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THERMOSOLUTAL CONVECTION IN A ROTATING VISCOELASTIC WALTERS FLUID

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Abstract. In this paper we study the nonlinear Lyapunov stability of the thermodiffusive equilibrium of a viscoelastic rotating Walters fluid, in a horizontal rotating layer heated and salted from below. We reformulate the nonlinear stability problem, projecting the initial perturbation evolution equations on some suitable subspaces of the space of kinematically admissible functions. In this way we preserve the contribution of the Coriolis term and, jointly, all nonlinear terms vanish. When the viscoelasticy parameter vanishes, that is for the classical rotating Bénard problem, if instability occurs as stationary convection, the linear and nonlinear stability bounds are equal.

1. Introduction

For a Newtonian fluid the classical Bénard problem, of a big importance in astrophisics, geophysics, oceanography, meteorology, has been largely investigated (Joseph 1965, 1966; Chandrasekhar 1968; Joseph 1970, 1976b; Galdi and Straughan 1985; Georgescu 1985; Rionero and Mulone 1987, 1988; Georgescu and Palese 1996; Mulone and Rionero 1997; Georgescu *et al.* 2000, 2001; Straughan 2004; Palese 2005; Georgescu and Palese 2009, 2010, 2011; Palese 2014a,b,c) in the Oberbeck-Boussinesq approximation too, including the influence of effects such as a rotation field, a magnetic field, chemical reactions of reactive fluids.

The problem of the coincidence of linear and nonlinear stability boundaries is largely studied, since the point of loss of linear stability theory is usually also a bifurcation point (Prodi 1962; Yudovich 1965; Sattinger 1970; Yudovich 1970a,b, 1989; Galdi and Padula 1990; Georgescu and Oprea 1994), at which subcritical instabilities may occur explaining unusual phenomena.

For Newtonian fluids, the problem of the coincidence of the linear and nonlinear stability bounds is studied in Mulone and Rionero (1997), including fully ionized fluids (Palese 2005), thermoanisotropic fluid mixtures in a horizontal layer heated from below (Georgescu and Palese 1996; Georgescu *et al.* 2000, 2001; Georgescu and

Palese 2009), chemical surface reactions (Georgescu and Palese 2010, 2011; Palese 2014b), rotation fields (Palese 2014c).

In Georgescu and Palese (1996), Georgescu *et al.* (2000, 2001), and Georgescu and Palese (2009, 2010, 2011) the equality of linear and nonlinear stability bounds was obtained, in the region of stationary convection of linear instability theory, without any restriction on initial data.

It is well known that the theory of Newtonian fluids in predicting the behaviour of some fluids, *e.g.* those with high molecular weight, leads to development of non Newtonian mechanics, of a great interest due to a wide range of engineering applications, such as ground pollutions by chemicals which were non Newtonian fluids, in biomedical applications, in agricolture.

In this framework there are many models of viscoelastic fluids in literature (Beard and Walters 1964; Baris 2002) and their extensive bibliography.

In this paper we consider a model of a second order viscoelastic Walters fluid, that involves only a non-Newtonian parameter. In such a fluid the constitutive equation for the Cauchy stress tensor, in the case of a short memory, is given by Baris (2002):

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\eta_0\boldsymbol{D} - k_0\left[\frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{D} - \boldsymbol{D}\cdot\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T\cdot\boldsymbol{D}\right],$$

where p is the pressure, $\boldsymbol{I}$ is the identity tensor, $\boldsymbol{D}$ is the rate of strain tensor, $\boldsymbol{v}$ is the velocity, ∇ is the gradient operator,

$$\eta_0 = \int_0^\infty N(\tau)d\tau, \qquad n \geq 2,$$

is a limiting viscosity at a small rate of shear (Baris 2002), and

$$k_0 = \int_0^\infty \tau N(\tau)d\tau$$

is a short memory coefficient, where $N(\tau)$ is the distribution function of the relaxation time τ. In this idealized model of a Walters fluid all terms involving

$$\int_0^\infty \tau^n N(\tau)d\tau, \qquad n \geq 2,$$

are neglected.

