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THERMODYNAMICAL DESCRIPTION AND VORTEX FORMATION AT THE SUPERFLUID TRANSITION IN ^{4}HE

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Abstract. In this paper, a system of field equations is written, able to describe the superfluid transition, under the hypothesis that the liquid is at rest, in the presence of heat flux and of a small production of vortices below the superfluid transition temperature. The proposed model chooses as fundamental fields the energy density, the heat flux, the averaged vortex line length per unit volume and an order parameter that controls the transition, having dimension of density. Using Liu procedure, restrictions for the constitutive quantities are obtained.

1. Introduction

The boiling point of ^{4}He is at $T_b = 5.2$ K. The reluctance of helium to condense is a consequence of the low mass of the helium atoms and of a very weak intermolecular interactions between them. Consider the p–T phase diagram shown schematically in Figure 1, there are several unique features that should be noted. At first, one sees that the phase diagram lacks a triple point of coexistence between liquid, vapor and solid, because the solid state cannot exist at any temperature, unless an external pressure of 2.5 MPa is applied. Another exceptional property is that in ^{4}He there are two liquid phases: helium I (He I) and helium II (He II). He I is a normal liquid, thus its hydrodynamical behaviour is described by the classical balance equations of mass, momentum and energy. He II has exceptional physical features, due to its quantum nature. The most remarkable aspects are its vanishingly small viscosity, and its extremely high thermal conductivity, several order of magnitude larger than high-conductivity liquids or solids. Further, He II is unable to boil and temperature waves can propagate in it.

There are two models to describe the macroscopic behaviour of this quantum liquid: the most known two-fluid model and the one-fluid model derived from extended thermodynamics.

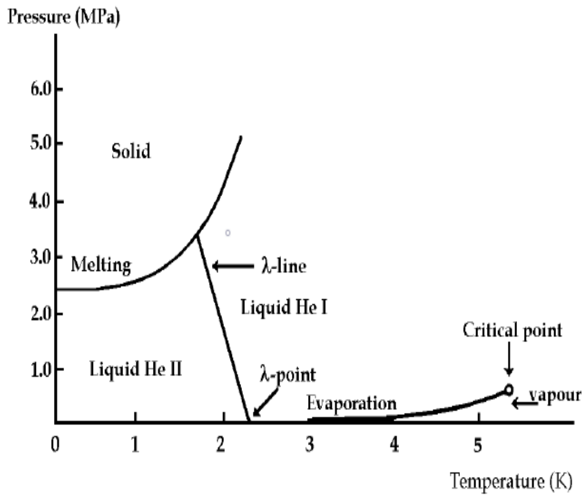

FIGURE 1. Phase diagram of ^{4}He.

The two-fluid model (Tisza 1938; Landau 1941) assumes that liquid helium II consists of two fluids: a normal fluid (with density ρ_n and velocity $\mathbf{v}_n$) which behaves like any other normal liquid with viscosity, and a superfluid (with density ρ_s and velocity $\mathbf{v}_s$) that does not carries entropy. It assumes that the amount of the superfluid component is zero at the λ–point and increases to unity at the absolute zero. The relative velocity $\mathbf{V}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ between normal and superfluid component is named "counterflow velocity".

The velocities of superfluid and normal component are linked to the barycenter velocity $\mathbf{v}$ and to the heat flux $\mathbf{q}$ by the relations:

$$\rho \mathbf{v} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n, \tag{1}$$

$$\mathbf{q} = \rho_s T s \mathbf{V}_{ns} + k \nabla T, \tag{2}$$

where $\rho = \rho_n + \rho_s$ is the total density, k the thermal conductivity of the normal component, and s the specific entropy, that in this model is ascribed only to the normal component. As one sees in equation (2), following the two fluid model, the energy transport is essentially of convective nature: the superfluid fraction flows from the low temperature situs to the high temperature situs, the normal fraction in the opposite direction.

A semimicroscopic physical basis for the anomalous behaviour of this quantum liquid was developed by Landau (1941). According to Landau, He II corresponds to a flowing ground state in which quasiparticles move. This ground state is a highly coherent macroscopic quantum state and is described by a complex macroscopic wave function $\psi(\mathbf{x})$. The excitations can flow separately from the ground state (the "condensate") and form a rarefied gas which is responsible for the thermal and viscous effects. The motion of the quasiparticles produces the "normal-superfluid counterflow", the relative motion of the "normal component" with respect to the ground state.

Another model able to describe the anomalous behaviour of liquid helium II is the one-fluid model (Mongiovì 1993, 2000; Saluto *et al.* 2014), derived from Extended Thermodynamics, a thermodynamic theory, formulated in the 1970's, that uses dissipative fluxes, besides the traditional variables, as independent fields (Müller and Ruggeri 1998; Jou *et al.* 2010, 2011). In a classical fluid, as liquid helium I, the heat flux $\mathbf{q}$ is given by the classical Fourier law; instead, in He II the extremely high thermal conductivity indicates that the heat flux must be considered as an independent variable, with its own evolution equation (Jou and Restuccia 2011). Therefore, the one-fluid model of liquid helium II selects as fundamental fields the density $\rho = \rho(\mathbf{x}, t)$, the velocity $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$, the absolute temperature $T = T(\mathbf{x}, t)$ and the heat flux $\mathbf{q} = \mathbf{q}(\mathbf{x}, t)$. In fact, in the framework of the one-fluid model, the transfer of energy due to the motion of the quasi-particles with respect to the ground state, is described by an internal vectorial variable, that, if one neglects non linear terms, can be identified with the heat flux.

