the equality holds. For the other cases, it is easy to verify the assertion. □

Lemma 2.2. *Let $I = (m_1, \dots, m_t) \subset K[x_1, x_2, \dots, x_n]$ be a monomial ideal. Let $1 \leq i_1 < i_2 < i_3 \leq t$ and $1 \leq j_1 < j_2 < j_3 \leq t$. Put:*

$$m_{i_1,i_2i_3} = \frac{lcm(m_{i_1}, m_{i_2}, m_{i_3})}{lcm(m_{i_2}, m_{i_3})}.$$

Suppose:

$$lcm(m_{i_1}, m_{i_2}) = lcm(m_{i_1}, m_{j_2}),$$
$$lcm(m_{i_1}, m_{i_3}) = lcm(m_{i_1}, m_{j_3}),$$
$$lcm(m_{i_2}, m_{i_3}) = lcm(m_{j_2}, m_{j_3}).$$

Then

$$m_{i_1,i_2i_3} = m_{i_1,j_2j_3}.$$

Proof. Using the definition, we have:

$$m_{i_1,i_2i_3} = \frac{lcm(m_{i_1}, m_{i_2}, m_{i_3})}{lcm(m_{i_2}, m_{i_3})} = \frac{lcm(lcm(m_{i_1}, m_{i_2}), lcm(m_{i_1}, m_{i_3}))}{lcm(m_{i_2}, m_{i_3})} =$$
$$= \frac{lcm(lcm(m_{i_1}, m_{j_2}), lcm(m_{i_1}, m_{j_3}))}{lcm(m_{j_2}, m_{j_3})} = \frac{lcm(m_{i_1}, m_{j_2}, m_{j_3})}{lcm(m_{j_2}, m_{j_3})} = m_{i_1,j_2j_3}$$

□

Lemma 2.3. *Let $I = (m_1, \dots, m_s) \subset K[x_1, x_2, \dots, x_n]$ be a monomial ideal. Let $1 \leq i < j < k \leq s$ and $1 \leq l < h < g \leq s$. Put:*

$$m_{i,jk} = \frac{m_i}{lcm(gcd(m_i, m_j), gcd(m_i, m_k))}.$$

Suppose that:

$$gcd(m_i, m_j) = gcd(m_i, m_l)$$

and

$$gcd(m_i, m_k) = gcd(m_i, m_h).$$

Then $m_{i,jk} = m_{i,lh}$.

Proof. Using the definition of $m_{i,jk}$, we have:

$$m_{i,jk} = \frac{m_i}{lcm(gcd(m_i, m_j), gcd(m_i, m_k))} = \frac{m_i}{lcm(gcd(m_i, m_l), gcd(m_i, m_h))} = m_{i,lh}$$

□

Theorem 2.3. *Let $I = (m_1, \dots, m_s) \subset K[x_1, x_2, \dots, x_n]$ be a monomial ideal.*
Put:

$$m_{i,jk} = \frac{m_i}{lcm(gcd(m_i, m_j), gcd(m_i, m_k))}.$$

Suppose that:

(a) *the second syzygy module, $Syz_2(I)$, is generated by trinomials:*

$$\{s_{ijk} = m_{i,jk}g_{jk} - m_{j,ik}g_{ik} + m_{k,ij}g_{ij}, 1 \le i < j < k \le s\}$$

where g_{ij} is the free basis of $\bigwedge^2 R^s$;

(b) *for $j \neq h$ or $k \neq g$, $gcd(m_{i,jk}, m_{l,hg}) = 1$, $1 \le i < j < k \le s$, $1 \le l < h < g \le s$;*

(c) *for $h = j$, $g = k$, $i \neq l$*

(1) *$gcd(m_i, m_j) = gcd(m_i, m_l) = gcd(m_j, m_l)$;*
$gcd(m_k, m_l) = gcd(m_i, m_k) = gcd(m_j, m_k)$;

(2) *if $q = gcd(m_{i,jk}, m_{l,jk})$, then $q/m_{j,ik}$ and $q/m_{k,ij}$.*

Then $Syz_1(I)$ is generated by an s-sequence

Proof. It is sufficient to prove that the relation ideal of $Syz_1(I)$,

$$Z = (s_{ijk} = m_{i,jk}T_{jk} - m_{j,ik}T_{ik} + m_{k,ij}T_{ij}, 1 \le i < j < k \le s) \subset K[x_1, \dots, x_n][T_{ij},$$
$$1 \le i < j < j \le n]$$

has a linear Gröbner basis on the variables T_{ij}. We fix a term order compatible with the following order on the variables:

$$T_{12} < T_{13} < \dots < T_{s-1,s}$$

and $x_i < T_{ij}$, $\forall i, j$. Then

$$in_<(s_{ijk}) = m_{i,jk}T_{jk}.$$

Let s_{ijk} and s_{lhg} where $1 \le i < l < h < g \le n$ be two second syzygies, $in_<(s_{lhg}) = m_{l,hg}T_{hg}$. We consider the $S-$polynomial $S(s_{ijk}, s_{lhg}) = \frac{m_{l,jk}}{n}s_{ijk} - \frac{m_{i,jk}}{n}s_{ljk}$.

