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A 3-DIMENSIONAL SINGULAR KERNEL PROBLEM IN VISCOELASTICITY: AN EXISTENCE RESULT

SANDRA CARILLO *

ABSTRACT. Materials with memory, namely those materials whose mechanical and/or thermodynamical behavior depends on time not only via the present time, but also through its past *history*, are considered. Specifically, a three dimensional viscoelastic body is studied. Its mechanical behavior is described via an integro-differential equation, whose kernel represents the *relaxation modulus*, characteristic of the viscoelastic material under investigation. According to the classical model, to guarantee the thermodynamical compatibility of the model itself, such a kernel satisfies regularity conditions which include the integrability of its time derivative. To adapt the model to a wider class of materials, this condition is relaxed; that is, conversely to what is generally assumed, no integrability condition is imposed on the time derivative of the relaxation modulus. Hence, the case of a relaxation modulus which is unbounded at the initial time $t = 0$, is considered, that is a *singular kernel* integro-differential equation, is studied. In this framework, the existence of a weak solution is proved in the case of a three dimensional singular kernel initial boundary value problem.

1. Introduction

This note is concerned about viscoelasticity problems. The interest on viscoelastic materials and their mechanical response to external actions is testified by a wide literature which goes from pioneering works due to Boltzmann (1878) and Volterra (1928) to a theoretical study developed to understand and model new materials also artificially devised. An overview on models which refer to materials requiring a *non-classical memory kernel* is provided by Carillo and Giorgi (2016) and in the references therein. Specifically, the model adopted throughout is due to Giorgi and Morro (1992), and, later, reconsidered by Gentili (1994). The crucial feature of all viscoelasticity models is that the stress response at time t linearly depends on the whole *past history* of the strain up to the present time t; hence, an integral term appears in the governing equation. For sake of simplicity, the body is generally assumed to be homogeneous and isotropic: this assumption allows to consider that all the physically relevant quantities do not depend on the space variable. Currently, there is a growing interest in viscoelastic materials due to new materials, such as viscoelastic polymers and/or bio-materials. Thus, recent models are concerned about modelling living tissues (Craiem and Armentano 2006; Zhuravkov and Romanova 2014; Yu *et al.* 2016) via suitable viscoelasticity models; however, often, to provide a mathematical description which might capture the behaviour in the case of new materials, the classical regularity requirements imposed on the relaxation modulus are not appropriate. In particular, less restrictive conditions correspond to relaxation moduli which are unbounded at the initial time $t = 0$ and, hence, correspond to a *singular kernel* model. Thus, Berti (2006), Grasselli and Lorenzi (1991) and, more recently, Conti *et al.* (2016) et al. have considered non-standard kernel models. Singular kernel problems, are widely investigated by many authors in various different applicative contexts (Miller and Feldstein 1971; Renardy *et al.* 1987; Desch and Grimmer 1989; Gentili 1994; Janno and Von Wolfersdorf 1998; Enelund *et al.* 1999; Enelund and Olsson 1999; Hanyga 2001; Hanyga and Seredyńska 2002, 2007; Ciambella *et al.* 2011; Messaoudi and Khenous 2015). Among the many, the book by Borcherdt (2009) represents an overview on new materials paying attention to applications of viscoelastic models to seismic problems. Fractional derivatives models are shown by Fabrizio (2014) to represent, in the case of a singular kernel, a possibility to investigate viscoelasticity problems. The interrelation between fractional derivatives and viscoelasticity (Koeller 1984; Adolfsson *et al.* 2005; Mainardi 2010, 2012; Hristov 2015) seems to be promising also under the perspective of bio-materials (Craiem and Armentano 2006; Deseri *et al.* 2014; Zhuravkov and Romanova 2014) or anisotropic homogeneous or non-homogeneous materials (Hilton 2012). Further to investigations which are concerned about the model to describe a physical behaviour, there are corresponding studies aiming to establish existence and, possibly, also uniqueness of solutions (Berti 2006; Carillo *et al.* 2013, 2017).