For the sake of simplicity, in this paper, we will suppose the fluid at rest and at constant mass density. In this simplified case, to describe the behaviour of ^{4}He, sufficiently below the lambda point, it is sufficient to consider as fundamental variables only the energy density $E = \rho\epsilon$ and the heat flux density $\mathbf{q}$. The evolution equations for these fields can be easily obtained from Mongiovì (1993) and Saluto *et al.* (2014) and are:

$$\begin{cases} \rho \partial_t \epsilon + \nabla \cdot \mathbf{q} = 0, \\ \partial_t \mathbf{q} + \zeta^* \nabla T = \sigma^{\mathbf{q}}. \end{cases} \tag{3}$$

In these equations, $\zeta^* = \lambda_1/\tau_0$ is the ratio between the thermal conductivity λ_1 of superfluid helium and its relaxation time τ_0. In laminar helium II both these quantities are extremely high, but their ratio is finite and determines the second sound velocity V_2^* indeed it is $V_2^* = \sqrt{\zeta^*/\rho c}$, being c the specific heat (Mongiovì 1993). The production term $\sigma^{\mathbf{q}}$ will be specified below.

An important issue to be addressed in the problems of superfluid transition is the creation of quantized vortex lines, just below λ transition temperature. In fact, it is known that a relatively high production of quantized vortex lines forms at the superfluid λ-transition (Hendry *et al.* 1993). Further, experiments show that the presence of an applied heat flow involves a change in transition temperature (Duncan *et al.* 1988). In fact, the presence of heat flow can produce the formation of microscopic quantized vortices, and the consequent onset of turbulence, thus modifying the critical temperature of the phase transition.

The picture of superfluid turbulence is a disordered tangle of quantized vortices. The vortex array is described by introducing a scalar quantity L, the average vortex line length per unit volume (briefly called *vortex line density* and whose dimensions are $(length)^{-2}$). In counterflow experiments, vortices form a disordered tangle; in stationary situations the vortex line density L depends on the modulus of the heat flux q (or of the counterflow velocity V_{ns}). The production term $\sigma^{\mathbf{q}}$ in the heat flux equation (3) has two contributions and it is $\sigma^{\mathbf{q}} = -\frac{1}{\tau_{\mathbf{q}}}\mathbf{q} + \zeta^* \nabla \cdot \Pi_{\mathbf{q}}$. The first term is zero in the absence of vortices, while it is given by $-\frac{1}{\tau_{\mathbf{q}}}\mathbf{q} = -K_q L \mathbf{q}$, in counterflow situations, in the presence of vortices, (L is vortex line density and $K_q = \frac{1}{3}\kappa B_{HV}$, with B_{HV} the dissipative Hall-Vinen coefficient (Hall and Vinen 1956; Russell 1991; Barenghi *et al.* 2001; Nemirovskii 2013). The second term is the divergence of a tensor and describes the dissipation of thermal origin, due to the friction between the elementary excitations and is given by Mongiovì (1993):

$$\Pi_{\mathbf{q}} = \xi_1 \nabla \cdot \mathbf{q} \mathbf{U} + 2\xi_2 \langle \nabla \mathbf{q} \rangle. \tag{4}$$

in this expression $\mathbf{U}$ is the unit matrix, angular brackets denote the deviatoric part of a two dimensional tensor (*i.e.* $[\langle \nabla \mathbf{q} \rangle]_{ij} = \frac{1}{2}\frac{\partial q_i}{\partial x_j} + \frac{1}{2}\frac{\partial q_j}{\partial x_i} - \frac{1}{3}\frac{\partial q_k}{\partial x_k}\delta_{ij}$). $\xi_1 = \lambda_0 \beta'^2 T^3$ and $\xi_2 = \lambda_2 \beta^2 T^3$ are related to λ_0 and λ_2 (respectively, bulk and shear viscosity of He II), β and β' (coefficients related to the moments of fluctuations, as to non-classical contributions to the entropy flux (Jou *et al.* 2010)).

In some previous works (Fabrizio and Mongiovì 2013; Mongiovì and Saluto 2014), we have just considered the transition from He I to He II, by formulating a phase-field model that chooses as order parameter that controls the transition a phase field F, linked to the macroscopic coherent wave function of the condensate $\psi(\mathbf{x}, t)$, by the relation

$$F^2(\mathbf{x}, t) = \rho m_4 |\psi(\mathbf{x}, t)|^2. \tag{5}$$

with m_4 the mass of an helium atom and ρ the density of the liquid.

