DOI: 10.1478/AAPP.97S1A16

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 97, No. S1, A16 (2019)

EFFECTS OF VORTEX TANGLE ANISOTROPY IN COUNTERFLOW SUPERFLUID TURBULENCE

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ABSTRACT. The physical picture of counterflow superfluid turbulence is a disordered tangle of quantized vortex lines. This tangle is produced and sustained by a heat flux crossing the system. When such a heat flux is high enough, vortices appear and evolve. In the former literature the vortex tangle is described by using only a scalar quantity, dependent on the imposed heat flux, the average vortex line density per unit volume. Recent experiments and numerical simulations show that the tangle is not homogeneous nor isotropic, especially in non steady states, or in the simultaneous presence of counterflow and rotation. In general situations the vortex tangle must be considered as a dynamical quantity, for which an evolution equation must be given. Here, we deepen a thermodynamical model of inhomogeneous counterflow superfluid turbulence worked out in a previous paper (Z. Angew. Math. Phys. (2018), **69**: 2), considering the particular case of counterflow (presence of heat flow but absence of mass flow). The model chooses as fundamental fields the energy density, the heat flux, and a complete vorticity tensor, including its scalar part, its symmetric trace-less part and its antisymmetric part. Restriction for constitutive relations are obtained using Liu's method of Lagrange multipliers. Assuming that the dynamical evolution of the vorticity tensor is much faster than that of the heat flux, we derive an evolution equation for the heat flux of Guyer-Krumhansl type. This leads to a new physical interpretation of some terms responsible for the attenuation of second sound.

1. Introduction

Liquid helium (isotope ^{4}He) is called helium I (He I) in the range of temperature from its boiling point to the lambda transition T_λ, and helium II (He II) from T_λ down to melting point. The temperature T_λ depends on the pressure; at vapor pressure $T_\lambda = 2.172$K. Helium I has classical fluid properties. Helium II has extraordinary physical features. In particular heat is propagated in it by an entirely new mechanism. In fact a peculiar property of liquid He II is the ability of propagating second sound: a quantum mechanical phenomenon in which heat transfer occurs by wave-like motion, rather than by the more usual mechanism of diffusion. A second example of the non-classical behavior of liquid He II is heat transfer in counterflow experiments, characterized by no net matter flow but only heat transport. In these experiments, if the heat flux $\mathbf{q}$ inside a considered channel is not too high ($q < q_c$, with q_c a critical value), the temperature gradient is so small that it can not be measured, so indicating that the liquid has an extremely high thermal conductivity (three million times larger than that of He I). This is confirmed by the fact that He II is unable to boil.

Two different macroscopic theories have been proposed to describe the behaviour of liquid ^{4}He below the λ-point: the most popular is the two-fluid model formulated independently by Tisza (1938) and Landau (1941) and the one-fluid model (Mongiovì 1993, 2001; Jou *et al.* 2002; Mongiovì and Jou 2007; Saluto *et al.* 2014, 2015; Mongiovì *et al.* 2018) which uses the heat flux as an independent variable, deduced in the framework of extended thermodynamics (Muller and Ruggeri 1993, 1998; Jou *et al.* 2010, 2011a; Jou and Restuccia 2011). Extended thermodynamics is a theory of nonequilibrium processes which goes beyond the local-equilibrium approximation, as it chooses the fluxes as fundamental fields. In the study of nonequilibrium phenomena, an extended thermodynamic approach is required when one is interested in sufficiently rapid phenomena, or when the relaxation times of some flux is long; in such cases, a constitutive description of this flux in terms of the traditional field variables is impossible, so that it must be treated as independent field in the description of the thermodynamic process.

An important issue to be addressed to the study of superfluid helium is the onset of turbulence (Donnelly 1991; Nemirovskii and Fiszdon 1995; Barenghi *et al.* 2001; Vinen and Niemela 2002; Van Sciver 2012; Nemirovskii 2013; Tsubota *et al.* 2013). Indeed, when the heat flux exceeds the critical value q_c, a disordered tangle of microscopic quantized vortices is formed, thus modifying the thermal conductivity of the superfluid (Van Sciver 2012; Sciacca *et al.* 2015). In some situations, to describe the vortex array it is sufficient to introduce the average vortex line length per unit volume L, briefly called "vortex line density ". But, in other situations, as in counterflow in channels of high aspect ratio, one expects a partially polarized vortex tangle, due to the presence of numerous vortices pinned to the walls of the container. Indeed, since the vortex lines can not have free ends, they will be free closed loops (moving, breaking and reconnecting in the volume) or pinned to the surface. Thus, in these situations it is necessary to obtain a more complete geometrical description of the tangle.

