DOI: 10.1478/AAPP.97S1A17

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 97, No. S1, A17 (2019)

VORTEX DIFFUSION IN SUPERFLUID TURBULENCE: HYSTERESIS AND DECAY

MARIA STELLA MONGIOVÌ [a], LIDIA SALUTO [b*] AND DAVID JOU [c]

ABSTRACT. In this paper we consider the effects of vortex diffusion in two paradigmatic inhomogeneous physical situations; radial heat flow between two concentric cylinders, and vortex decay in a finite cylindrical container. Diffusion effects may strongly modify the expected results for the vortex length density L as a function of the heat flux $\mathbf{q}$, leading also to dynamical unexpected features, relating L to time in decay, or to hysteresis behaviour of L for increasing/decreasing heat flux.

1. Introduction

One of the most typical effects in liquid helium II is the so-called *counterflow superfluid turbulence*, whose physical picture is a tangle of vortices of equal circulation $\kappa = h/m$, with h the Planck constant and m the mass of helium atom. This kind of turbulence is generated thermally, applying a heat flux, exceeding a critical value q_c (Barenghi, Donnelly, and Vinen 2001; Donnelly 1991; Nemirovskii 2013). The simplest way to describe the vortex tangle is in terms of a single scalar quantity L, the total length of vortex lines per unit volume (*vortex line density*, for short). In the framework of the two-fluid model, Vinen, first, proposed an evolution equation for L, under constant counterflow velocity $V_{ns} = \langle|\mathbf{v}_n - \mathbf{v}_s|\rangle$ ($\mathbf{v}_n$ and $\mathbf{v}_s$ being the velocities of normal and superfluid component, respectively). Such an equation is (Vinen and Shoenberg 1957) :

$$\frac{dL}{dt} = \alpha_v V_{ns} L^{3/2} - \beta_v \kappa L^2, \tag{1}$$

with α_v and β_v dimensionless parameters. This equation assumes homogeneous turbulence, i.e. that the value of L is the same everywhere in the system.

The progress in experiments and computer simulations has allowed for deeper observation, which shows that the tangles are not completely homogeneous. Our aim here is to study the effects of vortex diffusion in inhomogeneous vortex tangles. We use the one-fluid model deduced from extended thermodynamics (Mongiovì and Jou 2007). The basic variables we will use are temperature T, vortex length density L, and heat flux $\mathbf{q}$. We consider the fluid incompressible and at rest, so that neither the mass density ρ nor the barycentric velocity $\mathbf{v}$ are considered as variables here.

For L, we will consider the following generalization of Vinen equation, written in terms of $\mathbf{q}$:

$$\frac{\partial L}{\partial t} = \gamma_1 L^{3/2} q \left(1 - \mu L^{-1/2} q\right) - \beta_v \kappa L^2 + D\nabla^2 L - \nu \nabla \cdot \mathbf{q}. \tag{2}$$

The first term, that depends on the modulus of the heat flux q, describes the production of vortices, the second ($-\beta_v \kappa L^2$) their destruction; the term $D\nabla^2 L$ takes into account of vortex diffusion, and the term $-\nu\nabla \cdot \mathbf{q}$ relates a convective flux of vortices to the heat flux.

Recalling that in the two-fluid model the heat flux $\mathbf{q}$, is given by $\mathbf{q} = \rho_s T s \mathbf{V}_{ns}$ (where ρ_s is the density of the superfluid component, T the temperature, and s the specific entropy of the fluid), one can relate coefficient γ_1 in (2) to coefficient α_v in Vinen's equation as $\gamma_1 \rho_s T s = \alpha_v$ (in the following we will use α and β instead of α_v and β_v). The correction $-\mu L^{-1/2} q$ to the Vinen's production term can be explained observing that for very high heat flux the production of vortices is reduced of a small amount. Probably, for very high values it would reach a saturation value, in this case one would have $q/(1 + \mu L^{-1/2} q)$, whose expansion up to first order in $L^{-\frac{1}{2}} q$ yields the expression used in (2).

This term does not modify the steady solution of equation (2), but, as we will see, allows us to describe the vortex decay in counterflow superfluid turbulence in better agreement with experiments.