As we have said, an amount of quantized vortex lines is formed at the superfluid λ-transition (Hendry *et al.* 1993). Therefore, if one wants to describe also this phenomenon and the subsequent diffusion of vortex lines into the sample, we must formulate a model of phase transition that uses vortex line density L as an additional state variable. The proposed model chooses as field variables the phase field F, the temperature θ, the heat flux $\mathbf{q}$, and the line density L. Nonlocal terms to describe inhomogeneities in the field variables are also taken into account.

In Section 2, we determine some restrictions on the fluxes of the fundamental fields imposed by second law of thermodynamics, while in Section 3 we explain the physical meaning of the constitutive quantities and of the Lagrange multipliers. In Section 4 we obtain the expressions for the source terms in the evolution equation of the fundamental fields. Finally, in Section 5 we summarise the field equations, in particular in system (61) we obtain the equations that describe turbulent superfluid helium in the framework of the one-fluid model.

2. Balance equations and constitutive theory

The starting point will be a general set of balance equations for the order density F, the energy density $E = \rho\epsilon$, the heat flux density $\mathbf{q}$ and the line density L. They are written as:

$$\begin{cases} \partial_t F(\mathbf{x},t) + \nabla \cdot \mathbf{J}^F(\mathbf{x},t) = \sigma^F(\mathbf{x},t), \\ \partial_t E(\mathbf{x},t) + \nabla \cdot \mathbf{q}(\mathbf{x},t) = 0, \\ \partial_t \mathbf{q}(\mathbf{x},t) + \nabla \cdot \mathbf{J}^{\mathbf{q}}(\mathbf{x},t) = \sigma^{\mathbf{q}}(\mathbf{x},t), \\ \partial_t L(\mathbf{x},t) + \nabla \cdot \mathbf{J}^L(\mathbf{x},t) = \sigma^L(\mathbf{x},t), \end{cases} \tag{6}$$

In these equations $\mathbf{J}^F$ is the flux of the order parameter F, $\mathbf{J}^{\mathbf{q}}$ the flux of the heat flux, and $\mathbf{J}^L$ the flux of vortex lines. In this system, σ^F, $\sigma^{\mathbf{q}}$ and σ^L are terms describing the net supplies of the order, of heat flux and of vortices. Now, we will particularize the constitutive relations in order to describe the material in consideration.

We will determine restrictions on the fluxes by using the second law of thermodynamics. This law states that there exists a convex function S, the entropy per unit volume, and a vector function $\mathbf{J}^S$, the entropy flux density, such that the rate of entropy production per unit volume σ^S is non-negative for every thermodynamic process:

$$\sigma^S = \partial_t S + \nabla \cdot \mathbf{J}^S \geq 0. \tag{7}$$

The constitutive equations for the fluxes are assumed to depend linearly on the first derivative of the fundamental fields. As a consequence of the material objectivity principle they are expressed as:

$$\mathbf{J}^F = \gamma \mathbf{q} + \gamma_1 \nabla F + \gamma_2 \nabla E + \gamma_3 \nabla L, \tag{8}$$

$$\mathbf{J}^L = \nu \mathbf{q} + \nu_1 \nabla F + \nu_2 \nabla E + \nu_3 \nabla L, \tag{9}$$

$$\mathbf{J}^{\mathbf{q}} = \beta \mathbf{U}, \tag{10}$$

where the coefficients γ, γ_h, ν, ν_h and β are functions of F, E and L.

We will assume $S = S(F, E, L, q^2)$ and choose for $\mathbf{J}^S$ the following constitutive relation:

$$\mathbf{J}^S = \phi \mathbf{q} + \phi_1 \nabla F + \phi_2 \nabla E + \phi_3 \nabla L, \tag{11}$$

where ϕ and ϕ_h depend on F, E and L.

Note that inequality (7) does not hold for any value of the fundamental variables, but only for the thermodynamic processes, *i.e.* only for those values which are solutions of the system (6). This means that we can consider equations (6) as constraints for the entropy inequality to hold. We will exploit the consequences of second law using the Liu method of Lagrange multipliers (Liu 1972). This method affirms that the entropy inequality becomes totally arbitrary provided that we complement it by the evolution equations for the fields F, E, L and $\mathbf{q}$ affected by multiplying factors Λ^F, Λ^E, Λ^L and $\mathbf{\Lambda}^{\mathbf{q}}$, called Lagrange multipliers. Thus, we obtain the following inequality, which is satisfied for arbitrary values of the field variables:

$$\begin{aligned} \partial_t S &+ \nabla \cdot \mathbf{J}^S - \Lambda^F \left[\partial_t F + \nabla \cdot \mathbf{J}^F - \sigma^F\right] - \Lambda^E \left[\partial_t E + \nabla \cdot \mathbf{q}\right] \\ &- \Lambda^{\mathbf{q}} \cdot \left[\partial_t \mathbf{q} + \nabla \cdot \mathbf{J}^{\mathbf{q}} - \sigma^{\mathbf{q}}\right] - \Lambda^L \left[\partial_t L + \nabla \cdot \mathbf{J}^L - \sigma^L\right] \geq 0. \end{aligned} \tag{12}$$

The Lagrange multipliers are also supposed objective functions of the fundamental variables.