Also in the context of non-Newtonian fluids there are in literature many theoretical and experimental results on thermal instability of the thermodiffusive equilibrium in a fluid layer, including effects such as porosity (Baris 2002), Hall current (Gupta and Kumar 2010), dusty particles (Thirumurugan and Vasanthakumari 2013), and superposed fluids (Kumar and Singh 2010).

In this paper we reformulate the nonlinear Lyapunov stability problem of the thermodiffusive equilibrium for a second order viscoelastic Walters fluid in a rotating plane layer heated and salted from below.

Projecting the initial perturbation evolution equations on some suitable subspaces of the space of kinematically admissible functions, we preserve the contribution of skewsymmetric terms, such as Coriolis term, and, jointly, all the nonlinear terms vanish.

We derive (Sec. **2**) the perturbation evolution equations in terms of poloidal and toroidal fields, suitable to represent solenoidal fields in a plane layer, subsequently we project them on some orthogonal subspaces, obtaining (Sec. **3**) an energy relation where all nonlinear terms vanish, the contribution of the skewsymmetric term is instead preserved.

For a Newtonian fluid we recover, in a simpler way, the same result obtained in Palese (2014c), *i.e.*, in the region of the parameter space where the principle of exchange of stabilities holds, the coincidence of the nonlinear stability parameter with the critical Rayleigh number of the linear instability, without any restriction on initial data.

2. The initial/boundary value problem for perturbation

Let us consider, in the framework of mechanics of continua and in the Oberbeck-Boussinesq approximation, a Walters fluid mixture in a horizontal layer S, bounded by the surfaces $z = 0$ and $z = 1$ in a frame of reference $\{O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, with $\boldsymbol{k}$ unit vector in the vertical upwards direction, in rotation around the fixed vertical axis z with a constant angular velocity $\boldsymbol{\Omega} = \Omega\boldsymbol{k}$.

Let us now perturb the zero solution corresponding to a motionless state, $\{\mathbf{0}, \overline{T}, \overline{C}, \overline{P}\}$, where $\mathbf{0}$, $\overline{T}$, $\overline{C}$, $\overline{P}$, represent, respectively, the velocity, the temperature, the concentration and the pressure fields.

The perturbation $(\boldsymbol{u}, \theta, \gamma, p')$ of velocity, temperature, concentration and pressure fields satisfy the following nondimensional equations

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \nabla p' + \mathcal{R}\theta\boldsymbol{k} - \mathcal{R}_c\gamma\boldsymbol{k} + 2\boldsymbol{u}\times\boldsymbol{\Omega} + \left(1 - \mathcal{F}\dfrac{\partial}{\partial t}\right)\Delta\boldsymbol{u}, \\ P_r\left(\dfrac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta\right) = \Delta\theta + \mathcal{R}w, \qquad (t,\boldsymbol{x}) \in [0,\infty[\times V \\ S_c\left(\dfrac{\partial\gamma}{\partial t} + \boldsymbol{u}\cdot\nabla\gamma\right) = \Delta\gamma - \mathcal{R}_c w, \\ \nabla\cdot\boldsymbol{u} = 0, \end{cases} \tag{1}$$

in the space $\mathcal{N}$ of the functions $\boldsymbol{u}(\cdot,t), \Delta\boldsymbol{u}(\cdot,t), p'(\cdot,t), \theta(\cdot,t)$ and $\gamma(\cdot,t)$ belonging to $W^{2,2}(V)$, $\forall t \in [0,\infty[$, with $\boldsymbol{u}(\boldsymbol{x},\cdot) \in C^1[0,\infty[$, $\forall \boldsymbol{x} \in V$, verifying the following boundary conditions

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \theta = \gamma = 0 \quad \text{on } \partial V, \tag{2}$$

suitable for strees free and perfectly temperature and concentration conductor planes.