The tangle of quantized vortices produces a damping force on the heat flux (or on the counterflow velocity) known as mutual friction force, resulting from the collision of the quasiparticles (phonons and rotons) with the vortex lines. This force appears in the evolution equation for the heat flux, that describes the dynamics of quasiparticles inside the superfluid. The evolution equation for the heat flux that we will consider is:

$$\frac{\partial \mathbf{q}}{\partial t} + \zeta \nabla T + \chi \nabla L = -KL\mathbf{q}. \tag{3}$$

In this equation ζ is a coefficient that determines the second sound velocity V_2 trough $V_2^2 = \frac{\zeta}{\rho c_V}$ as proved in Mongiovì 1993, χ a coupling coefficient between heat flux and the gradient of L, and K a coefficient related to the mutual friction between heat flux and vortices. Equations (2) and (3) should be complemented by the temperature equation.

In this paper we focus our attention on the effects of vortex diffusion and of the reduction coefficient μ in two inhomogeneous situations of physical interest. First, we analyze steady radial counterflow between two concentric cylinders at different temperatures, and consider the possibility of hysteresis in the vortex line density under cyclic variation of the heat flux as in (Saluto, Jou, and Mongiovì 2014). Second, the anomalous decay of superfluid turbulence observed in (Skrbek, Gordeev, and Soukup 2003; Varga, Babuin, and Skrbek 2015) when the heater is switched off, and exhibiting an unexpected non–monotonic evolution as a function of time is analyzed.

2. Application to turbulent radial counterflow

In this section we apply equation (2) to a steady-state radial counterflow of HeII between two concentric cylindrical walls at different temperatures, which provide a natural way to have a strongly inhomogeneous system. The inner cylinder (the hotter one) is at fixed

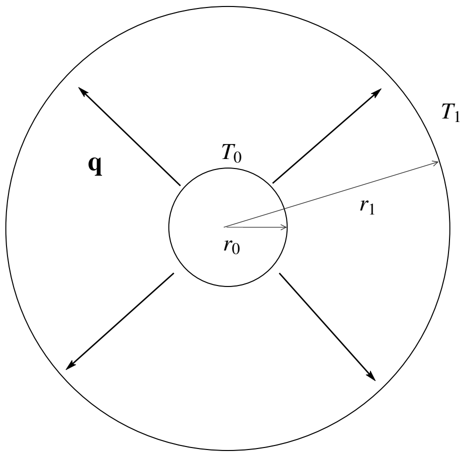

FIGURE 1. Heat flows radially from the inner cylinder (the hotter one), at fixed temperature T_0, to the outer colder one. The radial temperature profile $T(r)$ depends on the heat flow and the vortex density radial distribution (Saluto, Jou, and Mongiovì 2014).

temperature T_0, the temperature profile $T(r)$ between both cylinders will depend on the heat flux, the thermal conductivity, and the vortex line density profile $L(r)$ (see Fig. 1).

The steady state situation requires $\nabla \cdot \mathbf{q} = 0$. Since the heat flux has only a radial component $q_r = q$, this implies that $q = \Gamma/r$, with Γ being the heat supplied per unit time and unit length of the cylinder and r the distance with respect to the axis of the cylinders. Thus, the situation is clearly inhomogeneous, because $\mathbf{q}$ depends strongly on r, especially for r small, and the process may be dominated by diffusion.

Here we study the behavior of $L(r)$, which could be obtained experimentally from the attenuation coefficient of second sound. Introduction of $q = \Gamma/r$ in (2), and taking into account the axial symmetry of the problem, allows one to obtain $L(r)$ from:

$$\gamma_1 \frac{\Gamma}{r} L^{3/2} \left(1 - \mu L^{-1/2} \frac{\Gamma}{r}\right) - \beta_v \kappa L^2 + D \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial L}{\partial r} \right) = 0. \tag{4}$$

Assuming γ_1, μ, β and D constant, differential equation (4) admits the solution:

$$L(r) = \gamma^2 \frac{\Gamma^2}{r^2}, \tag{5}$$

where γ is given by

$$\gamma = \frac{\gamma_1}{2\beta\kappa} \left(1 + \sqrt{1 - 4\frac{\mu\beta\kappa}{\gamma_1} + \frac{16D}{\gamma_1 \Gamma^2} \frac{\mu\beta\kappa}{\gamma_1}} \right). \tag{6}$$