The results here presented are part of a research project, the author is involved in, which concerns materials with memory, their behavior as well as the study of related initial boundary value problems. Hence, further to isothermal viscoelasticity (Amendola *et al.* 2010; Carillo *et al.* 2013) also rigid heat conduction with memory (Amendola and Carillo 2004; Carillo 2010, 2011a,b; Carillo *et al.* 2014; Carillo 2019) as well as, similarities between the two different cases under the analytical viewpoint (Carillo 2005, 2015) are investigated. More recently, magneto-viscoelasticity problems are studied by Carillo *et al.* (2011, 2012, 2017), motivated by new materials which are devised incorporating magnetically sensible nanoparticles in a viscoelastic gel.

The material is organized as follows. The opening Section 2 provides an overview on the classical model. Thus, the strain tensor, the stress tensor and the relaxation modulus are introduced together with the functional requirements they are assumed to satisfy. Then, the linear integro-differential equation which models the viscoelasticity problem is written. Finally, the classical Dirichlet problem studied by Dafermos (1970) is recalled. In the next Section 3 the singular kernel problem under investigation is stated. Then, on introduction of the integrated relaxation tensor, an equivalent formulation of the problem is obtained. An approximation strategy, as in the 1-dimensional case (Carillo *et al.* 2013), is devised to construct a sequence of problems whose solution approximates the solution to the problem under investigation. A Lemma which gives an estimate, crucial for subsequent results, is proved. Section 4 contains the main existence result: the singular problem under investigation is proved to admit solution. Specifically, via a suitable weak formulation of the problem, which takes into account the prescribed initial data and boundary conditions, a sequence of approximated solutions is proved to admit a limit which turns out to solve the singular problem. The closing Section 5 is concerned about some perspectives and open problems as well as connections with other works or related subjects.

2. A regular viscoelasticity problem

This Section is concerned about the introduction of the model of viscoelastic body; then, to a regular viscoelasticity problem. specifically, a 3-dimensional *smooth* body whose reference configuration is a smooth compact set $\Omega \subset \mathbb{R}^3$ is considered. Accordingly, the key features of the model of *viscoelastic body*, following Fabrizio and Morro (1992), Giorgi and Morro (1992), and Gentili (1994), are briefly recalled. The material is assumed to be a *material with memory* to stress that its mechanical response depends on time not only through the *present* time t but also on the whole *past history*. Hence, when the viscoelastic body is assumed homogeneous and isotropic, then the spatial dependence can be omitted in all the quantities of interest, that is, let $\mathbb{E}$ be the symmetric tensor

$$\mathbb{E} := \frac{1}{2}\left[\nabla \mathbf{u} + \nabla \mathbf{u}^T\right] \tag{1}$$

then

$$\mathbb{E} = \mathbb{E}(t) \ , \ \mathbb{T} = \mathbb{T}(t) \ , \ \mathbb{G} = \mathbb{G}(t) \tag{2}$$

represent, in turn, the *strain tensor* $\mathbb{E} \in Sym$, the *stress tensor* $\mathbb{T} \in Sym$ and the *relaxation modulus* $\mathbb{G} \in Sym$. According to Gentili (1994), when $\mathbb{G}_0 := \mathbb{G}(0)$ denotes the *instantaneous elastic modulus*, the following constitutive assumption links strain and stress tensors

$$\mathbb{T}(t) = \mathbb{G}_0\mathbb{E}(t) + \int_0^\infty \mathbb{G}(\tau)\, \dot{\mathbb{E}}^t(\tau)\, d\tau \ , \qquad \mathbb{E}^t(\tau) := \mathbb{E}(t-\tau) \tag{3}$$

where $\mathbb{E}^t$ is termed strain past history. Equation (3) can be, equivalently, written as

$$\mathbb{T}(t) = \mathbb{G}_0\mathbb{E}(t) + \int_0^\infty \dot{\mathbb{G}}(\tau)\, \mathbb{E}^t(\tau)\, d\tau \ . \tag{4}$$