(b) If $j \neq h$ or $k \neq g$, $i \neq l$, the variables T_{jk} and T_{hg} are different and $gcd(m_{i,jk}T_{jk}, m_{l,hg}T_{hg}) = gcd(m_{i,jk}, m_{l,hg}) = 1$ by hypothesis. Then the $S-$polynomial $S(s_{ijk}, s_{lhg})$ reduces to zero.
If $h \neq j$ or $g \neq k$, $i = l$, then $s_{lhg} = s_{ihg}$, $in_<(s_{ihg}) = m_{i,hg}T_{hg}$,

$$gcd(m_{i,hg}T_{hg}, m_{i,jk}T_{jk}) = gcd(m_{i,hg}, m_{i,jk}) = 1$$

by hypothesis and the $S-$polynomial $S = (s_{ijk}, s_{ihj})$ reduces to zero.

(c) Let $i \neq l$,$h = j$, $g = k$, then $T_{hg} = T_{jk}$ and

$$gcd(m_{i,jk}T_{jk}, m_{l,jk}T_{jk}) = gcd(m_{i,jk}, m_{l,jk})T_{jk}$$

Put $q = gcd(m_{i,jk}, m_{l,jk})$, where q is a monomial, then the $S-$polynomial is

$$S(s_{ijk}, s_{ljk}) = \frac{m_{l,jk}}{q}s_{ijk} - \frac{m_{i,jk}}{q}s_{ljk} =$$

$$= \frac{m_{l,jk}}{q}(m_{i,jk}T_{jk} - m_{j,ik}T_{ik} + m_{k,ij}T_{ij}) - \frac{m_{i,jk}}{q}(m_{l,jk}T_{jk} - m_{j,lk}T_{lk} + m_{k,lj}T_{lj}) =$$

$$= \frac{m_{l,jk}}{q} m_{i,jk} T_{jk} - \frac{m_{l,jk}}{q} m_{j,ik} T_{ik} + \frac{m_{l,jk}}{q} m_{k,ij} T_{ij} - \frac{m_{i,jk}}{q} m_{l,jk} T_{jk} + \frac{m_{i,jk}}{q} m_{j,lk} T_{lk} +$$
$$- \frac{m_{i,jk}}{q} m_{k,lj} T_{lj} = - \frac{m_{l,jk}}{q} m_{j,ik} T_{ik} + \frac{m_{l,jk}}{q} m_{k,ij} T_{ij} + \frac{m_{i,jk}}{q} m_{j,lk} T_{lk} - \frac{m_{i,jk}}{q} m_{k,lj} T_{lj}.$$

Consider the set $\{i,l,j,k\}$. We can suppose $i < l$ and we have to consider only the case $i < l < j < k$. In the following $gcd(m_i, m_j) = [m_i, m_j]$.
By $[m_j, m_k] = [m_l, m_k]$, we have $m_{k,ij} = m_{k,lj}$ and by $[m_i, m_j] = [m_j, m_l]$, we have $m_{j,ik} = m_{j,lk}$ (Lemma 2.3).
By $[m_k, m_j] = [m_l, m_k]$, we have $m_{k,il} = m_{k,ij}$ (Lemma 2.3). Now we are exactly in the case (c) of the Theorem **2.2**, by Lemma 2.1 then the assertion holds. □

Remark 2.1. *One deduces easily that the conditions (c) in the last theorem imply the conditions (c) required in Theorem **2.2**, but their formulation is more explicit and we do not know if there is an equivalence.*

Example 2.3. *The sequence of monomials xy, xz, xu, xv in $K[x,y,z,u,v]$ satisfies all the conditions of Theorem **2.2** and Theorem **2.3**, as we can easy verify. If $I = (xy, xz, xu, xv)$, $Syz_2(I)$ is generated by the cyclic syzygies:*

$$ye_2 \wedge e_3 - ze_1 \wedge e_3 + ue_1 \wedge e_2,$$
$$ye_3 \wedge e_4 - ue_1 \wedge e_4 + ve_1 \wedge e_3,$$
$$ye_2 \wedge e_4 - ze_1 \wedge e_4 - ve_1 \wedge e_2,$$
$$ze_3 \wedge e_4 - ue_2 \wedge e_4 + ve_2 \wedge e_3$$