In $(1)_1$-$(1)_3$ $\boldsymbol{u} = (u, v, w)$, $V = \mathcal{V}\times[0,1]$ denotes the three dimensional box over the rectangle $\mathcal{V} \equiv [0, 2\pi/k_1]\times[0, 2\pi/k_2]$, whose boundary is denoted by ∂V, after assuming the perturbation fields, depending on the time t and space $\boldsymbol{x} = (x, y, z)$, doubly periodic functions in x and y, of period $2\pi/k_1$ and $2\pi/k_2$.

$\mathcal{R}^2$, $\mathcal{R}_c^2$, are the Rayleigh and solute Rayleigh numbers, P_r and S_c are the Prandtl and Schmidt numbers, respectively, and $\mathcal{F}$ is a dimensionless viscoelasticity parameter.

If we multiply $(1)_1$ by $\boldsymbol{u}$, the contribute of the Coriolis term is lost, namely

$$(\boldsymbol{u} \times \boldsymbol{\Omega}, \boldsymbol{u}) = 0,$$

whence, to avoid the loss of the contribution of the Coriolis term, and, consequently, a weaker resulting stability criterion, we modify the perturbation evolution equations by projecting them on some suitable orthogonal subspaces of $W^{2,2}(V)$.

Alternatively, we are forced to consider more complicated Lyapunov functions to evaluate the contribution of the skewsymmetric rotation term (Galdi and Straughan 1985).

Because of the representation theorem of solenoidal vectors (Joseph 1976a) in a plane layer, if the mean values of u, v, w vanish over $\mathcal{V}$ (Schmitt and von Wahl 1992), that is if the conditions

$$\int_{\mathcal{V}} u(x,y,z)dxdy = \int_{\mathcal{V}} v(x,y,z)dxdy = \int_{\mathcal{V}} w(x,y,z)dxdy = 0, \quad \forall z \in [0,1],$$

hold, the velocity perturbation $\boldsymbol{u}$ has the unique decomposition (Joseph 1976a; Schmitt and von Wahl 1992)

$$\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2, \tag{3}$$

with

$$\nabla \cdot \boldsymbol{u}_1 = \nabla \cdot \boldsymbol{u}_2 = \boldsymbol{k} \cdot \nabla \times \boldsymbol{u}_1 = \boldsymbol{k} \cdot \boldsymbol{u}_2 = 0, \tag{4}$$

$$\boldsymbol{u}_1 = \nabla \frac{\partial \chi}{\partial z} - \boldsymbol{k}\Delta\chi \equiv \nabla \times \nabla \times (\chi \boldsymbol{k}), \qquad \boldsymbol{u}_2 = \boldsymbol{k} \times \nabla\psi = -\nabla \times (\boldsymbol{k}\psi), \tag{5}$$

where $\nabla\times$ is the curl operator, the poloidal and toroidal potentials χ and ψ are doubly periodic functions satisfying the equations (Joseph 1976a)

$$\Delta_1 \chi \equiv \frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} = -\boldsymbol{k} \cdot \boldsymbol{u}, \qquad \Delta_1 \psi = \boldsymbol{k} \cdot \nabla \times \boldsymbol{u}. \tag{6}$$

From now on, we denote $\dfrac{\partial f}{\partial x} \equiv f_x$, where f is an arbitrary function. The boundary conditions for χ and ψ, for free planar surfaces, are (Joseph 1976a):

$$\chi = \chi_{zz} = \psi_z = 0, \qquad z = 0, 1. \tag{7}$$

From (3)-(4) it follows that

$$\boldsymbol{u} \cdot \boldsymbol{k} = \boldsymbol{u}_1 \cdot \boldsymbol{k} = -\Delta_1 \chi. \tag{8}$$

In order to project the perturbation equation $(1)_1$ on some suitable subspaces of the space of kinematically admissible functions, we observe that

$$\boldsymbol{u} = \nabla \times \nabla \times (\chi \boldsymbol{k}) - \nabla \times (\psi \boldsymbol{k}), \tag{9}$$

because of $\nabla \cdot \boldsymbol{u} = 0$,

$$\Delta \boldsymbol{u} = -\nabla \times \nabla \times \boldsymbol{u}. \tag{10}$$