If $D = 0$ (i.e. for vanishing diffusion) one finds:

$$L(r) = L_0(r) = \frac{\gamma_1^2}{4\beta^2\kappa^2}\left(1+\sqrt{1-4\frac{\mu\beta\kappa}{\gamma_1}}\right)^2\frac{\Gamma^2}{r^2}. \tag{7}$$

Coefficient μ is small, with respect to $\gamma_1/(\beta\kappa)$, and therefore represents a small correction to the stationary solution $L_0(r) = (\gamma_1^2/(\beta^2\kappa^2))(\Gamma^2/r^2)$ found using Vinen equation (1).

In contrast, the diffusion contribution will be specially relevant when $\gamma_1/(\beta\kappa)$ is small as compared to $D/\gamma_1\Gamma^2$ (or in other terms when D is higher than $\beta\kappa/\gamma_1\Gamma^2$) because in that case the third term inside the root in equation (6) will be high. This increase of L may be intuitively understood as the result of a diffusive flow of vortices from the inner cylinder (where $\mathbf{q}$ is higher and therefore the vortex production is higher) to the outer cylinder, where they disappear when colliding against the external wall. Thus, the term in $\gamma_1\Gamma$ accounts for the "native" vortices (formed in the site at a given point), and the term in D accounts for the "migrating" vortices in that point carried by diffusion from other points. For small heat flux or for high diffusion, the "migrating" vortex population is higher than the "native" vortex population.

When the vortex diffusion effects become dominating, the vortex length density profile (5) takes the form:

$$L(r) = L_D(r) = \frac{4D}{\beta\kappa r^2}. \tag{8}$$

In this regime, L is independent on the heat flux, but it keeps the same dependence $1/r^2$ on the radius than in (7) for $D = 0$. Thus, the influence of the vortex diffusion is focused on the local values of $L(r)$ rather than on the form of the spatial steady distribution. In the limit of low Γ, (4) with $D = 0$ would lead to $L = 0$, but for $D \neq 0$ it leads for L to (8).

In the case $\mu = 0$ the possibility of the different values of L in the presence or absence of diffusion suggests the possibility of hysteresis as shown in Saluto, Jou, and Mongiovì (2014), see also Fig. 2. In Fig. 2, the initial situation of the cycle corresponds to $L = 0$ and $\Gamma = 0$ (state A). A quick increase of Γ produces the corresponding increase of L (from A to B), but if such increase is fast enough, vortices do not have time to migrate. From B to C the heat flux is no longer changed and diffusion plays a central role, allowing vortices of high L region to move to small L regions. From C to D, Γ (i.e. the heat flux $\mathbf{q}$) slowly decreases. The path from D to A corresponds to the decay of L.

This situation is also found for $\mu \neq 0$, provided the heat flux is not too high or L is not too low, in such a way that $\mu\mathbf{q}L^{-\frac{1}{2}} < 1$, in order that the vortex production is positive everywhere otherwise. More general expression for the production term, including higher order terms in $\mathbf{q}L^{-\frac{1}{2}}$, should be incorporated or obtained, for instance, from the saturation expression we have discussed in the introduction below Eq. (2). Thus, hysteresis shown in Fig. 2 is present also for $\mu \neq 0$, a situation which was not considered in our previous analysis (Saluto, Jou, and Mongiovì 2014).

3. Decay of counterflow superfluid turbulence

We will see now that equations (2) and (3) allow us to explain the anomalous non-monotonic decay observed in some experiments in counterflow superfluid helium when