The constitutive theory is obtained imposing that the coefficients of all derivatives must vanish. Imposing the coefficients of time derivatives of the fundamental fields to be zero, we obtain: $\partial_t S - \Lambda^F \partial_t F - \Lambda^E \partial_t E - \Lambda^{\mathbf{q}} \cdot \partial_t \mathbf{q} - \Lambda^L \partial_t L = 0$. From which we obtain:

$$dS = \Lambda^F dF + \Lambda^E dE + \Lambda^L dL + \lambda \mathbf{q} \cdot d\mathbf{q}, \tag{13}$$

where we have assumed $\boldsymbol{\Lambda}^{\mathbf{q}} = \lambda \mathbf{q}$. From this relation we get:

$$\frac{\partial S}{\partial F} = \Lambda^F, \qquad \frac{\partial S}{\partial E} = \Lambda^E, \tag{14}$$

$$\frac{\partial S}{\partial L} = \Lambda^L, \qquad \frac{\partial S}{\partial q^2} = \frac{\lambda}{2}. \tag{15}$$

Imposing the coefficients of space derivatives to be zero, we obtain:

$$\nabla \cdot \mathbf{J}^S - \Lambda^F \nabla \cdot \mathbf{J}^F - \Lambda^E \nabla \cdot \mathbf{q} - \Lambda^{\mathbf{q}} \cdot \nabla \cdot \mathbf{J}^{\mathbf{q}} - \Lambda^L \nabla \cdot \mathbf{J}^L = 0. \tag{16}$$

Substituting relations (8)–(11) in (16), one can get some useful relations between coefficients ϕ, ϕ_h appearing in the entropy flux equation and coefficients γ, γ_h, ν, ν_h and β appearing in the expressions of the fluxes of fundamental variables. In particular imposing the coefficients of first-order space derivatives of $\mathbf{q}$ to be zero, we obtain:

$$\phi - \Lambda^E - \Lambda^F \gamma - \Lambda^L \nu = 0. \tag{17}$$

While considering the vanishing of the linear terms in $\mathbf{q}$, we get:

$$d\phi = \lambda d\beta + \Lambda^F d\gamma + \Lambda^L d\nu. \tag{18}$$

In the next section, we will determine the implications on the constitutive quantities imposed by relations (13), (17) and (18).

3. Physical interpretation of the constitutive quantities and of the Lagrange multipliers

As it is known, the use of the Lagrange multipliers as independent variables, results very useful from a mathematical point of view (see for example Ardizzone *et al.* (2009), and also Müller and Ruggeri (1998)). In order to single out the physical meaning of the constitutive quantities and of the Lagrange multipliers, we analyze now in detail the relations obtained in the previous section.

3.1. The non-equilibrium temperature. We recall now that in classical thermodynamics the inverse of the temperature T is defined as the derivative of entropy with respect to internal energy. We shall denote by T the temperature of liquid helium I, that is a classical fluid

$$\frac{1}{T} = \left[\frac{\partial S}{\partial E}\right]_{(F=0,L=0)} \tag{19}$$

Here we admit, by analogy with (19), that the temperature θ of liquid helium (in the normal and in the superfluid phase) will be given by

$$\frac{1}{\theta} = \left[\frac{\partial S}{\partial E}\right]_{F,L,\mathbf{q}} \tag{20}$$

In fact, the definition of temperature far from equilibrium is a debated topic; there are several different definitions for the non-equilibrium temperature (Casas-Vázquez and Jou 2003; Jou and Restuccia 2011), but the definition (20) based on the definition of a non-equilibrium entropy is the most fundamental from a theoretical point of view. Therefore θ will not only depend on E, but also on the order parameter F, on the heat flux density $\mathbf{q}$ and on the line density L. In more general models, where also high order non-local terms in the constitutive equations for the fluxes are taken into account, θ depends also on the gradients of the fundamental variables.

3.2. The free energy density. In what follows, we will choose the temperature θ as independent variable (that is the parameter that controls the superfluid transition), instead of the energy density E. Therefore, we introduce the free energy G (in the normal and in superfluid phases) by

$$G(\mathbf{x}, t) = E - \theta S \tag{21}$$

that is a function of fields F, θ, L and q^2.

Following the Ginzburg-Landau (GL) general theory of second-order phase transitions, we will suppose that the free-energy density G can be expanded in powers of F^2 in the following way:

$$G = G_1 + G^F = G_1 + \frac{1}{2}\tilde{a}F^2 + \frac{1}{4}\tilde{b}F^4, \tag{22}$$

where $G_1 = G_1(T)$ is the free-energy density of He I, and coefficients $\tilde{a}$ and $\tilde{b}$ are general functions of the field variables

$$\tilde{a} = \tilde{a}(\theta, L, q^2), \qquad \tilde{b} = \tilde{b}(\theta, L, q^2). \tag{23}$$