We project the perturbation equation $(1)_1$ on the plane orthogonal to $\boldsymbol{k}$, and we obtain

$$\frac{\partial \boldsymbol{u}^\perp}{\partial t} = -(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}^\perp - \nabla_1 p' + 2\boldsymbol{u}\times\boldsymbol{\Omega} + \left(1-\mathcal{F}\frac{\partial}{\partial t}\right)\Delta\boldsymbol{u}^\perp. \tag{11}$$

Taking into account (9) and (10) it follows that

$$\boldsymbol{\Omega}\times\boldsymbol{u}^\perp = \boldsymbol{\Omega}\times(\nabla_1\chi_z - \nabla\times(\boldsymbol{k}\psi)) = -\Omega\nabla\times(\chi_z\boldsymbol{k}) - \Omega\nabla_1\psi. \tag{12}$$

Let us recall the Weyl decomposition theorem (Mikhlin 1970; Georgescu 1985)

$$L^2(V) = G(V)\oplus N(V), \tag{13}$$

with $G(V)$ and $N(V)$ spaces of generalized solenoidal and potential vectors respectively.

The advective term in (11) can be uniquely written as

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u}^\perp = \nabla U + \nabla\times\boldsymbol{A}, \tag{14}$$

where U is a scalar function and $\boldsymbol{A}$ a vector field we specify later.

If we define the scalar and vector fields

$$\Phi = \nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}^\perp), \quad \boldsymbol{W} = \nabla\times(\boldsymbol{u}\cdot\nabla\boldsymbol{u}^\perp),$$

the imbedding Sobolev theorems of $W^{2,2}(V)$ in the space of continuous functions $C(\overline{V})$ (Sobolev 1963) allows us to prove the following identity

$$\nabla\times(\boldsymbol{u}\cdot\nabla\boldsymbol{u}^\perp) \equiv \nabla\times(\boldsymbol{u}\cdot\nabla\boldsymbol{u}^\perp - \nabla U).$$

Let us define

$$\boldsymbol{B} = \boldsymbol{u}\cdot\nabla\boldsymbol{u}^\perp - \nabla U,$$

by choosing $\nabla\cdot\boldsymbol{B} = 0$, the scalar function U is (up to a constant) the solution of the interior Neumann problem (Mikhlin 1970) in the periodicity cell V

$$\begin{cases} \Delta U = \Phi, \\ \dfrac{\partial U}{\partial \boldsymbol{n}} = \Gamma, \end{cases} \tag{15}$$

where $\dfrac{\partial U}{\partial \boldsymbol{n}}$ is the normal derivative of U on the boundary ∂V of the periodicity cell V and $\Gamma = -\boldsymbol{B}\cdot\boldsymbol{n}$.

The relation

$$\int_V \Phi dV - \int_{\partial V}\Gamma dV = \int_{\partial V}\boldsymbol{u}\cdot\nabla(\boldsymbol{u}_1^\perp + \boldsymbol{u}_2)\cdot\boldsymbol{n}d\sigma + \int_V \nabla\cdot\boldsymbol{B}dV = 0,$$

which is a necessary condition for the existence of a solution of (15) is fulfilled, otherwise the interior Neumann problem in the general case has no solution.

Taking into account the solenoidality of $\boldsymbol{B}$, it follows that exists a vector field $\boldsymbol{A}$ such that $\boldsymbol{B} = \nabla\times\boldsymbol{A}$, *i.e.* (14).