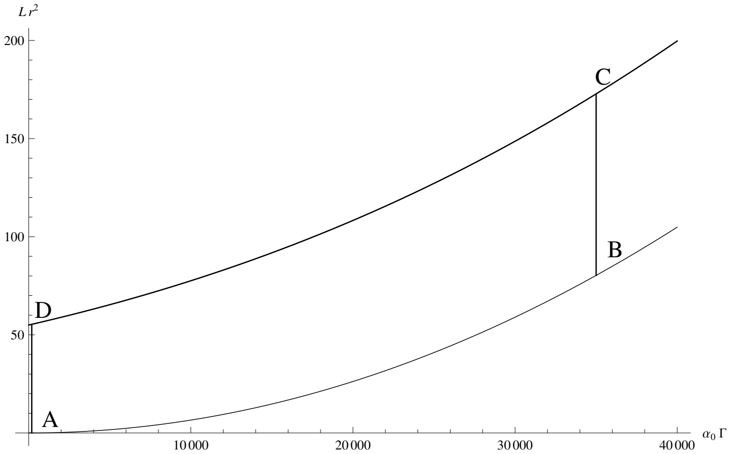

FIGURE 2. From Saluto, Jou, and Mongiovì (2014). Hysteresis cycle. The value of Lr^2 (vertical axis) as a function of the dimensionless expression for the heat flux $\alpha_0\Gamma$. The value of L for fastly increasing heat flux (A to B), corresponding to eq. (7), is different from that for slowly decreasing heat flux (C to D), corresponding to eq. (5)-(6) with $D \neq 0$. The values of $\alpha_0 = 1.28 * 10^{-4}$ s^3/(cm*g) for $T = 1.5$ K and $D = 2.2\kappa$ cm^2/s have been taken from Nemirovskii 2010 for D, from Sciacca, Mongiovì, and Jou 2008 for β, and from Martin and Tough 1983 for the ratio γ_1/β.

the heater is switched off (Schwarz and Rozen 1991; Skrbek, Gordeev, and Soukup 2003; Varga, Babuin, and Skrbek 2015) and **q** decays to zero. Such experiments show that the line density L decays as t^{-1} after the switching off the heather until a local minimum value of L is reached. After this time, L unexpectedly grows forming a elbow, and later turns again to decay but as $t^{-3/2}$.

To describe experiments (Skrbek, Gordeev, and Soukup 2003; Varga, Babuin, and Skrbek 2015), instead of the full (2) and (3) we will use only the simpler system:

$$\begin{cases} \frac{dq}{dt} = -KLq \\ \\ \frac{dL}{dt} = \gamma_1 L^{\frac{3}{2}} q \left(1 - \mu L^{-1/2} q\right) - \beta\kappa L^2 + D\nabla^2 L. \end{cases} \tag{9}$$

In the first part of the decay we can suppose the tangle homogeneous. Note also that, in Skrbek experiments, the measurements of line density L were made through second sound measurements in the central region of the channel. In this central region the effects of the walls can be neglected, so in equations (2)-(3) the terms dependent on the gradients of the field variables may be neglected.

If we suppose the turbulence homogeneous, from (9) we obtain the following simple generalization of Vinen equation (1):

$$\frac{dL}{dt} = \gamma_1 L^{\frac{3}{2}} q \left(1 - \mu L^{-1/2} q\right) - \beta\kappa L^2. \tag{10}$$

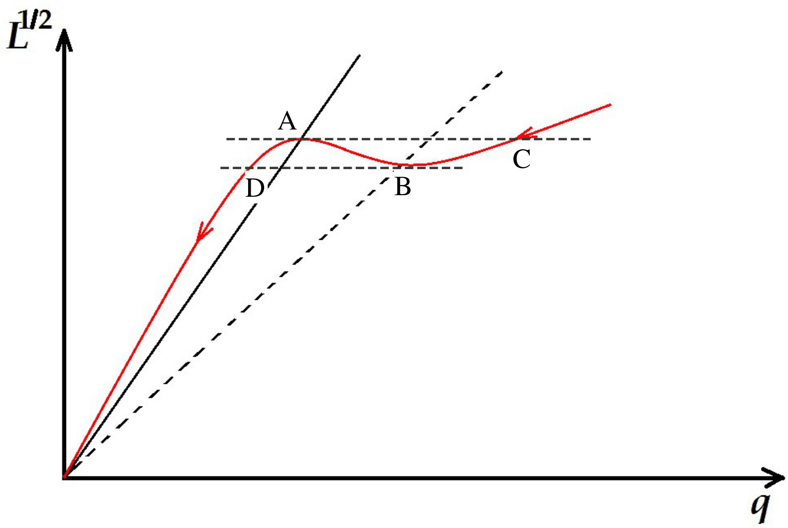

FIGURE 3. Qualitative behaviour of the orbits of system (13). Note the possibility of hysteresis in the case that when lowering q the system goes from B to D, and when increasing q the system goes from A to C.