In a previous paper (Mongiovì and Restuccia 2018), the authors developed a (coarse grained) hydrodynamical model of liquid helium II, taking into account the presence of a polarized anisotropic tangle of quantized vortices, in order to better study the internal structure of a turbulent superfluid. The model chooses as thermodynamic state vector the mass density, the velocity, the energy density, the heat flux, and a complete vorticity tensor field, including its trace, its symmetric trace-less part and its antisymmetric part.

In this paper, we restrict our attention to the study of heat transport in liquid helium II at rest, and generalize in this treated case the expressions for the fluxes of the fundamental fields chosen in the previous paper (Mongiovì and Restuccia 2018). Assuming that the dynamical evolution of the vorticity tensor is much faster than that of the heat flux, one obtains an evolution equation for the heat flux that is reminiscent of the Guyer and Krumhansl model (Guyer and Krumhansl 1966a,b). This allow us a new physical interpretation of the terms responsible for the attenuation of second sound.

2. Local model of anisotropic polarized counterflow superfluid turbulence

In a previous paper (Mongiovì and Restuccia 2018), in the framework of extended thermodynamics with internal variables, we developed a (coarse grained) hydrodynamical model of liquid helium II, taking into account the presence of a polarized and anisotropic tangle of quantized vortices.

In this Section, we focus our attention to the study of heat transport in liquid helium II at rest, and modify the results obtained, considering more general expressions for the fluxes, see Mongiovì and Restuccia (2018). We assume that the thermodynamic state vector of the independent variables is as follows:

$$\mathscr{C} = \{E, \mathbf{q}, \mathbf{P}_\omega\}, \tag{1}$$

where E is the energy density, $\mathbf{q}$ the heat flux and $\mathbf{P}_\omega$ the vorticity tensor, defined by Jou and Mongiovì (2006):

$$\mathbf{P}_\omega = \frac{1}{3}\kappa L\left[B_{HV}\mathbf{\Pi}^s + B'_{HV}\mathbf{\Pi}^a\right], \tag{2}$$

where κ is the quantum of vorticity, B_{HV} and B'_{HV} are the Hall–Vinen experimental parameters (Hall and Vinen 1956; Vinen 1957) and

$$\mathbf{\Pi}^s \equiv \frac{3}{2}\langle \mathbf{U} - \mathbf{s}'\otimes\mathbf{s}'\rangle = \frac{3}{2}(\mathbf{U}-\mathbf{c}), \qquad \mathbf{\Pi}^a \equiv \frac{3}{2}\langle \boldsymbol{\varepsilon}\cdot\mathbf{s}'\rangle = \frac{3}{2}\boldsymbol{\varepsilon}\cdot\mathbf{p}, \tag{3}$$

with $\mathbf{s}'$ the unit vector tangent to a vortex line, $\mathbf{U} \equiv (\delta_{ij})$ the identity tensor and $\boldsymbol{\varepsilon} \equiv (\varepsilon_{ijk})$ the antisymmetric Ricci tensor. In Eq. (2) L is the average *vortex line density*, defined by Donnelly (1991):

$$L = \frac{1}{\Lambda}\int d\xi, \tag{4}$$

being ξ the arc-length and the integral is calculated on all vortex lines in the elementary volume Λ; in (3) $\mathbf{p} = (p_i)$ is the *polarity vector* defined by Jou and Mongiovì (2006):

$$\mathbf{p} = \langle \mathbf{s}'\rangle = \frac{1}{\Lambda L}\int \mathbf{s}' d\xi, \tag{5}$$

where the symbol $\langle..\rangle$ represents the average calculated on the ensemble of vortex lines inside the elementary particle (coarse grained model). Finally, $\mathbf{c} = (c_{ij})$ is the anisotropic tensor, modeling the anisotropy of the vortex tangle inside the considered particle, and is defined by Jou and Mongiovì (2006) and Jou *et al.* (2011b):

$$\mathbf{c} = \langle \mathbf{s}'\otimes\mathbf{s}'\rangle = \frac{1}{\Lambda L}\int \mathbf{s}'\otimes\mathbf{s}' d\xi, \tag{6}$$

with $\mathbf{s}'\otimes\mathbf{s}' = \mathbf{s}'(\xi,t)\otimes\mathbf{s}'(\xi,t)$ and $\otimes$ the dyadic product. We observe that the tensor $\mathbf{c}$ has only five independent components, because its trace is equal to 1: $c_{ll} = {s'_x}^2 + {s'_y}^2 + {s'_z}^2 = 1$.