By using equation (21), the entropy inequality (7) assumes the following form:

$$\sigma^S = \theta^{-1}\partial_t E - S\theta^{-1}\partial_t\theta - \theta^{-1}\partial_t G + \nabla \cdot \mathbf{J}^S \geq 0. \tag{24}$$

From (13) we get:

$$dG = -Sd\theta - \theta\Lambda^F dF - \theta\Lambda^L dL - \theta\mathbf{\Lambda}^q \cdot d\mathbf{q}. \tag{25}$$

from which we obtain the physical meaning of the Lagrange multipliers Λ^F, Λ^L and $\mathbf{\Lambda^q}$. Indeed it results:

$$\Lambda^F = -\frac{1}{\theta}\left[\frac{\partial G}{\partial F}\right]_{\theta,L,\mathbf{q}} = -\frac{F}{\theta}\left[\tilde{a}(\theta,L,q^2)+\tilde{b}(\theta,L,q^2)F^2\right], \tag{26}$$

$$\Lambda^L = -\frac{1}{\theta}\left[\frac{\partial G}{\partial L}\right]_{\theta,F,\mathbf{q}} = -\frac{F^2}{2\theta}\left[\frac{\partial\tilde{a}}{\partial L}+\frac{1}{2}F^2\frac{\partial\tilde{b}}{\partial L}\right], \tag{27}$$

$$\mathbf{\Lambda}^q = -\frac{1}{\theta}\left[\frac{\partial G}{\partial \mathbf{q}}\right]_{\theta,F,L} = -\frac{F^2}{\theta}\left[\frac{\partial\tilde{a}}{\partial q^2}+\frac{1}{2}F^2\frac{\partial\tilde{b}}{\partial q^2}\right]\mathbf{q}. \tag{28}$$

More detailed expressions for these quantities will be obtained in the following subsection, where expressions for coefficients $\tilde{a}$ and $\tilde{b}$ will be given.

3.3. Consequences for the fluxes. Considering now equations (17) and (18), one obtains the expression for $d\beta$:

$$\lambda d\beta = d(\theta^{-1}) + \gamma d\Lambda^F + \nu d\Lambda^L \tag{29}$$

With the following positions:

$$\zeta = \frac{F^2}{\rho^2}\frac{\partial\beta}{\partial\theta} = -\frac{F^2}{\rho^2\lambda\theta^2}+\frac{F^2}{\rho^2\lambda}\left[\gamma\frac{\partial\Lambda^F}{\partial\theta}+\nu\frac{\partial\Lambda^L}{\partial\theta}\right], \tag{30}$$

$$\varphi_1 = \frac{F^2}{\rho^2}\frac{\partial\beta}{\partial F} = \frac{F^2}{\rho^2\lambda}\left[\gamma\frac{\partial\Lambda^F}{\partial F}+\nu\frac{\partial\Lambda^L}{\partial F}\right], \tag{31}$$

$$\varphi_2 = \frac{F^2}{\rho^2}\frac{\partial\beta}{\partial L} = \frac{F^2}{\rho^2\lambda}\left[\gamma\frac{\partial\Lambda^F}{\partial L}+\nu\frac{\partial\Lambda^L}{\partial L}\right], \tag{32}$$

$$\tag{33}$$

equation (29) is written:

$$\frac{F^2}{\rho^2}d\beta = \zeta d\theta + \varphi_1 dF + \varphi_2 dL. \tag{34}$$

In particular, if we assume that coefficients γ and ν vanish, it results $\varphi_1 = 0$, $\varphi_2 = 0$, and coefficient ζ assumes the simplified expression

$$\zeta = -\frac{F^2}{\rho^2}\frac{1}{\lambda\theta^2}, \tag{35}$$

that, when $F = \rho$ i.e. in the superfluid phase, reduces to that found in Mongiovì (2000) in the study of laminar flows of non viscous fluids in the presence of heat flux.

Note that above the λ-line, where it is $F = 0$, all coefficients ζ, φ_1, and φ_2 vanish.

4. Expressions of the source terms

4.1. A generalized GL equation for the order parameter. We will assume here for the free-energy density G the expression (22). Following the GL–theory of second-order phase transitions, we will assume that the stable stationary state of the system is that which minimizes the total free energy with respect to the order parameter F. One obtains the following Euler-Lagrange equation:

$$\frac{\delta G}{\delta F} = \tilde{a}F + \tilde{b}F^3 = 0. \tag{36}$$

Our aim, in this paper, is to write an evolution equation for the phase field F that reduces to equation (36) in homogeneous stationary states. Therefore, we will assume that the production term in the equation of the order parameter is proportional to $\delta G/\delta F$. We get:

$$\sigma^F = -K\frac{\delta G}{\delta F} = -K\left[\tilde{a}(\theta, L, q^2)F + \tilde{b}(\theta, L, q^2)F^3\right], \tag{37}$$

where K is a positive coefficient. Then we obtain the following evolution equation for the phase field F:

$$\partial_t F + \nabla\cdot[\gamma\mathbf{q} + \tilde{\gamma}_1\nabla F + \tilde{\gamma}_2\nabla\theta + \tilde{\gamma}_3\nabla L] = -K\left[\tilde{a}(\theta, L, q^2)F + \tilde{b}(\theta, L, q^2)F^3\right], \tag{38}$$

that reduces to equation (36) in homogeneous stationary states.