Projecting the perturbation equation (11), on the subspace of solenoidal vectors, taking into account (9), (12), (13) and (14) we have

$$-\frac{\partial}{\partial t}\nabla\times(\psi\boldsymbol{k}) = -\nabla\times\boldsymbol{A} + 2\Omega\nabla\times(\chi_z\boldsymbol{k}) - \nabla\times\Delta(\psi\boldsymbol{k}) + \mathcal{F}\frac{\partial}{\partial t}\nabla\times\Delta(\psi\boldsymbol{k}). \tag{16}$$

From (16) it follows that exists a scalar field F such that

$$-\frac{\partial}{\partial t}(\psi \boldsymbol{k}) = -\boldsymbol{A} + 2\Omega\chi_z \boldsymbol{k} - \Delta(\psi \boldsymbol{k}) + \mathcal{F}\frac{\partial}{\partial t}\Delta(\psi \boldsymbol{k}) + \nabla F. \tag{17}$$

Since the vector field $\boldsymbol{A}$ is defined up to the gradient of a scalar function, we can write (17) in the following form

$$-\frac{\partial}{\partial t}(\psi \boldsymbol{k}) = -\boldsymbol{A} + 2\Omega\chi_z \boldsymbol{k} - \Delta(\psi \boldsymbol{k}) + \mathcal{F}\frac{\partial}{\partial t}\Delta(\psi \boldsymbol{k}). \tag{18}$$

From (16) it follows that $\nabla \times \boldsymbol{A}$ is a two-dimensional vector, indeed

$$\boldsymbol{k} \cdot \nabla \times \boldsymbol{A} = 0,$$

then we can assume the third component of $\boldsymbol{A}$ equal to zero.

The third component of (18) is

$$\frac{\partial \psi}{\partial t} = -2\Omega\chi_z + \Delta\psi - \mathcal{F}\frac{\partial}{\partial t}\Delta\psi. \tag{19}$$

This equation allows us to consider the contribution of the Coriolis term in nonlinear Lyapunov stability, as we shall see later.

3. Lyapunov stability

If we multiply $(1)_1$ by $\boldsymbol{u}$, $(1)_2$ by $\dfrac{b}{P_r}\theta$ and $(1)_3$ by $\dfrac{d}{S_c}\gamma$, where b, d are some positive parameters, adding the resulted equations and integrating over V, we obtain

$$\frac{d}{dt}\frac{1}{2}\int_V \left[\boldsymbol{u}^2 + b\theta^2 + d\gamma^2\right] dV = \mathcal{R}\left(1 + \frac{b}{P_r}\right)(\theta, w) - \mathcal{R}_c\left(1 + \frac{d}{S_c}\right)(\gamma, w) + \\ + \frac{b}{P_r}(\theta, \Delta\theta) + \frac{d}{S_c}(\gamma, \Delta\gamma) + (\boldsymbol{u}, \Delta\boldsymbol{u}) - \mathcal{F}\left(\boldsymbol{u}, \frac{\partial}{\partial t}\Delta\boldsymbol{u}\right). \tag{20}$$

Let us introduce the functions

$$E_{\mathcal{L}}(t) = \frac{1}{2}\int_V \left[\boldsymbol{u}^2 + b\theta^2 + d\gamma^2\right] dV, \qquad E^*(t) = \frac{1}{2}\int_V \nabla^2 \boldsymbol{u} dV.$$

If the condition

$$\frac{dE^*(t)}{dt} \leq 0 \tag{21}$$

is satisfied, then $E^*(t)$ is a decreasing function of t, whence the thermodiffusive equilibrium is nonlinearly globally stable.

In fact, along the solutions of the boundary value problem, the inequality (21) is equivalent to the following one

$$\frac{dE_{\mathcal{L}}(t)}{dt} \leq \mathcal{R}\left(1 + \frac{b}{P_r}\right)(\theta, w) - \mathcal{R}_c\left(1 + \frac{d}{S_c}\right)(\gamma, w) + \tag{22}$$

$$+\frac{b}{P_r}(\theta, \Delta\theta) + \frac{d}{S_c}(\gamma, \Delta\gamma) + (\boldsymbol{u}, \Delta\boldsymbol{u}).$$