The vorticity tensor can be split in its scalar, deviatoric and antisymmetric parts as follows (Mongiovì and Restuccia 2018):

$$\overline{\mathbf{P}}_\omega = \frac{1}{3}\mathscr{P}^\omega_{ll}\mathbf{U} = \kappa L B_{HV}\mathbf{U}, \tag{7}$$

$$\mathbf{P}^{ss}_\omega = (\mathscr{P}^{ss}_{<ij>}) = \frac{1}{2}\kappa L B_{HV}\left(\frac{1}{3}\delta_{ij} - c_{ij}\right), \tag{8}$$

$$\mathbf{P}^{a}_\omega = (\mathscr{P}^{a}_{[ij]}) = \kappa L B'_{HV}\varepsilon_{ijk}p_k. \tag{9}$$

In Eqs. (8) and (9) and in the following angular and square brackets around the indices denote the deviatoric and the antisymmetric parts of a second order tensor (*i.e.*, $A_{<ij>} = (1/2)(A_{ij}+A_{ji}) - (1/3)A_{ll}\delta_{ij}$ and $A_{[ij]} = (1/2)(A_{ij}-A_{ji})$). Summation convention on repeated indices will be used throughout this work.

In stationary counterflow superfluid turbulence, when the imposed heat flux $\mathbf{q}_0$ is constant, the tensor $\mathbf{P}_\omega$ is often assumed a constant quantity, while $\mathbf{P}_\omega$ must be considered a field tensor in an unsteady inhomogeneous situation. In the following subsections, we construct evolution equations for the three internal variables (7), (8), (9) describing $\mathbf{P}_\omega = \mathbf{P}_\omega(\mathbf{x},t)$, with the aim to study their effects on counterflow superfluid turbulence.

2.1. Balance equations. We postulate the following set of balance equations, for the determination of the evolution equations of the state variables:

$$\begin{cases} \frac{\partial E}{\partial t} + \frac{\partial q_j}{\partial x_j} = Q^E, \\ \frac{\partial q_i}{\partial t} + \frac{\partial J^q_{ij}}{\partial x_j} = Q^q_i, \\ \frac{\partial L}{\partial t} + \frac{\partial J^L_k}{\partial x_k} = Q^L, \\ \frac{\partial \mathscr{P}^{ss}_{<ij>}}{\partial t} + \frac{\partial J^{ss}_{<ij>k}}{\partial x_k} = Q^{ss}_{<ij>}, \\ \frac{\partial \mathscr{P}^{a}_{[ij]}}{\partial t} + \frac{\partial J^{a}_{[ij]k}}{\partial x_k} = Q^{a}_{[ij]}. \end{cases} \tag{10}$$

The first equation is the energy balance equation, with Q^E the energy source; for the heat flux $\mathbf{q} = (q_i)$ we have postulated a Maxwell-Vernotte-Cattaneo type equation with J^q_{ij} the flux of the vector q_i, and Q^q_i the source term, that are constitutive quantities and must be expressed as function of the state vector $\mathscr{C}$. The three latter equations are the rate equations that describe the dynamics of the vortex tangle; in them the fluxes J^L_k, $J^{ss}_{<ij>k}$ and $J^a_{[ij]k}$ are constitutive quantities; Q^L, $Q^{ss}_{<ij>}$ and $Q^a_{[ij]}$ are source terms.

For the heat flux production term $\mathbf{Q}^q$ we will use the following constitutive relation proposed by Jou *et al.* (2002) (see also Jou and Mongiovì (2006)), that takes into account the interaction between vortex lines and heat flux in a non isotropic tangle:

$$\mathbf{Q}^q = -\mathbf{P}_\omega \cdot \mathbf{q}. \tag{11}$$

To determine the line density production term Q^L, we recall the evolution equation for L formulated by Vinen (1957), in spatially homogeneous counterflow superfluid turbulence. Vinen assumes that the time derivative of L is composed of two opposite contributions, the first ($[dL/dt]_g$) responsible for the growth of L, the second ($[dL/dt]_d$) for its decay. Vinen assumes that the first term depends on the instantaneous value of L and the force **f** between the vortex line and the gas of excitations (phonons and rotons) which is proportional to the modulus of the heat flux $q = |\mathbf{q}|$, obtaining $[dL/dt]_g = AqL^{3/2}$. The form of the term responsible for the vortex decay was determined assuming that Feynman's model of vortex breakup is analogous to Kolmogorov's cascade in classical turbulence, obtaining $[dL/dt]_d = -BL^2$. Thus, we assume for the production term in the evolution equation for the line density L, the Vinen's expression:

$$Q^L = AqL^{3/2} - BL^2, \tag{12}$$

where the coefficients A and B depend on the temperature T of helium II. A more general expression of (12), which includes a term linked to the dry friction force, was proposed by Jou *et al.* (2011b), but here we will use the source term (12), for sake of simplicity. At last, we formulate the production terms of $\mathbf{P}^{ss}_\omega$ and $\mathbf{P}^a_\omega$, assuming that they relax to their equilibrium values:

$$\mathbf{Q}^{ss}_\omega = -\frac{1}{\tau_{ss}}\left(\mathbf{P}^{ss}_\omega - (\mathbf{P}^{ss}_\omega)^{\mathbf{eq}}\right), \tag{13}$$

$$\mathbf{Q}^{a}_\omega = -\frac{1}{\tau_{a}}\left(\mathbf{P}^{a}_\omega - (\mathbf{P}^{a}_\omega)^{\mathbf{eq}}\right), \tag{14}$$

where τ_{ss} and τ_a are the respective relaxation times. Note that the equilibrium values of the symmetric traceless $(\mathbf{P}^{ss}_\omega)^{\mathbf{eq}}$ part and antisymmetric part $(\mathbf{P}^{a}_\omega)^{\mathbf{eq}}$ of tensor $\mathbf{P}_\omega$ depend on the experiments in considerations.

2.2. Restrictions imposed by the entropy principle. We start our analysis imposing the validity of the second law of thermodynamics, which states that the rate of entropy production per unit volume Q^S is a positive definite quantity, *i.e.*,

$$Q^S = \frac{\partial S}{\partial t} + \frac{\partial J^s_k}{\partial x_k} \geq 0, \tag{15}$$

where S and $\mathbf{J}^S$ are entropy density and entropy flux density, respectively.

For the fluxes of the fundamental fields we choose the following expressions:

$$J^q_{ij} = \beta_0(E,L)\delta_{ij} + \beta_1(E,L)\mathscr{P}^{ss}_{<ij>} + \beta_2(E,L)\mathscr{P}^a_{[ij]}, \tag{16}$$

$$J^L_i = \nu_0(E,L)q_i, \tag{17}$$

$$J^{ss}_{<ij>k} = \chi_0(E,L)q_{<i}\delta_{j>k}, \tag{18}$$

$$J^a_{[ij]k} = \xi_0(E,L)q_{[i}\delta_{j]k}, \tag{19}$$

which generalize those that we chose in a previous paper (Mongiovì and Restuccia 2018), where the coefficients β_1, β_2, χ_0 and ξ_0 were assumed constant. These are the most general expressions, linear in the fields $\mathbf{q}$, $\mathbf{P}^{ss}_\omega$, $\mathbf{P}^a_\omega$, with scalar coefficients, compatible with the material objectivity principle (Smith 1971; Muschik and Restuccia 2008).

Restrictions to these equations can be obtained using Liu's procedure (Liu 1972). This method considers the equations (10) as constraints for the validity of entropy inequality (15). Thus, this inequality becomes totally arbitrary provided that we complement it by the evolution equations for the fields E, L, q_i, $\mathscr{P}^{ss}_{<ij>}$ and $\mathscr{P}^{a}_{[ij]}$ multiplied by Lagrange factors Λ^E, Λ^L, Λ^q_i, $\Lambda^{ss}_{<ij>}$ and $\Lambda^a_{[ij]}$, which in turn are also supposed objective functions of the fundamental fields.