For the 2-syzygy generated by four elements of the basis, we have:

$$uve_1 \wedge e_2 + yve_2 \wedge e_3 + yze_3 \wedge e_4 + zue_4 \wedge e_1 =$$
$$u(ye_2 \wedge e_4 - ze_1 \wedge e_4 - ve_1 \wedge e_2) - y(ze_3 \wedge e_4 - ue_2 \wedge e_4 + ve_2 \wedge e_3),$$

then $Syz_2(I)$ is generated by trinomials.

In the following example, the second module of syzygies is generated by an s-sequence, but the sequence does not satisfy the conditions of Theorem **2.2** and Theorem **2.3**, showing the sufficiency of the conditions.

Example 2.4. *Let $I = (x_1x_2x_4, x_1x_3x_4, x_2x_3x_4, x_1x_2x_5, x_1x_3x_5, x_2x_3x_5)$ be a monomial ideal of the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. Let*

$$0 \to S^2(-5) \to S^7(-4) \to \oplus_{i=1}^{6} R_i e_i = S^6(-3) \to I \to 0$$

be the minimal graded free resolution.

$Syz_1(I) = \langle x_4e_6 - x_5e_3, x_4e_5 - x_5e_2, x_4e_4 - x_5e_1, x_1e_6 - x_2e_5, x_2e_5 - x_3e_4, x_1e_3 - x_2e_2, x_2e_2 - x_3e_1 \rangle$

$$J = (x_4T_6 - x_5T_3, x_4T_5 - x_5T_2, x_4T_4 - x_5T_1, x_1T_6 - x_2T_5, x_2T_5 - x_3T_4, x_1T_3 - x_2T_2, x_2T_2 - x_3T_1)$$

$$Syz_2(I) = \langle x_1g_7 - x_2g_6 - x_4g_4 + x_5g_2, x_2g_6 - x_3g_5 - x_4g_3 + x_5g_1 \rangle$$

$$Z = (x_1Y_7 - x_2Y_6 - x_4Y_4 + x_5Y_2, x_2Y_6 - x_3Y_5 - x_4Y_3 + x_5Y_1)$$

J does not admit a linear Gröbner basis for any term order in $R[T_1,\dots,T_6]$ with $x_i < T_j$, for all i, j, and $T_1 < T_2 < \dots < T_6$. In fact, if we compute the Gröbner basis G for J, we obtain:

$$G = \{T_2x_2 - T_1x_3, T_3x_1 - T_1x_3, T_5x_2 - T_4x_3, T_6x_1 - T_4x_3, T_4x_4 - T_1x_5,$$

$$T_5x_4 - T_2x_5, T_6x_4 - T_3x_5, T_3T_5x_5 - T_2T_6x_5, T_3T_4x_5 - T_1T_6x_5,$$

$$T_2T_4x_5 - T_1T_5x_5, T_2T_4x_3 - T_1T_5x_3, T_3T_4x_3 - T_1T_6x_3, -T_1T_3T_5x_3 + T_1T_2T_6x_3\}$$

Nevertheless, Z admits a linear Gröbner basis in the variables Y_i's:

$$G^{(1)} = \{Y_6x_2 - Y_5x_3 - Y_3x_4 + Y_1x_5, Y_7x_1 - Y_5x_3 - Y_3x_4 - Y_4x_4 + Y_1x_5 + Y_2x_5\}$$

so J is generated by a (not-strong) $s-$sequence:

$$in_<(Z) = ((x_2)Y_6, (x_1)Y_7)$$

with

$$I_1 = I_2 = I_3 = I_4 = I_5 = (0), \quad I_6 = (x_2), \quad I_7 = (x_1)$$

3. Syzygy modules of the maximal ideal

Let K be a field, $R = K[x_1, x_2, \dots, x_n]$ a polynomial ring and $\mathfrak{m} = (x_1, x_2, \dots, x_n)$ the maximal homogeneous ideal. We want to study the symmetric algebra of the first syzygy module of $\mathfrak{m}$, $Syz_1(\mathfrak{m})$ and of the $(n-1)$-th syzygy module of $\mathfrak{m}$, $Syz_{n-1}(\mathfrak{m})$. Consider the minimal free resolution of $K = R/(x_1, x_2, \dots, x_n)$ given by the $K.(x_1, x_2, \dots, x_n)$, the Koszul complex of S with respect to the set of elements $\{x_1, x_2, \dots, x_n\} = \{\underline{x}\}$

$$K(\underline{x}) := 0 \to K_n \to K_{n-1} \to \dots \to K_2 \to K_1 \to K_0 \to \mathfrak{m} \to 0$$