Assuming q constant, the stationary solutions of this equation are $L = 0$ and:

$$L^{1/2} = \frac{\gamma_1}{2\beta\kappa}\left(1 \pm \sqrt{1 - 4\mu\frac{\beta\kappa}{\gamma_1}}\right) q, \tag{11}$$

in accord with Gorter-Mellink law.

One sees easily that the solutions $L^{1/2} = 0$ and $L^{1/2} = \frac{\gamma_1}{2\beta\kappa}\left(1 + \sqrt{1 - 4\mu\frac{\beta\kappa}{\gamma_1}}\right) q$ are stable, whereas the solution $L^{1/2} = \frac{\gamma_1}{2\beta\kappa}\left(1 - \sqrt{1 - 4\mu\frac{\beta\kappa}{\gamma_1}}\right) q$ is unstable.

We consider now the system:

$$\begin{cases} \frac{dq}{dt} = -KLq \\ \frac{dL}{dt} = \gamma_1 L^{\frac{3}{2}} q \left(1 - \mu L^{-1/2} q\right) - \beta\kappa L^2. \end{cases} \tag{12}$$

Putting $L^{1/2} = p$ we obtain the following dynamical system

$$\begin{cases} \frac{dq}{dt} = -Kp^2 q \\ \frac{dp}{dt} = \frac{p}{2}\left(\gamma_1 q(p - \mu q) - \beta\kappa p^2\right). \end{cases} \tag{13}$$

The qualitative behaviour of the orbits of this system is shown in Fig. 3. Note that while the heat flux q (in the x-axis) always decreases, the line density L presents an elbow, as in Skrbek experiment.

It remains to determine the dependence of L on t in these two different traits of this decay. In the decay of superfluid turbulence, two different regimes are present. The starting point of this decay is a situation in which turbulence is homogeneous, i.e. q and L have respectively the same values in every region in the container. Furthermore, in this situation both q and L are high, and the diffusion does not have influence, so q and L remain coupled (i.e. it is $L \propto q^2$) and decay together.

The equation that must be considered in this regime is (10). Being $q \propto AL^{1/2}$, we obtain the equation

$$\frac{dL}{dt} \simeq (\gamma_1 A - \gamma_1 \mu A^2 - \beta\kappa)L^2 \simeq -\beta'\kappa L^2, \tag{14}$$

whose solution is

$$L(t) = \frac{1}{\beta'\kappa t + L_0^{-1}}. \tag{15}$$

Therefore, in this regime L decreases as t^{-1} (if L_0^{-1} is sufficiently small). In finite containers, the decay is not homogeneous, because decay is faster near the walls, where the vortices pin on, or simply disappear. Thus, as time goes on, the difference between L at the center of the channel and L near the walls increases and the diffusion term becomes more and more relevant.

If the right–hand side of (10) is considered in detail in the steady–state, one has $\beta\kappa L^{\frac{1}{2}} = \mathbf{q}(1-\mu\mathbf{q}L^{-\frac{1}{2}})$ in such a way that $L^{\frac{1}{2}}$ as a function of $\mathbf{q}$ has an extremum for $L^{\frac{1}{2}} = 2\mu\mathbf{q}$, where $dL^{\frac{1}{2}}/d\mathbf{q} = 0$. This could be identified as point B in Fig. 3.

After the elbow shown in Fig. 3 the tangle is sufficiently diluted, in such a way the diffusion term is dominant. In this case the production and destruction may be neglected and the evolution equation for L from (10) becomes

$$\frac{dL}{dt} \simeq D\nabla^2 L, \tag{16}$$

whose solution, assuming x as the axis of the cylindrical container, in the limit of sufficiently long containers, is:

$$L(x,t) = \frac{1}{2\sqrt{\pi Dt}} \int_{-\infty}^{+\infty} L_0(\xi)\, e^{-\frac{(x-\xi)^2}{4Dt}}\, d\xi, \tag{17}$$

where $L_0(x)$ is the initial value of L.