Having in mind to describe λ-transition in liquid helium, now we will determine some constrains for coefficients $\tilde{a}$ and $\tilde{b}$. The first observation is that the stationary solutions of equation (38), neglecting spatial inhomogeneities of the field variables, are

$$F = 0, \qquad\qquad F^2 = -\frac{\tilde{a}(\theta, L, q^2)}{\tilde{b}(\theta, L, q^2)}. \tag{39}$$

The first of them describes the normal phase, and therefore this solution must be stable for θ greater than a critical temperature θ_c, instead, the second stationary solution describes the superfluid phase, and must be stable for $\theta < \theta_c$.

In the following, we will suppose that $\tilde{a}$ depends on L, q^2 and on the difference $\theta - \theta_c$, while $\tilde{b}$ is independent of the temperature; then we assume $\tilde{a}$ linear in the difference $\theta - \theta_c$, obtaining:

$$\tilde{a} = \mathcal{A}(L, q^2)(\theta - \theta_c) \quad \text{and} \quad \tilde{b} = \mathcal{B}(L, q^2). \tag{40}$$

Experiments show that the critical temperature θ_c depends on the pressure and on the heat current, thus we have:

$$\theta_c = \theta_c(p, q^2). \tag{41}$$

At constant pressure p_0, when heat flux is absent, equation (41) must reduce to

$$\theta_c = \theta_c(p_0, 0) = \theta_\lambda\left[1 - ap_0\right]. \tag{42}$$

where a is the slope of the λ-line in the phase diagram in Figure 1. If we suppose q not too high, equation (41) can be approximated by:

$$\theta_{c,p_0} = \theta_{c,p_0}(q^2) \simeq \theta_\lambda\left[1 - ap_0 - bq^2\right], \tag{43}$$

where b can be determined from experimental data. Noting that the presence of the heat flux can create vortices, so destroying superfluidity, we deduce that coefficient b is a positive coefficient. In the space of the variables p, q^2 and θ, equation (43) represents the plane of critical points of equation $p = p_0$, that we will call $\lambda_{(p=p_0)}$-plane.

Recalling the microscopic meaning of the order parameter F ($F^2 = \rho m_4 |\psi(\mathbf{x},t)|^2 = \rho\rho_s$), we can determine the link between coefficients $\mathcal{A}$ and $\mathcal{B}$. In fact, with the assumptions (40) for $\tilde{a}$ and $\tilde{b}$, if we neglect spatial inhomogeneities of the field variables, the non zero stationary solution of equation (38) is

$$F^2 = \frac{\mathcal{A}(L,q^2)}{\mathcal{B}(L,q^2)} \left(\theta_{c,p_0}(q^2) - \theta\right). \tag{44}$$

Since $\rho_s \to \rho$, when $\theta \to 0$, then F must be equal to ρ when $\theta \to 0$. So we infer that:

$$\mathcal{B} = \frac{\mathcal{A}\theta_{c,p_0}}{\rho^2}. \tag{45}$$

The order parameter dynamical equation (38) becomes

$$\begin{aligned} &\partial_t F + \nabla \cdot [\gamma \mathbf{q} + \tilde{\gamma}_1 \nabla F + \tilde{\gamma}_2 \nabla\theta + \tilde{\gamma}_3 \nabla L] = \\ &= -K \left[\mathcal{A}(L,q^2)\left(\theta - \theta_{c,p_0}(q^2)\right) F + \frac{1}{\rho^2}\mathcal{A}(L,q^2)\theta_{c,p_0}(q^2) F^3\right], \end{aligned} \tag{46}$$

while the free energy G becomes

$$G(\mathbf{x},t) = G_1 + \frac{1}{2}\mathcal{A}(\theta - \theta_{c,p_0})F^2 + \frac{1}{4\rho^2}\mathcal{A}\theta_{c,p_0}F^4. \tag{47}$$

This choice of G allows us to obtain the expressions for the entropy density S^F and the energy density E^F that depend on the order parameter

$$S^F = -\frac{\partial G}{\partial \theta} = -\frac{1}{2}\mathcal{A}F^2, \tag{48}$$

$$E^F = G^F + \theta S^F = \mathcal{A}\theta_{c,p_0}\left(\frac{1}{4\rho^2}F^4 - \frac{1}{2}F^2\right). \tag{49}$$

By using (26)–(28), we finally get the following expressions for the Lagrange multipliers:

$$\Lambda^F = -\mathcal{A}F\left[1 - \left(1 - \frac{F^2}{\rho^2}\right)\frac{\theta_{c,p_0}}{\theta}\right], \tag{50}$$

$$\Lambda^L = -\frac{\partial \mathcal{A}}{\partial L}\frac{F^2}{2}\left[1 - \left(1 - \frac{F^2}{2\rho^2}\right)\frac{\theta_{c,p_0}}{\theta}\right], \tag{51}$$

$$\boldsymbol{\Lambda}^{\mathbf{q}} = -F^2\left[\frac{\partial \mathcal{A}}{\partial q^2} - \frac{1}{\theta}\left(1 - \frac{F^2}{2\rho^2}\right)\frac{\partial(\mathcal{A}\theta_{c,p_0})}{\partial q^2}\right]\mathbf{q}. \tag{52}$$