We are going to study $Syz_1(\mathfrak{m})$. By the morphism

$$\dots \to K_2 \to Syz_1 \to$$

we have the following presentation of the symmetric algebra

$$Sym_R(Syz_1(\mathfrak{m})) \cong Sym_R(K_2/Z)$$

where Z is the relation ideal of $Syz_1(\mathfrak{m})$. Since
$Syz_1(\mathfrak{m}) = \langle x_ie_j - x_je_i, 1 \le i < j \le n \rangle$ and $Syz_1(\mathfrak{m}) \subset K_1$, if we put $\sigma_{ij} = x_ie_j - x_je_i$, we

obtain

$$Sym_R(Syz_1(\mathfrak{m})) = R[Y_{ij}, 1 \le i < j \le n]/Z$$
$$Y_{ij} \to \sigma_{ij}$$

where Z is the relation ideal.

$$Z = (x_iY_{jk} - x_jY_{ik} + x_kY_{ij}, 1 \le i < j < k \le n)$$

Theorem 3.1. *The first syzygy module, $Syz_1(\mathfrak{m})$, of the maximal ideal $(x_1, x_2, \ldots, x_n)$ of the polynomial ring $K[x_1, x_2, \ldots, x_n]$ is generated by an s-sequence if and only if $n = 3$.*

Proof. $\Leftarrow$ For $n = 3$, the relation ideal Z of $Sym_R(Syz_1(\mathfrak{m}))$ is generated by the trinomial

$$Z = (x_1Y_{23} - x_2Y_{13} + x_3Y_{12})$$

and $in_<(Z) = (x_1Y_{23})$ for the order on the variables $Y_{23} > Y_{13} > Y_{12}$, hence the assertion hold.

$\Leftarrow$ Suppose $n > 3$, than the first syzygy module of the ideal $\mathfrak{m}$ cannot be generated by an s-sequence and then the assertion holds. Let $\sigma_{ijk} = x_iY_{jk} - x_jY_{ik} + x_kY_{ij}$, $1 \le i < j < k \le n$, and $\sigma_{ghl} = x_gY_{hl} - x_hY_{gl} + x_lY_{gh}$, $1 \le g < h < l \le n$ two elements of the set σ. For the admissible order on the variables Y_{ij}, $in_<(\sigma_{ijk}) = x_iY_{jk}$ and $in_<(\sigma_{ghl}) = x_gY_{hl}$ and each $S-$polynomial must to be zero by definition of s-sequence.

(i) If $g \ne i$, $h \ne j$ or $l \ne k$, $gcd(x_iY_{jk}, x_gY_{hl}) = gcd(x_i, x_g) = 1$ and then since the leading terms of σ_{ijk} and σ_{ghl} are coprime, the $S-$polynomial $S(\sigma_{ijk}, \sigma_{ghl})$ reduces to zero. If $g = i$, $h \ne j$ or $l \ne k$, $in_<(\sigma_{ijk}) = x_iY_{hl}$, $in_<(\sigma_{ghl}) = x_iY_{jk}$, $gcd(x_iY_{jk}, x_iY_{hl}) = x_i$

$$S(\sigma_{ijk}, \sigma_{ghl}) = Y_{jk}(x_iY_{jk} - x_jY_{ik} + x_kY_{ij}) - Y_{hl}(x_iY_{hl} - x_hY_{il} + x_lY_{ih}) =$$
$$= -x_jY_{ik}Y_{hl} + x_kY_{ij}Y_{hl} + x_hY_{il}Y_{jk} - x_lY_{ih}Y_{jk}$$

It is easy to verify that no term of this sum divides the initial term of all the elements of the set $\{\sigma_{ijk}, 1 \le i < j < k \le n\}$, for no admissible term order on the monomials in the variables Y_{ij}. Consider the monomial $x_jY_{ik}Y_{hl}$. For $i = g$, $h \ne j$, $l \ne k$, we have: $i = g < j < k$, $i = g < h < l$ and in correspondence we have the monomials x_iY_{jk}, x_iY_{hl}, initial terms of the trinomials $x_iY_{jk} - x_jY_{ik} + x_kY_{ij}$, $x_iY_{hl} - x_hY_{il} + x_lY_{ih}$ and these monomials do not divide $x_jY_{ik}Y_{hl}$. Since they not divide none of the remaing monomials, $x_kY_{ij}Y_{hl}$, $x_hY_{il}Y_{jk}$, $x_lY_{ih}Y_{jk}$, the assertion follows. Then $S(\sigma_{ijk}, \sigma_{gkl})$ cannot be reduced by the set $\{\sigma_{ijk}, \le i < j < k \le n\}$.