As it is known, the asymptotic behaviour of this decay is

$$L(x,t) = \frac{L_0(x)}{4\sqrt{\pi D}} t^{-3/2}, \tag{18}$$

in accord with the second decay regime of Skrbek experiment Varga, Babuin, and Skrbek 2015. The behaviour of $L(t)$ is shown in Fig. 4.

A salient physical feature in Fig. 3 and 4 is the raising elbow region between the two decaying regions. According to our analysis, it is due to the contribution of the term in μ in (2). According to this term, and if L is sufficiently small, a decrease in $\mathbf{q}$ produces an increase in the production term. This, in turn, enhances diffusion, because it increases the gradient of L between the center of the container and the walls. Thus, this increasing elbow will be truncated when diffusion becomes dominant with respect to production/destruction effects. In other terms, the elbow represents the intermediate zone when a decrease of $\mathbf{q}$

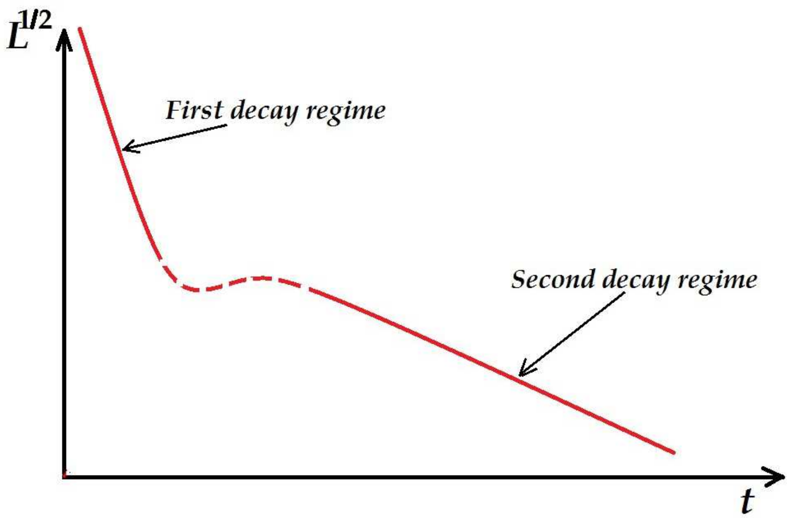

FIGURE 4. Qualitative behaviour of $L(t)$. Note two different regimes for the decay, separated by an elbow region.

increases vortex production in the volume, and diffusion effects toward the walls. When the first effect is dominating, L increases for smaller $\mathbf{q}$, but this increase becomes a decrease when diffusion turns dominant. In contrast for $\mu = 0$, one would have two decay regimes, but both regimes of decay would not be separated by a raising elbow zone, but they would connect to each other by means of a region with decreasing L.

4. Conclusions

In this paper we have considered two aspects of the dynamics of L beyond the usual Vinen's equation, **a)** the presence of vortex diffusion, and **b)** a reduction of the rate of vortex formation at high values of $\mathbf{q}$ (coefficient μ).

We have explored the effects of μ and D in two physical inhomogeneous situations, **1)** radial heat flow between two cylinders at different temperatures, **2)** decay of turbulence after switching off the heat flux. In **1** we have generalized our previous results (obtained for $\mu = 0$, but which keep their validity for $\mu \neq 0$ (Saluto, Jou, and Mongiovì 2014)). In **2** we have explored the crossover between two different regimes of decay, namely $L \sim t^{-1}$, dominated by the destruction of L in the volume, and $L \sim t^{-\frac{3}{2}}$, dominated by vortex diffusion towards the walls. These two regions are to be expected in general, because they reflect two different physical aspects of the process. What is surprising is the raising elbow (L increasing with t) between both regions instead of having a monotonical decay. We have proposed an interpretation of this elbow in terms of the competition of the effects coming from the terms in μ and in D. Thus, whereas the term in μ has minor consequence in situation 1, it is crucial in situation 2.

A topic for future analysis would be to explore the proposed saturation behaviour of the vortex production term as a function of **q** discussed in the introduction below Eq. (2).