One obtains the following inequality, which must be satisfied for arbitrary values of the field variables:

$$\begin{aligned}\frac{\partial S}{\partial t}+\frac{\partial J^s_k}{\partial x_k} &- \Lambda_E\left[\frac{\partial E}{\partial t}+\frac{\partial q_k}{\partial x_k}\right]\\ &- \Lambda^q_i\left[\frac{\partial q_i}{\partial t}+\frac{\partial J^q_{ik}}{\partial x_k}-Q^q_i\right]\\ &- \Lambda_L\left[\frac{\partial L}{\partial t}+\frac{\partial J^L_k}{\partial x_k}-Q^L\right]\\ &- \Lambda^{ss}_{<ij>}\left[\frac{\partial \mathscr{P}^{ss}_{<ij>}}{\partial t}+\frac{\partial J^{ss}_{<ij>k}}{\partial x_k}-Q^{ss}_{<ij>}\right]\\ &- \Lambda^a_{[ij]}\left[\frac{\partial \mathscr{P}^{a}_{[ij]}}{\partial t}+\frac{\partial J^{a}_{[ij]k}}{\partial x_k}-Q^{a}_{[ij]}\right]\geq 0. \end{aligned} \tag{20}$$

In order to make the theory internally consistent, we must choose for S and $\mathbf{J}^S$ constitutive expressions approximated to second order in the variables q_i, $\mathscr{P}^{ss}_{<ij>}$ and $\mathscr{P}^{a}_{[ij]}$. Generalizing the expression considered by Mongiovì and Restuccia (2018) for S and J^s_k, we assume:

$$S = S_0(E,L)+S_1(E,L)q_kq_k+S_2(E,L)\mathscr{P}^{ss}_{<ik>}\mathscr{P}^{ss}_{<ik>}+S_3(E,L)\mathscr{P}^{a}_{[ik]}\mathscr{P}^{a}_{[ik]}, \tag{21}$$

$$J^s_k = \phi_0(E,L)q_k+\phi_1(E,L)\mathscr{P}^{ss}_{<ik>}q_i+\phi_2(E,L)\mathscr{P}^{a}_{[ik]}q_i. \tag{22}$$

Imposing in (20) that the coefficients of the time derivatives of E, q_i, L, $\mathscr{P}^{ss}_{<ij>}$ and $\mathscr{P}^{a}_{[ij]}$ vanish, one gets:

$$dS=\Lambda_E dE+\Lambda_L dL+\Lambda^q_i\, dq_i+\Lambda^{ss}_{<ij>}d\mathscr{P}^{ss}_{<ij>}+\Lambda^a_{[ij]}d\mathscr{P}^{a}_{[ij]}. \tag{23}$$

Differentiating the entropy density expression (21) and comparing it with (23), we get

$$\Lambda_E = \frac{\partial S}{\partial E}=\frac{\partial S_0}{\partial E}+\frac{1}{2}\frac{\partial \lambda_0}{\partial E}q_kq_k+\frac{1}{2}\frac{\partial \lambda_1}{\partial E}\mathscr{P}^{ss}_{<ik>}\mathscr{P}^{ss}_{<ik>}+\frac{1}{2}\frac{\partial \lambda_2}{\partial E}\mathscr{P}^{a}_{[ik]}\mathscr{P}^{a}_{[ik]}, \tag{24}$$

$$\Lambda_L = \frac{\partial S}{\partial L}=\frac{\partial S_0}{\partial L}+\frac{1}{2}\frac{\partial \lambda_0}{\partial L}q_kq_k+\frac{1}{2}\frac{\partial \lambda_1}{\partial L}\mathscr{P}^{ss}_{<ik>}\mathscr{P}^{ss}_{<ik>}+\frac{1}{2}\frac{\partial \lambda_2}{\partial L}\mathscr{P}^{a}_{[ik]}\mathscr{P}^{a}_{[ik]}, \tag{25}$$

$$\Lambda^q_i = \frac{\partial S}{\partial q_i}=\lambda_0(E,L)q_i, \tag{26}$$

$$\Lambda^{ss}_{<ij>} = \frac{\partial S}{\partial \mathscr{P}^{ss}_{<ij>}}=\lambda_1(E,L)\mathscr{P}^{ss}_{<ij>}, \tag{27}$$

$$\Lambda^{a}_{[ij]} = \frac{\partial S}{\partial \mathscr{P}^{a}_{[ij]}}=\lambda_2(E,L)\mathscr{P}^{a}_{[ij]}, \tag{28}$$

where we have made the positions:

$$2S_1(E,L) = \lambda_0(E,L), \qquad 2S_2(E,L) = \lambda_1(E,L), \qquad 2S_3(E,L) = \lambda_2(E,L). \tag{29}$$

Imposing in (20) that the coefficients of space derivatives of E, q_i, L, $\mathscr{P}^{ss}_{<ij>}$ and $\mathscr{P}^{a}_{[ij]}$ vanish, one finds the following relation, see also Mongiovì and Restuccia (2018):

$$dJ^S_k = \Lambda^E dq_k + \Lambda^q_i dJ^q_{ik} + \Lambda_L dJ^L_k + \Lambda^{ss}_{<ij>} dJ^{ss}_{<ij>k} + \Lambda^a_{[ij]} dJ^a_{[ij]k}, \tag{30}$$

that must be satisfied for arbitrary values of the field variables.