(*ii*) If $i \neq g$, $h = j$, $l = k$, $in_<(\sigma_{ijk}) = x_i Y_{jk}$, $in_<(\sigma_{ghl}) = x_g Y_{jk}$, $gcd(x_i Y_{jk}, x_g Y_{jk}) = Y_{jk}$

$$S(\sigma_{ijk}, \sigma_{gjk}) = \frac{x_g Y_{jk}}{Y_{jk}}(x_i Y_{jk} - x_j Y_{ik} + x_k Y_{ij}) - \frac{x_i Y_{jk}}{Y_{jk}}(x_g Y_{jk} - x_j Y_{gk} + x_k Y_{gj}) =$$
$$= \quad -x_g x_j Y_{ik} + x_g x_k Y_{ij} + x_i x_j Y_{gk} - x_i x_k Y_{gj}.$$

Consider the set $\{i, g, j, k\}$. We can suppose $i < g$ and we have to consider only the case $i < g < j < k$.

We can write

$$S(\sigma_{ijk}, \sigma_{gjk}) = x_j(-x_g Y_{ik} + x_i Y_{gk} + x_k Y_{ig}) - x_k(x_j Y_{ig} - x_g Y_{ij} + x_i Y_{gj}) =$$

$$= x_j(x_i Y_{gk} - x_g Y_{ik} + x_k Y_{ig} - x_k(x_i Y_{gj} - x_g Y_{ij} + x_j Y_{ij}) = x_j \sigma_{igk} - x_k \sigma_{igj}$$

and $S(\sigma_{ijk}, \sigma_{gjk})$ reduces to zero by $\{\sigma_{igk}, \sigma_{igj}\}$.

□

Corollary 3.1. *For $n > 3$, the set $\sigma = \{x_i Y_{jk} - x_j Y_{ik} + x_k Y_{ij}, 1 \leq i < j < k \leq n\}$ is not a Gröbner basis of Z, for any term order on the monomials Y_{ij} and such that*

$$x_i < Y_{12} < Y_{13} < \ldots < Y_{n-1,n}$$

Theorem 3.2. *Let I_{ij}, $1 \leq i < j \leq n$, be the annihilator ideals of the sequence $\{\sigma_{ij}, 1 \leq i < j \leq n\}$ generating $Syz_1(\mathfrak{m})$. Then $I_{ij} = (x_1, \ldots, x_{i-1})$, $1 \leq i < j \leq n$, $i = 2, \ldots, n$, and $I_{1j} = 0$, for all $j \leq n$.*

Proof. Put $I_{ij} = (\sigma_{12}, \sigma_{13}, \ldots, \sigma_{i,j-1}) : (\sigma_{ij})$. Then

$$I_{12} = (0) : (\sigma_{12}) = (0)$$

$$I_{13} = (0, \sigma_{12}) : (\sigma_{13}) = (0)$$

In fact $\sigma_{12} = x_1 e_2 - x_2 e_1$, $\sigma_{13} = x_1 e_3 - x_3 e_1$. Let $\lambda \in I_{13}$ such that $\lambda \sigma_{13} = \mu \sigma_{12}$, $\lambda, \mu \in S$,

$$\lambda(x_1 e_3 - x_3 e_1) = \mu(x_1 e_2 - x_2 e_1)$$

that is

$$\lambda x_1 e_3 - \lambda x_3 e_1 = \mu x_1 e_2 - \mu x_2 e_1$$

Then

$$\begin{cases} \lambda x_1 = 0 \\ \mu x_1 = 0 \\ \lambda x_3 - \mu x_2 = 0 \end{cases}$$

hence $\lambda = \mu = 0$ and $I_{13} = (0)$

$$I_{14} = (\sigma_{12}, \sigma_{13}) : (\sigma_{14})$$

From the sequence $1 < 2 < 3 < 4$, we have the forms where σ_{14} appears:

$$f_{124} = x_1\sigma_{24} - x_2\sigma_{14} + x_4\sigma_{12}, \quad in_<(f_{124}) = x_1\sigma_{24}$$

$$f_{134} = x_1\sigma_{34} - x_3\sigma_{14} + x_4\sigma_{13}, \quad in_<(f_{134}) = x_1\sigma_{34}$$