4.2. A generalized Cattaneo equation for the heat flux. For the production terms $\sigma^{\mathbf{q}}$ in the equation for the heat flux, we will choose the expression:

$$\sigma^{\mathbf{q}} = -\frac{1}{\tau_q(F,L)}\mathbf{q} + \nabla \cdot \mathbf{\Pi_q}, \tag{53}$$

with $\mathbf{\Pi_q}$ given by (4), while $\tau_q(F,L)$ is a scalar coefficient of the dimension of time, that can be interpreted as relaxation time. This coefficient depends on the order parameter F, and it results zero above the $\lambda_{(p=p_0)}$-plane, while it becomes extremely high below the $\lambda_{(p=p_0)}$-plane. A possible expression for τ_q could be (Ardizzone *et al.* 2017):

$$\tau_q(F,L) = \tau_0 \frac{\frac{F^2}{\rho^2}}{1-\frac{F^2}{\rho^2}(1-\tau_0 KL)} = \tau_0 \frac{F^2}{\rho^2 - F^2(1-\tau_0 KL)} \qquad (54)$$

where τ_0 is a constant having the dimension of time. In this way the evolution equation for the heat flux is

$$\frac{F^2}{\rho^2}\partial_t \mathbf{q} + [\zeta\nabla\theta + \varphi_1\nabla F + \varphi_2\nabla L] = \; = -\left(\frac{\rho^2 - F^2}{\tau_0\rho^2} + \frac{F^2}{\rho^2}KL\right)\mathbf{q} + \frac{F^2}{\rho^2}\nabla\cdot\mathbf{\Pi_q}.$$

Assuming β dependent only on θ, equation (55) can be written as:

$$\frac{F^2}{\rho^2}\partial_t \mathbf{q} + \zeta\nabla\theta = -\left(\frac{\rho^2 - F^2}{\tau_0\rho^2} + \frac{F^2}{\rho^2}KL\right)\mathbf{q} + \frac{F^2}{\rho^2}\nabla\cdot\mathbf{\Pi_q}. \qquad (55)$$

In the normal phase, when $F=0$, this expression reduces to the well-known Fourier's law, from which we deduce that $\tau_0\zeta$ must be identified with the heat conductivity k of He I.

Recalling equation (30) we deduce

$$\lambda = -\frac{F^2}{\rho^2}\frac{1}{\zeta\theta^2}\left[1 + \gamma\theta_{c,p_0}F\mathcal{A}\left(1-\frac{F^2}{\rho^2}\right) + \nu\theta_{c,p_0}\frac{F^2}{2}\frac{\partial\mathcal{A}}{\partial L}\left(1-\frac{F^2}{2\rho^2}\right)\right], \qquad (56)$$

that under the hypothesis $\beta = \beta(\theta)$ reduces to

$$\lambda = -\frac{F^2}{\rho^2}\frac{1}{\zeta\theta^2} < 0. \qquad (57)$$

4.3. A generalized Vinen equation for the line density. A simple expression for the source term σ^L in line density evolution equation, is:

$$\sigma^L = \alpha_1 L^{3/2}\mathbf{q} - \frac{\rho^2}{F^2}\beta\kappa L^2, \qquad (58)$$

In this way the evolution equation for the vortex line density can be written:

$$\frac{F^2}{\rho^2}\left[\partial_t L + \nabla\cdot(\gamma\mathbf{q} + \tilde{\gamma}_1\nabla F + \tilde{\gamma}_2\nabla\theta + \tilde{\gamma}_3\nabla L)\right] = \frac{F^2}{\rho^2}\alpha_1 L^{3/2}\mathbf{q} - \beta\kappa L^2. \qquad (59)$$

In such a way that, when $F=0$, i.e. in the normal phase, it results $L=0$, while, when $F=\rho$, *i.e.* sufficiently below the superfluid transition, we obtain a suitable generalization of the well known Vinen equation to inhomogeneous situations (Vinen 1957).

5. Field equations

Substituting the expressions of the production terms σ^F, σ^L and $\sigma^{\mathbf{q}}$ obtained in the previous Section in equations (6), the following system of field equations is

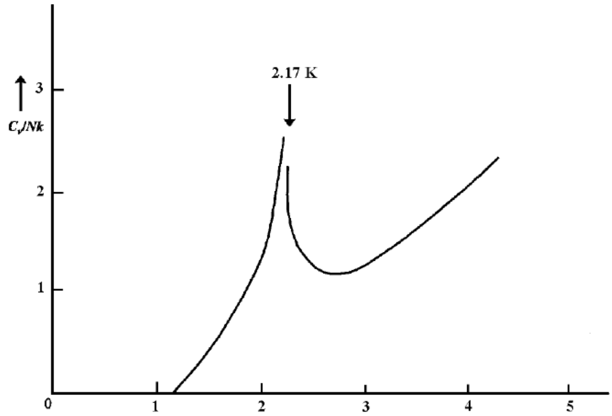

FIGURE 2. The specific heat of liquid ^{4}He vs the temperature.