Using the constitutive relations for the fluxes (16)–(19) and for the Lagrange multipliers (24)–(28) the right-hand-side of the latter equation furnishes:

$$\begin{aligned} dJ^S_k &= q_i \left[(\lambda_0 d\beta_0 + \Lambda_L d\nu_0)\delta_{ik} + \lambda_1 \mathscr{P}^{ss}_{<ik>} d\chi_0 + \lambda_2 \mathscr{P}^{a}_{[ik]} d\xi_0 + \lambda_0(\beta_1 d\mathscr{P}^{ss}_{<ik>} + \beta_2 d\mathscr{P}^{a}_{[ik]}) \right] \\ &+ dq_i \left[(\Lambda_E + \Lambda_L \nu_0)\delta_{ik} + \lambda_1 \chi_0 \mathscr{P}^{ss}_{<ik>} + \lambda_2 \xi_0 \mathscr{P}^{a}_{[ik]} \right], \end{aligned} \tag{31}$$

while using (22) we get:

$$\begin{aligned} dJ^S_k &= q_i \left[d\phi_0 \delta_{ik} + \mathscr{P}^{ss}_{<ik>} d\phi_1 + \mathscr{P}^{a}_{[ik]} d\phi_2 + \phi_1 d\mathscr{P}^{ss}_{<ik>} + \phi_2 d\mathscr{P}^{a}_{[ik]} \right] \\ &+ dq_i \left[\phi_0 \delta_{ik} + \phi_1 \mathscr{P}^{ss}_{<ik>} + \phi_2 \mathscr{P}^{a}_{[ik]} \right]. \end{aligned} \tag{32}$$

Equating Eqs. (31) and (32), we obtain the following relations:

$$d\phi_0 = \lambda_0 d\beta_0 + \Lambda_L d\nu_0, \tag{33}$$
$$d\phi_1 = \lambda_1 d\chi_0, \tag{34}$$
$$d\phi_2 = \lambda_2 d\xi_0, \tag{35}$$
$$\phi_0 = \Lambda_E + \Lambda_L \nu_0, \tag{36}$$
$$\phi_1 = \beta_1 \lambda_0 = \lambda_1 \chi_0, \tag{37}$$
$$\phi_2 = \beta_2 \lambda_0 = \lambda_2 \xi_0. \tag{38}$$

Differentiating (37)–(38) and comparing them with (34)–(35), we deduce $\chi_0 d\lambda_1 = 0$ and $\xi_0 d\lambda_2 = 0$. Because χ_0 and ξ_0 are assumed here different than zero, we deduce that λ_1 and λ_2 (and therefore S_2 and S_3) must be constant quantities, in this order of approximation.

A physical interpretation of the constitutive relations and of Lagrange multipliers was made by Mongiovì and Restuccia (2018), in linear regime. In particular, we have obtained the following Gibbs equation:

$$dS_0 = \frac{1}{T} dE - \frac{\mu^L}{T} dL, \tag{39}$$

where the temperature T and the chemical potential of vortex lines μ^L were identified as:

$$\Lambda^E = \left[\frac{\partial S_0}{\partial E} \right]_L = \frac{1}{T}, \tag{40}$$

$$-T\Lambda^L = -T \left[\frac{\partial S_0}{\partial L} \right]_L = \mu^L. \tag{41}$$

We have also introduced the following quantities (Mongiovì and Restuccia 2018):

$$\zeta_0 = \frac{\partial \beta_0}{\partial T} = -\frac{1}{T^2\lambda_0}\left[1 + \nu_0 T^2 \frac{\partial}{\partial T}\left(\frac{\mu^L}{T}\right)\right], \tag{42}$$

$$\zeta_1 = \frac{\partial \beta_0}{\partial L} = -\frac{\nu_0}{T\lambda_0}\frac{\partial \mu^L}{\partial L}. \tag{43}$$

Finally, it was shown that the following residual inequality for the entropy production (Mongiovì and Restuccia 2018)

$$Q^S = \Lambda^L Q^L + \lambda_0 q_i Q_i^q + \lambda_1 \mathscr{P}^{ss}_{<ij>} Q^{ss}_{<ij>} + \lambda_2 \mathscr{P}^a_{[ij]} Q^a_{[ij]} \geq 0 \tag{44}$$

must be satisfied.