Consequently $I_{14} = 0$. On the other hand, by direct calculations:
Let $\lambda \in I_{14}$ such that $\lambda\sigma_{14} = \mu\sigma_{12} + \nu\sigma_{13}$, $\lambda, \mu, \nu \in S$,

$$\lambda(x_1e_4 - x_4e_1) = \mu(x_1e_2 - x_2e_1) + \nu(x_1e_3 - x_3e_1)$$

$$\lambda x_1e_4 - \lambda x_4e_1 = \mu x_1e_2 - \mu x_2e_1 + \nu x_1e_3 - \nu x_3e_1$$

Then

$$\begin{cases} \lambda x_1 = 0 \\ \nu x_1 = 0 \\ -\lambda x_4 - \mu x_2 = 0 \\ \mu x_1 - \nu x_3 = 0 \end{cases}$$

hence $\lambda = \mu = \nu = 0$ and $I_{14} = (0)$
Likewise for the other ideals until I_{1n}. The first non zero annihilator ideal is I_{23}. In fact, for I_{23} we have:

$$I_{23} = (\sigma_{12}, \sigma_{13}, \ldots, \sigma_{1n}) : (\sigma_{23}) = (x_1)$$

From the sequence $1 < 2 < 3$, we have the forms in which σ_{23} appears:

$$f_{123} = x_1\sigma_{23} - x_2\sigma_{13} + x_3\sigma_{12}, \quad in_<(f_{123}) = x_1\sigma_{23}.$$

$$I_{24} = (\sigma_{12}, \sigma_{13}, \ldots, \sigma_{1n}, \sigma_{23}) : (\sigma_{24}) = (x_1).$$

In fact, from the sequence $1 < 2 < 3 < 4$, we have the forms in which σ_{24} is the leading term:

$$f_{124} = x_1\sigma_{24} - x_2\sigma_{14} + x_4\sigma_{12}, \quad in_<(f_{124}) = x_1\sigma_{24}$$

And so on. In general, we have:

$$I_{2j} = (x_1) \text{ for } j > 2$$

For the ideal I_{34}, from the sequence $1 < 2 < 3 < 4$, we have the forms in which σ_{34} is the leading term:

$$f_{134} = x_1\sigma_{34} - x_3\sigma_{14} + x_4\sigma_{13}, \quad in_<(f_{134}) = x_1\sigma_{34}$$

$$f_{234} = x_2\sigma_{34} - x_3\sigma_{24} + x_4\sigma_{23}, \quad in_<(f_{234}) = x_2\sigma_{34}$$

then $I_{34} = (x_1, x_2)$ and in general

$$I_{3j} = (x_1, x_2) \text{ for } j > 3$$

By induction

$$I_{n-1,n} = (x_1, x_2, \ldots, x_{n-2})$$

Therefore

$$I_{ij} = (x_1, \dots, x_{i-1}), \text{ for } 1 < i < j \leq n \quad \text{ and } I_{1j} = (0) \quad \forall j \leq n$$

□

Proposition 3.1. *Let $Syz_1(\mathfrak{m})$ the first syzygy module of $\mathfrak{m} = (x_1, x_2, x_3) \subset R = K[x_1, x_2, x_3]$. Then we have:*

(i) $dim(Sym_R(Syz_1(\mathfrak{m}))) = 4$
(ii) $e(Sym_R(Syz_1(\mathfrak{m}))) = 4$
(iii) $depth(Sym_R(Syz_1(\mathfrak{m}))) = 4$
(iv) $reg(Sym_R(Syz_1(\mathfrak{m}))) = 1$

Proof. Since $Syz_1(\mathfrak{m})$ is generated by the $s-$sequence s_{12}, s_{13}, s_{23},

$$Sym_R(Syz_1(\mathfrak{m})) = R[Y_{ij}, 1 \leq i < j \leq 3]/Z,$$

being $Z = (x_1Y_{23} - x_2Y_{13} + x_3Y_{12})$. Then we have:
$in_<(Z) = (x_1T_{23})$

(i)

$$\begin{aligned} dim(Sym_R(Syz_1(\mathfrak{m})) &= dim(R[Y_{ij}, 1 \leq i < j \leq 3]/in_<(Z)) = \\ &= dim(R/I_{23}) + 2 = dim(R/(x_1)) + 2 = 4. \end{aligned}$$

(ii)

$$e(Sym_R(Syz_1(\mathfrak{m})) = e(R[Y_{ij}, 1 \leq i < j \leq 3]/in_<(Z)) = e(R/I_{23}) = e(K[x_2, x_3]) = 4$$

(iii)

$$\begin{aligned} depth(Sym_R(Syz_1(\mathfrak{m})) &\geq depth(R[Y_{ij}, 1 \leq i < j \leq 3]/in_<(Z)) = \\ &= depth(R/I_{23}) + 2 = 4 \end{aligned}$$

It follows the equality, since $depth(A) \leq dim(A)$ for any finitely generated $K-$algebra.