In terms of poloidal and toroidal fields (22) becomes

$$\frac{dE_{\mathcal{L}}(t)}{dt} \leq \mathcal{I} - \mathcal{E} \tag{23}$$

where

$$\mathcal{I} \equiv -\mathcal{R}\left(1+\frac{b}{P_r}\right)(\theta,\Delta_1\chi) + \mathcal{R}_c\left(1+\frac{d}{S_c}\right)(\gamma,\Delta_1\chi), \tag{24}$$

$$\begin{gathered}\mathcal{E} \equiv \frac{b}{P_r}\|\nabla\theta\|^2 + \frac{d}{S_c}\|\nabla\gamma\|^2 + \|\nabla\chi_{xz}\|^2 + \|\nabla\chi_{yz}\|^2 + \\ + \|\nabla\Delta_1\chi\|^2 + \|\nabla\psi_x\|^2 + \|\nabla\psi_y\|^2,\end{gathered} \tag{25}$$

and $\|f\|^2$ denotes the L^2 norm of a function f.

We introduce now the function

$$E^*_{\mathcal{L}}(t) = E_{\mathcal{L}}(t) + cE'_{\mathcal{L}}(t), \tag{26}$$

where

$$E'_{\mathcal{L}} = \frac{1}{2}\left[\|\psi_x\|^2 + \|\psi_y\|^2 - \mathcal{F}\left(\|\psi_{xz}\|^2 + \|\psi_{yz}\|^2 + \|\Delta_1\psi\|^2\right)\right],$$

and c is a parameter we determine later. From (19), (23) and (26) we have

$$\frac{dE^*_{\mathcal{L}}}{dt} \leq \mathcal{I}^* - \mathcal{E}^* \tag{27}$$

where

$$\begin{gathered}\mathcal{I}^* \equiv -\mathcal{R}\left(1+\frac{b}{P_r}\right)(\theta,\Delta_1\chi) + \mathcal{R}_c\left(1+\frac{d}{S_c}\right)(\gamma,\Delta_1\chi) + \\ + 2c\Omega(\chi_z,\Delta_1\psi) - c\alpha(\Delta\psi,\Delta_1\psi),\end{gathered} \tag{28}$$

$$\begin{gathered}\mathcal{E}^* \equiv \frac{b}{P_r}\|\nabla\theta\|^2 + \frac{d}{S_c}\|\nabla\gamma\|^2 + \|\nabla\chi_{xz}\|^2 + \|\nabla\chi_{yz}\|^2 + \|\nabla\Delta_1\chi\|^2 + \\ + [1 + c(1-\alpha)]\left(\|\nabla\psi_x\|^2 + \|\nabla\psi_y\|^2\right),\end{gathered} \tag{29}$$

and α is a parameter we determine later.

Using the Wirtinger inequality it follows that $E'_{\mathcal{L}}$ is negative definite if

$$c_1 \leq \mathcal{F}, \tag{30}$$

where

$$c_1 \leq \max\{1, k_1^2, k_2^2\},$$

and k_1, k_2 are the wave numbers in x and y directions, respectively.

If $c < 0$ and (30) is satisfied, the function $E^*_{\mathcal{L}}(t)$ is positive definite and satisfy (27).

Therefore, a sufficient condition for the nonlinear global stability of the basic motion is

$$1 \leq \sqrt{R^*_a}, \tag{31}$$

where

$$\frac{1}{\sqrt{R_a^*}} = \max \frac{\mathcal{I}^*}{\mathcal{E}^*}, \tag{32}$$

in the class of admissible functions.

4. The maximum problem and the stability bound

We will study the variational problem (32) and later determine the parameters b, d and c in terms of the physical quantities, such that $\sqrt{R_a^*}$ will be maximal.

The Euler- Lagrange equations associated with the maximum problem (32) are:

$$\begin{cases} -\mathcal{R}\left(1+\dfrac{b}{P_r}\right)\Delta_1\theta + \mathcal{R}_c\left(1+\dfrac{d}{S_c}\right)\Delta_1\gamma - 2\Omega c\Delta_1\psi_z + \dfrac{2}{\sqrt{R_a^*}}\Delta\Delta\Delta_1\chi = 0, \\ -\mathcal{R}\left(1+\dfrac{b}{P_r}\right)\Delta_1\chi + \dfrac{b}{P_r}\dfrac{2}{\sqrt{R_a^*}}\Delta\theta = 0, \\ \mathcal{R}_c\left(1+\dfrac{d}{S_c}\right)\Delta_1\chi + \dfrac{d}{S_c}\dfrac{2}{\sqrt{R_a^*}}\Delta\gamma = 0, \\ 2\Omega c\Delta_1\chi_z - 2c\alpha\Delta\Delta_1\psi - \dfrac{2}{\sqrt{R_a^*}}[1+c(1-\alpha)]\Delta\Delta_1\psi = 0. \end{cases} \tag{33}$$