written:

$$\begin{cases} \partial_t F + \nabla \cdot [\gamma \mathbf{q} + \gamma_1 \nabla F + \gamma_2 \nabla E + \gamma_3 \nabla L] = \\ \qquad = -K \left[\mathcal{A}(L, q^2) \left(\theta - \theta_{c,p_0}(q^2) \right) F + \frac{1}{\rho^2} \mathcal{A}(L, q^2) \theta_{c,p_0}(q^2) F^3 \right], \\ \frac{F^2}{\rho^2} \partial_t \mathbf{q} + [\zeta \nabla \theta + \varphi_1 \nabla F + \varphi_2 \nabla L] = \\ \qquad = -\left(\frac{\rho^2 - F^2}{\tau_0 \rho^2} + \frac{F^2}{\rho^2} KL \right) \mathbf{q} + \frac{F^2}{\rho^2} \nabla \cdot [\xi_1 \nabla \cdot \mathbf{q} \mathbf{U} + \xi_2 \langle \nabla \mathbf{q} \rangle], \\ \frac{F^2}{\rho^2} \partial_t L + \nabla \cdot [\gamma \mathbf{q} + \nu_1 \nabla F + \nu_2 \nabla E + \nu_3 \nabla L] = \frac{F^2}{\rho^2} \alpha_1 L^{3/2} \mathbf{q} - \beta \kappa L^2, \\ \rho c \partial_t \theta + \nabla \cdot \mathbf{q} = -\rho \frac{\partial \epsilon}{\partial F} \partial_t F - \rho \frac{\partial \epsilon}{\partial L} \partial_t L - 2\rho \frac{\partial \epsilon}{\partial q^2} \mathbf{q} \cdot \partial_t \mathbf{q}. \end{cases} \tag{60}$$

where ϵ is the specific entropy of helium and c the specific heat, whose plot near the λ point is shown in Figure 2. It can be approximated by $c = c_0 |\theta - \theta_\lambda|^{-\alpha}$, with $\alpha = 0.0127 \pm 0.0003$ (Lipa *et al.* 2003). A tentative expression for c, proposed in Mongiovì and Saluto (2014), could be $c = c_0 \left|\theta - \theta_{c,p_0}\right|^{-\alpha} \simeq c_0 \left|\theta - \theta_\lambda(1 - ap_0 - bq^2)\right|^{-\alpha}$.

From this system, when $F = 0$ we get the equations of a normal fluid (the liquid helium I), instead when $F = \rho$, we obtain the following system of field equations

$$\begin{cases} \partial_t \mathbf{q} + [\zeta \nabla \theta + \varphi_2 \nabla L] = -KL\mathbf{q} + \nabla \cdot [\xi_1 \nabla \cdot \mathbf{q} \mathbf{U} + \xi_2 \langle \nabla \mathbf{q} \rangle], \\ \partial_t L + \nabla \cdot [\gamma \mathbf{q} + \nu_2 \nabla E + \nu_3 \nabla L] = \alpha_1 L^{3/2} \mathbf{q} - \beta \kappa L^2 \\ \rho c \partial_t \theta + \nabla \cdot \mathbf{q} = -\rho \frac{\partial \epsilon}{\partial L} \partial_t L - 2\rho \frac{\partial \epsilon}{\partial q^2} \mathbf{q} \cdot \partial_t \mathbf{q}. \end{cases} \tag{61}$$

that generalizes to nonlocal situations the system of evolution equations of turbulent superfluid helium in counterflow situations found in Mongiovì and Jou (2007).

6. Conclusions

In the present work we have developed an inhomogeneous and nonlocal phase field model of turbulent superfluid in counterflow situations, in an inertial frame.

The study of superfluid transition in ^{4}He often does not consider the formation of quantized vortex lines at the λ–transition temperature, or assumes homogeneity of the vortex tangle line density L (Ardizzone *et al.* 2017). In several situations this homogeneity will not hold, and the vortex lines will diffuse from the most concentrated to the less concentrated zones.

To incorporate these effects has been the main motivation of this paper. Thus, we have used as fundamental variable the line density L, in addition to the temperature θ, to the heat flux $\mathbf{q}$ (that is the variable used in the one-fluid model of superfluid helium II) and to the order parameter F (that is the variable that controls the transition). We have worked in a macroscopic thermodynamic framework, which yields the several consequences of incorporating the additional terms to the evolution equations for the order parameter, the heat flux and the vortex line density. The thermodynamic consequences are exploited, using the Liu method of Lagrange multipliers.

We have written finally the field equations for the relevant quantities. The evolution equation for the order parameter F is a GL-like equation, the evolution equation for the heat flux is a proper modification of Cattaneo equation, the evolution equation for the line density L is a suitable modification of Vinen equation.

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