(iv)

$$\begin{aligned} reg(Sym_R(Syz_1(\mathfrak{m})) &= reg(R[Y_{ij}, 1 \leq i < j \leq 3]/Z) \leq \\ &\leq reg(R[Y_{ij}, 1 \leq i < j \leq 3]/in_<(Z)) \leq reg(R/I_{23}) = 1 \end{aligned}$$

□

We recall here a Groebner basis for the relation ideal of $Syz_1(\mathfrak{m})$, not linear in the variables Y_{ij}, obtained by Herzog, Tang, and Zarzuela (2003) for a suitable order on the variables and for the lexicographic order on the monomial in the variables Y_{ij}.

Put $P_2(Y) = \{Y_{ij}Y_{kl} - Y_{jk}Y_{il} + Y_{ik}Y_{jl}, 1 \leq\leq i < j < k < l \leq n\}$, then, we have:

Theorem 3.3. *The set* $G = \{x_iY_{jk} - x_jY_{ik} + x_kY_{ij}, 1 \le i < j < k \le n\} \cup \{x_rP_2(Y), 1 \le r \le n\}$ *is a Groebner basis of* Z *with respect to any term order with* $Y_{n-1,n} < \ldots < Y_{12} < x_1 < x_2 < \ldots < x_n$.

Proof. (see Herzog, Tang, and Zarzuela 2003, Lemma 3.1) □

Now we present a variation of the Groebner basis that appears in the previous Theorem, obtained by varying the order on the variables and the admissible order. The obtained Groebner basis is coherent with our definition of s-sequence.

Theorem 3.4. *The set* $G = \{x_iY_{jk} - x_jY_{ik} + x_kY_{ij}, 1 \le i < j < k \le n\} \cup \{x_rP_2(Y), 1 \le r \le n\}$ *is a Groebner basis of* Z *with respect to* $x_n > \ldots > x_1 > Y_{12} < \ldots < Y_{n-1,n}$

Proof. By symmetry, G is also a Groebner basis with respect to any term order with $x_n > \ldots > x_1 > Y_{12} < \ldots < Y_{n-1,n}$. Let $u \in in_<(Z)$. Then $u \in in_<(I)$, and so $u = vw$ where $w = in(g)$ for some $g \in G$. If g belongs to $\{x_iY_{jk} - x_jY_{ik} + x_kY_{ij}, 1 \le i < j < k \le n\}$, then there is nothing to prove. Otherwise $g \in P(Y)$, and w is a monomial only in the Y_{ij}. On the other hand, $u = in(h)$ for some $h \in Z$, and hence the leading term of h is divisible by some x_i. Therefore v is divisible by some x_i, and so $u = v'(x_iw)$ for some monomial v'. Since $x_iw = in(x_ig)$, and $x_ig \in \{x_rP_2(Y), 1 \le r \le n\}$, the conclusion follows. □

Remark 3.1. *We note that the Groebner basis given by Theorem* ***3.3*** *is linear in the variables* $x_1, x_2, \ldots, x_n$*. This fact will be crucial for proving that the Jacobian dual of* $Syz_1(\mathfrak{m})$ *is generated by the* $s-$*sequence* $x_1^*, x_2^*, \ldots, x_n^*$*, where* $x_1^*, x_2^*, \ldots, x_n^*$ *are the images of* $x_1, x_2, \ldots, x_n$ *in* $Sym(Syz_1(\mathfrak{m}))$

Now, we study two syzygy modules of $\mathfrak{m}$ whose symmetric algebra has a simple presentation, as shown by Restuccia, Utano, and Tang (2014).