Acknowledgments

M.S.M. acknowledges the financial support of the Università di Palermo (under Grant Fondi 60% 2012) and the support of "National Group of Mathematical Physics, GNFM-INdAM". L.S. acknowledges the support of "National Group of Mathematical Physics, GNFM-INdAM". D.J. acknowledges the financial support of the Spanish Ministry of Economy and Competitiveness under grant TEC 2015–67462–C2–2–R, and of the Direcció General de Recerca of the Generalitat of Catalonia under grant 2009 SGR–00164, and of the Consolider Program NanoTherm of the Spanish Ministry of Science and Innovation under grant SD–2010–0044507.

References

Barenghi, C. F., Donnelly, R. J., and Vinen, W. F. (2001). *Quantized vortex dynamics and superfluid turbulence*. Vol. 571. Springer Science & Business Media. DOI: 10.1007/3-540-45542-6.

Donnelly, R. J. (1991). *Quantized vortices in helium II*. Vol. 2. Cambridge University Press. DOI: 10.1063/1.2809626.

Martin, K. P. and Tough, J. T. (1983). "Evolution of superfluid turbulence in thermal counterflow". *Physical Review B* **27**(5), 2788. DOI: 10.1103/PhysRevB.27.2788.

Mongiovì, M. S. (1993). "Extended irreversible thermodynamics of liquid helium II". *Physical Review B* **48**, 6276. DOI: 10.1103/PhysRevB.48.6276.

Mongiovì, M. S. and Jou, D. (2007). "Thermodynamical derivation of a hydrodynamical model of inhomogeneous superfluid turbulence". *Physical Review B* **75**(2), 024507. DOI: 10.1103/PhysRevB.75.024507.

Nemirovskii, S. K. (2010). "Diffusion of inhomogeneous vortex tangle and decay of superfluid turbulence". *Physical Review B* **81**(6), 064512. DOI: 10.1103/PhysRevB.81.064512.

Nemirovskii, S. K. (2013). "Quantum turbulence: Theoretical and numerical problems". *Physics Reports* **524**(3), 85–202. DOI: 10.1016/j.physrep.2012.10.005.

Saluto, L., Jou, D., and Mongiovì, M. S. (2014). "Vortex diffusion and vortex-line hysteresis in radial quantum turbulence". *Physica B: Condensed Matter* **440**, 99–103. DOI: 10.1016/j.physb.2014.01.041.

Schwarz, K. W. and Rozen, J. R. (1991). "Transient behavior of superfluid turbulence in a large channel". *Physical Review B* **44**(14), 7563. DOI: 10.1103/PhysRevB.44.7563.

Sciacca, M., Mongiovì, M. S., and Jou, D. (2008). "Alternative Vinen equation and its extension to rotating counterflow superfluid turbulence". *Physica B: Condensed Matter* **403**(13-16), 2215–2224. DOI: 10.1016/j.physb.2007.12.001.

Skrbek, L., Gordeev, A., and Soukup, F. (2003). "Decay of counterflow He II turbulence in a finite channel: Possibility of missing links between classical and quantum turbulence". *Physical Review E* **67**(4), 047302. DOI: 10.1103/PhysRevE.67.047302.

Varga, E., Babuin, S., and Skrbek, L. (2015). "Second-sound studies of coflow and counterflow of superfluid 4He in channels". *Physics of Fluids* **27**(6), 065101. DOI: 10.1063/1.4921816.

Vinen, W. F. and Shoenberg, D. (1957). "Mutual friction in a heat current in liquid helium II I. Experiments on steady heat currents". *Proc. R. Soc. Lond. A* **240**(1220), 114–127. DOI: 10.1098/rspa.1957.0071.

[a] Università degli Studi di Palermo
Dipartimento di Innovazione Industriale e Digitale (DIID)
90128 Palermo, Italy

[b] Istituto Nazionale di Alta Matematica "F. Severi" – Roma
Unità di ricerca di Palermo c/o DIID - Università degli Studi di Palermo

[c] Universitat Autònoma de Barcelona
Departament de Física
08193 Bellaterra, Catalonia, Spain

* To whom correspondence should be addressed | email: lidia.saluto@unipa.it

Paper contributed to the conference entitled "Thermal theories of continua: survey and developments (Thermocon 2016)", which was held in Messina, Italy (19–22 April 2016) under the patronage of the *Accademia Peloritana dei Pericolanti*

Manuscript received 6 December 2017; published online 20 May 2019