3. Field equations and non-local approximate model

Substituting in (10) the constitutive expressions obtained in previous Section, the following set of evolution equations is obtained:

$$\begin{cases} \frac{\partial E}{\partial t} + \nabla \cdot \mathbf{q} = 0, \\ \frac{\partial \mathbf{q}}{\partial t} + \zeta_0 \nabla T + \zeta_1 \nabla L + \beta_1 \nabla \cdot \mathbf{P}^{ss}_\omega + \beta_2 \nabla \cdot \mathbf{P}^a_\omega = -KL\mathbf{q}, \\ \frac{\partial L}{\partial t} + \nabla \cdot (\nu_0 \mathbf{q}) = -BL^2 + AqL^{3/2}, \\ \frac{\partial \mathbf{P}^{ss}_\omega}{\partial t} + \chi_0 \langle \nabla \mathbf{q} \rangle = -\frac{1}{\tau_{ss}}\left(\mathbf{P}^{ss}_\omega - (\mathbf{P}^{ss}_\omega)^{\mathbf{eq}}\right), \\ \frac{\partial \mathbf{P}^a_\omega}{\partial t} + \xi_0 [\nabla \mathbf{q}] = -\frac{1}{\tau_a}\left(\mathbf{P}^a_\omega - (\mathbf{P}^a_\omega)^{\mathbf{eq}}\right). \end{cases} \tag{45}$$

In the absence of a vortex tangle it results $L = 0$, $\mathbf{P}^{ss}_\omega = 0$ and $\mathbf{P}^a_\omega = 0$; therefore coefficients ν_0, χ_0 and ξ_0 vanish, and system (45) reduces to the following set of equations:

$$\begin{cases} \frac{\partial E}{\partial t} + \nabla \cdot \mathbf{q} = 0, \\ \frac{\partial \mathbf{q}}{\partial t} + \zeta_0 \nabla T = 0, \end{cases} \tag{46}$$

found by Mongiovì (1993) in the case of laminar counterflow ($q < q_c$), when the bulk and shear viscosity of the gas of excitations are neglected. When viscosity is considered, the evolution equations for laminar counterflow are (Mongiovì 1993; Saluto *et al.* 2014):

$$\begin{cases} \frac{\partial E}{\partial t} + \nabla \cdot \mathbf{q} = 0, \\ \frac{\partial \mathbf{q}}{\partial t} + \zeta_0 \nabla T - \lambda_b \frac{\zeta_0}{TS^2} \nabla(\nabla \cdot \mathbf{q}) - 2\lambda_{sh} \frac{\zeta_0}{TS^2} \nabla \cdot (\langle \nabla \mathbf{q} \rangle) = 0, \end{cases} \tag{47}$$

where λ_b and λ_{sh} are coefficients which (in a normal fluid) can be identified with the bulk and shear viscosity respectively, and $S = \rho(s - s_s)$ is the entropy density of gas of excitations (the normal component) in superfluid helium (Mongiovì *et al.* 2018), being s the total specific entropy of He II and s_s denotes the specific entropy associated to the ground state (the superfluid component) (Mongiovì 2000, 2001; Mongiovì *et al.* 2018).

The nonlocal terms in the heat flux equation in system (47) are responsible for the second sound attenuation, whose attenuation coefficient k_s is given by Mongiovì (1993) and Mongiovì *et al.* (2018):

$$k_s = \frac{\omega^2}{2u_2^3}\frac{\zeta_0}{TS^2}\left(\lambda_b + \frac{4}{3}\lambda_{sh}\right), \tag{48}$$

where u_2 is the second sound velocity (given by $u_2^2 = \zeta_0/(\rho c)$, with c the specific heat) and ω the frequency of the wave.