In the class of normal mode perturbations

$$\chi(\boldsymbol{x}) = W(z)\exp[i(k_1x_1+k_2x_2)+\sigma t], \qquad \psi(\boldsymbol{x}) = Z(z)\exp[i(k_1x_1+k_2x_2)+\sigma t],$$

$$\theta(\boldsymbol{x}) = \Theta(z)\exp[i(k_1x_1+k_2x_2)+\sigma t], \qquad \gamma(\boldsymbol{x}) = \Gamma(z)\exp[i(k_1x_1+k_2x_2)+\sigma t],$$

with $\sigma \in \mathbb{C}$, the equations (33) become

$$\begin{cases} k^2\mathcal{R}\left(1+\dfrac{b}{P_r}\right)\Theta - k^2\mathcal{R}_c\left(1+\dfrac{d}{S_c}\right)\Gamma + 2\Omega ck^2DZ - \dfrac{2}{\sqrt{R_a^*}}(D^2-k^2)^2k^2W = 0, \\ k^2\mathcal{R}\left(1+\dfrac{b}{P_r}\right)W + \dfrac{b}{P_r}\dfrac{2}{\sqrt{R_a^*}}(D^2-k^2)\Theta = 0, \\ -k^2\mathcal{R}_c\left(1+\dfrac{d}{S_c}\right)W + \dfrac{d}{S_c}\dfrac{2}{\sqrt{R_a^*}}(D^2-k^2)\Gamma = 0, \\ -\Omega ck^2DW + \left\{c\left[\alpha\left(1-\dfrac{1}{\sqrt{R_a^*}}\right)+\dfrac{1}{\sqrt{R_a^*}}\right]+\dfrac{1}{\sqrt{R_a^*}}\right\}(D^2-k^2)k^2Z = 0, \end{cases} \tag{34}$$

where $k^2 = k_1^2 + k_2^2$ is the wave number.

To (34) we add the following boundary conditions:

$$W = D^2W = \Theta = D^2\Theta = \Gamma = D^2\Gamma = DZ = 0. \tag{35}$$

Owing to (35) we choose (Georgescu 1985; Straughan 2004)

$$\begin{aligned} W(z) &= \sum_{n=1}^{\infty} W_n \sin(n\pi z), \qquad Z(z) = \sum_{n=1}^{\infty} Z_n \cos(n\pi z), \\ \Theta(z) &= \sum_{n=1}^{\infty} \Theta_n \sin(n\pi z), \qquad \Gamma(z) = \sum_{n=1}^{\infty} \Gamma_n \sin(n\pi z). \end{aligned} \tag{36}$$

From (34), (35) and (36) we have

$$\begin{aligned} k^4\mathcal{R}^2\frac{P_r}{b}\left(1+\frac{b}{P_r}\right)^2 + k^4\mathcal{R}_c^2\frac{S_c}{d}\left(1+\frac{d}{S_c}\right)^2 = 4\frac{1}{R_a^*}k^2(n^2\pi^2+k^2)^3 + \\ + 4\Omega^2 n^2\pi^2 k^2\frac{1}{\sqrt{R_a^*}}\frac{c^2}{c\left[\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}\right]-\frac{1}{\sqrt{R_a^*}}}. \end{aligned} \tag{37}$$