Theorem 3.5. *Let* $Syz_{n-1}(\mathfrak{m})$ *be the* $(n-1)-$*th syzygy module of* $\mathfrak{m}$*, then* $Syz_{n-1}(\mathfrak{m})$ *is generated by a s-sequence.*

Proof. By the complex $K.(\underline{x})$, $Syz_{n-1}(\mathfrak{m})$ has the presentation

$$0 \to K_n \to K_{n-1} \to Syz_{n-1}(\mathfrak{m}) \to 0$$

$$Sym_R(Syz_{n-1}(\mathfrak{m})) \cong Sym_R(K_{n-1})/Z,$$

where

$$Sym_R(K_{n-1}) = Sym_R(\bigwedge^{n-1} P^n)$$

Since

$$\bigwedge^{n-1} P^n \cong \bigwedge^{n-(n-1)} P^n \cong P^n = Pe_1 \oplus Pe_2 \oplus \ldots \oplus Pe_n$$

$$Sym_R(K_{n-1}) \cong Sym_R(R^n) \cong R[Y_1, \ldots, Y_n],$$

a polynomial ring in n variables on R, being $K_n = \bigwedge^n P^n$, the module K_n is generated by the unique generator,

$$\widetilde{t} = e_1 \wedge e_2 \wedge \ldots \wedge e_n.$$

Under the Koszul maps:

$$0 \to K_n - K_{n-1},$$

we map

$$\widetilde{t} \to \sum_{i=1}^{n} (-1)^{i+1} x_i e_1 \wedge e_2 \wedge \ldots \widehat{e_i} \wedge \ldots \wedge e_n$$

Concerning the generators of $Syz_{n-1}(\mathfrak{m})$, they are the image of the generators of $K_{n-1} \cong \bigwedge^{n-1} P^n$, under the map

$$\partial(e_1 \wedge e_2 \wedge \ldots \widehat{e_i} \wedge \ldots \wedge e_n) = \sum_{j=1}^{n-1} (-1)^{j+1} x_j e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \cdots \wedge e_n \tag{1}$$

Then $Syz_{n-1}(\mathfrak{m})$ has $n-1$ generators given by the elements of the right-hand member of (1). So, $Sym_R(Syz_{n-1}(\mathfrak{m})) \cong R[Y_1, \ldots, Y_n]/Z$ and $Z = (t)$, where $t = \sum_{i=1}^{n} (-1)^{i+1} x_i Y_i$, the unique linear form in the variables Y_i and $in_<(t) = (-1)^{n+1} x_n Y_n$, hence

$$in_<(Z) = \big(((-1)^{n+1} x_n) Y_n\big) = ((x_n) Y_n)$$

and it follows

$$I_n = (x_n)$$

for the annihilator ideal. □

Proposition 3.2. *Let $Syz_{n-1}(\mathfrak{m})$ the $(n-1)$-th syzygy module of $\mathfrak{m}$. Then we have:*

(i) $dim(Sym_R(Syz_{n-1}(\mathfrak{m}))) = 2n-1$
(ii) $e(Sym_R(Syz_{n-1}(\mathfrak{m}))) = 0$
(iii) $depth(Sym_R(Syz_{n-1}(\mathfrak{m}))) = 2n-1$
(iv) $reg(Sym_R(Syz_{n-1}(\mathfrak{m}))) = 1$

Proof. By the structure of $in_<(Z) = (x_n Y_n)$ and $I_n = (x_n)$, we can compute the invariants. We have $Sym_R(Syz_{n-1}(\mathfrak{m})) = R[Y_1, \ldots, Y_n]/Z$ □

It is easy to study when the $n-$th syzygy module of the ideal $\mathfrak{m}$, $Syz_n(\mathfrak{m})$, is generated by s-sequences. We recall that $Syz_n(\mathfrak{m}) = K_n = \bigwedge^n P^n \cong \bigwedge^{n-n} P^n \cong R$. Therefore for the symmetric algebra of $Syz_n(\mathfrak{m})$, we have $Sym_R(Syz_n(\mathfrak{m})) \cong Sym_R(R) \cong Sym_R(Re)$, where e is the unique free generator. Hence $Sym_R(R) = R[Y]$, a polynomial ring in a new variable Y on R, $Sym_R(R) = K[x_1, x_2, \ldots, x_n, Y]$ a polynomial ring in $n+1$ variables on K. The relation ideal of $Sym_R(R)$ is $Z = (0)$ and then the syzygy module $Syz_n(\mathfrak{m})$ is generated by the $s-$sequence e. We have only one annihilator ideal $I_1 = (0) = \langle 0 \rangle : \langle e \rangle = (0)$.

The result shows the claim:

Proposition 3.3. *Let* $Syz_n(\mathfrak{m})$ *the n-th syzygy module of* $\mathfrak{m}$*. Then one has:*

(i) $dim(Sym_R(Syz_n(\mathfrak{m}))) = n+1$

(ii) $e(Sym_R(Syz_n(\mathfrak{m}))) = 0$

(iii) $depth(Sym_R(Syz_n(\mathfrak{m}))) = n+1$

(iv) $reg(Sym_R(Syz_n(\mathfrak{m}))) = 0$

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