An interesting result is obtained by the following considerations. Note that if one assumes that the dynamical evolution of the deviatoric and antisymmetric parts of the vorticity tensor $\mathbf{P}_\omega$ is much faster than the dynamical evolution of energy and heat flux, the two latter equations in system (45) furnish:

$$\mathbf{P}_\omega^{ss} = (\mathbf{P}_\omega^{ss})^{\mathbf{eq}} - k_{ss}\langle\nabla\mathbf{q}\rangle, \tag{49}$$

$$\mathbf{P}_\omega^{a} = (\mathbf{P}_\omega^{a})^{\mathbf{eq}} - k_{a}[\nabla\mathbf{q}], \tag{50}$$

with $k_{ss} = \tau_{ss}\chi_0$ and $k_a = \tau_a\xi_0$. Substituting (49) and (50) in the rate equation for the heat flux in system (45), in the simplest case in which $\mathbf{P}_\omega^{ss}$ and $\mathbf{P}_\omega^{a}$ are zero at equilibrium, we get:

$$\frac{\partial\mathbf{q}}{\partial t} + \zeta_0\nabla T + \zeta_1\nabla L - k_{ss}\beta_1\nabla\cdot(\langle\nabla\mathbf{q}\rangle) - k_a\beta_2\nabla\cdot([\nabla\mathbf{q}]) = -KL\mathbf{q}. \tag{51}$$

This equation describes the evolution of heat flux $\mathbf{q}$ in presence of an anisotropic polarized vortex tangle, under the previous assumptions. If, in particular we consider the tangle not polarized (*i.e.*, $\mathbf{P}_\omega^{a} = 0$) and assume homogeneity in the line density L ($L(\mathbf{x},t) = L_0(t)$, *i.e.*, $v_0 = 0$), this equation can be written in the form:

$$\frac{\partial\mathbf{q}}{\partial t} + \zeta_0\nabla T - \frac{1}{6}k_{ss}\beta_1\nabla(\nabla\cdot\mathbf{q}) - \frac{1}{2}k_{ss}\beta_1\nabla^2\mathbf{q} = -KL_0(t)\mathbf{q}. \tag{52}$$

It is seen immediately that the nonlocal terms depending on the gradient of the heat flux in this latter equation are similar to the nonlocal terms in the heat flux equation of system (47), and therefore they contribute to the second sound attenuation too.

It is known that a small amount of quantized vortices formed at the λ–transition is always present in liquid helium II. Previous considerations show that the anisotropy of the small tangle of quantized vortices created at the lambda transition contributes to the attenuation of the second sound with a term proportional to the square of the frequency ω. This result is new and interesting. A more general model taking into account both dissipative contributions will be analyzed in a successive paper.

4. Conclusions

In this paper we have deepen a thermodynamical model of inhomogeneous anisotropic and polarized superfluid turbulence which was formulated by Mongiovì and Restuccia (2018). Here, the particular case of counterflow superfluid turbulence is considered, choosing as fundamental fields the energy density, the heat flux, and the complete vorticity tensor, including its scalar part, its symmetric trace-less part and its antisymmetric part. Restrictions for constitutive relations are obtained using Liu's method of Lagrange multipliers.

In this paper the results obtained in Mongiovì and Restuccia (2018) have been generalized, choosing more general expressions for the entropy density and for the fluxes, in the particular case of counterflow superfluid turbulence. This generalization is needed, as in previous paper (Mongiovì and Restuccia 2018) we supposed constant the coefficients appearing in Eqs. (16)–(19) and Eqs. (21)–(22). However, with the choice made by Mongiovì and Restuccia (2018) one can not describe with sufficiently accuracy the dependence on the temperature of the second sound velocity, that in this model depends on ζ_0, which in turn depends on S_1 (see Eqs. (29) and (42)). Thus, we have found that in the present model S_2 and S_3 are constant coefficients, while S_1 can depend on E and L, in the considered approximation.

Finally, we observe that Eqs. (52) and (46) are formally identical to the so-called Guyer-Krumhansl equation (Guyer and Krumhansl 1966a,b), that describes the evolution of heat flux taking into account of nonlocal effects. This kind of equation was also obtained in low-temperature crystals and in phonon hydrodynamics, where non-local effects were introduced by expressing the bulk and shear stress in terms of spatial derivatives of the heat flux (Jou *et al.* 2010). In this paper, we have shown that this kind of equation is also obtained in counterflow superfluid turbulence, when the vortex tangle is non isotropic.

Acknowledgments

M.S.M. acknowledges the hospitality of the Department of Mathematical and Computer Sciences, Physical Sciences and Earth Sciences of the University of Messina. L.R. acknowledges the hospitality of Palermo University. M.S.M and L.R. acknowledge the support of "National Group of Mathematical Physics, GNFM-INdAM.

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Paper contributed to the conference entitled "Thermal theories of continua: survey and developments (Thermocon 2016)", which was held in Messina, Italy (19–22 April 2016) under the patronage of the *Accademia Peloritana dei Pericolanti*

Manuscript received 8 January 2018; published online 20 May 2019