Differentiating with respect to c we obtain:

$$c = \frac{2}{\sqrt{R_a^*}}\frac{1}{\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}},$$

whence (37) becomes:

$$\begin{aligned} k^4\mathcal{R}^2\frac{P_r}{b}\left(1+\frac{b}{P_r}\right)^2 + k^4\mathcal{R}_c^2\frac{S_c}{d}\left(1+\frac{d}{S_c}\right)^2 = 4\frac{1}{R_a^*}k^2(n^2\pi^2+k^2)^3 + \\ + 16\Omega^2 n^2\pi^2 k^2\frac{1}{R_a^*}\frac{1}{\left[\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}\right]^2}. \end{aligned} \tag{38}$$

Substituting in (38) a value of α satisfying the equation

$$\left[\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}}\right]^2 = 1$$

we have

$$\frac{R_a^*}{4} = \frac{(n^2\pi^2+k^2)^3 + 4\Omega^2 n^2\pi^2}{k^2\mathcal{R}^2\frac{P_r}{b}\left(1+\frac{b}{P_r}\right)^2 + k^2\mathcal{R}_c^2\frac{S_c}{d}\left(1+\frac{d}{S_c}\right)^2}. \tag{39}$$

By choosing

$$\alpha\left(\frac{1}{\sqrt{R_a^*}}-1\right)-\frac{1}{\sqrt{R_a^*}} = -1,$$

that is

$$\alpha = 1,$$

it follows that $c < 0$ and $\mathcal{E}^*$ is positive definite.

Differentiating with respect to the parameters b and d we obtain

$$\frac{b}{P_r} = 1, \qquad \frac{d}{S_c} = 1. \tag{40}$$

Substituting (40) in (39) we obtain

$$R_a^* = \frac{(n^2\pi^2 + k^2)^3 + 4\Omega^2 n^2 \pi^2}{k^2\mathcal{R}^2 + k^2\mathcal{R}_c^2}. \tag{41}$$

We proved the following theorem:

Theorem 1. *If the principle of exchange of stabilities holds and (30) is satisfied, a sufficient condition for the zero solution of (1)-(2), corresponding to the basic conduction state, to be nonlinearly globally stable is*

$$1 \leq R_a^*. \tag{42}$$

This inequality is equivalent to the following one

$$k^2\mathcal{R}^2 + k^2\mathcal{R}_c^2 \leq (n^2\pi^2 + k^2)^3 + 4\Omega^2 n^2\pi^2,$$

where the function on the right hand side is just the critical function of the linear instability theory, if the principle of exchange of stabilities holds.

For an ordinary Newtonian fluid the previous inequality represents also a sufficient condition of nonlinear global stability, from which it follows the equality of linear and nonlinear stability bounds.

Conclusions

In this paper we have studied the nonlinear Lyapunov stability of the thermodiffusive equilibrium for a rotating Walters' fluid in a horizontal layer.

Projecting the initial perturbation evolution equations, written in terms of toroidal and poloidal fields, on some orthogonal subspaces of the kinematically admissible functions, we get an additional equation that allows us to preserve skewsymmetric terms, such as the Coriolis force, and, at the same time, all the nonlinear terms disappear, by using the standard L^2 norm.

We have studied the nonlinear Lyapunov stability first solving the Euler-Lagrange equations associated with the maximum problem, and then maximizing the stability domain with respect to the involved parameters. If the principle of exchange of stabilities holds, we recover, for an ordinary Newtonian fluid, *the equality between the linear and nonlinear critical parameters for the global stability.*

We observe that, anyhow, in this paper we applied an idea similar to that followed in Georgescu and Palese (1996), Georgescu *et al.* (2000, 2001), and Georgescu and Palese (2010, 2011), where the given problem governing the perturbation evolution was changed in order to obtain an optimum energy relation, that is *we modified the evolution equations obtaining equations with better symmetries, otherwise, if the initial evolution equations were used, the contribution of some terms would be null and, correspondingly, the stability criterion weaker.*

In a similar way in Labianca and Palese (2018) is obtained the equality of linear and nonlinear stability bounds for the classical magnetic Bénard problem.

Alternatively, we are forced to use some more complicated energies, to preserve the contribution of the skewsymmetric term.

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