The relaxation modulus in (3) and (4) satisfies the following regularity requirements

$$\mathbb{G} \in L^1(\mathbb{R}^+) \ , \ \dot{\mathbb{G}} \in L^1(\mathbb{R}^+, Lin(Sym)) \tag{5}$$

so that

$$\mathbb{G}(t) = \mathbb{G}_0 + \int_0^t \dot{\mathbb{G}}(s)\, ds \ , \qquad \mathbb{G}(\infty) = \lim_{t\to\infty} \mathbb{G}(t), \tag{6}$$

where $\mathbb{G}(\infty) \in Lin(Sym)$ is termed *equilibrium elastic modulus* (Fabrizio and Morro 1992). Hence, the relaxation modulus $\mathbb{G}$ enjoys the *fading memory property*, that is

$$\forall \varepsilon > 0\, \exists\, \tilde{a} = a(\varepsilon, \mathbb{E}^t) \in \mathbb{R}^+ s.t.\ \forall a > \tilde{a}, \left| \int_0^\infty \dot{\mathbb{G}}(s+a)\mathbb{E}^t(s)\, ds \right| < \varepsilon. \tag{7}$$

Now, the integro-differential equation which gives the displacement $\mathbf{u} : \Omega \times (0,T) \to \mathbb{R}^3$ in the case of a viscoelastic material, reads

$$\rho \mathbf{u}_{tt} - \operatorname{div}\left(\mathbb{G}(0)\nabla \mathbf{u} + \int_0^t (\dot{\mathbb{G}}(t-\tau)\nabla \mathbf{u}(\tau) d\tau \right) = \mathbf{f}, \tag{8}$$

wherein the parameter $\rho \in \mathbb{R}^+$ can be, without loss of generality, assumed to be 1, *i.e.*, $\rho = 1$ is the choice throughout. In particular, on introduction of the external force $\mathbf{f}$ in which, further to an external force (optional), also the history of the material is included, the following initial boundary value problem, wherein the integral is posed in Volterra form, is considered

$$\begin{cases} \mathbf{u}_{tt} - \mathbb{G}(0)\Delta\mathbf{u} + \int_0^t \dot{\mathbb{G}}(t-\tau)\Delta\mathbf{u}(\tau)d\tau = \mathbf{f} \\ \mathbf{u}(0) = \mathbf{u}^0, \quad \mathbf{u}_t(0) = \mathbf{u}^1 \text{ in } \Omega; \ \mathbf{u} = 0 \quad \text{on } \Sigma = \partial\Omega \times (0,T). \end{cases} \tag{9}$$

Under the classical regularity requirements (5) an existence and uniqueness of a weak solution is proved by Dafermos (1970). Crucial to establish the existence result is an *a priori* estimate: it relies on the regularity assumptions (5)-(6) the symmetric tensor $\mathbb{G}$ is classically assumed to satisfy; these assumptions, in particular, imply

$$\mathbb{G} \in L^1(0,T) \cap C^2(0,T), \ \forall T \in \mathbb{R}\ . \tag{10}$$

In addition, (see, for instance Fabrizio and Morro (1992) and Gentili (1994)):

$$\mathbb{G}(t) > 0, \qquad \dot{\mathbb{G}}(t) \le 0, \qquad \ddot{\mathbb{G}}(t) \ge 0, \qquad t \in (0,\infty), \tag{11}$$

that is, the tensor's entries of $\mathbb{G}$, are such that, for any symmetric tensor e_{kl}

$$\begin{aligned} &\bullet\ \mathbb{G}_{klmn} = \mathbb{G}_{mnkl} = \mathbb{G}_{lkmn} \\ &\bullet\ \mathbb{G}_{klmn}\, e_{kl} e_{mn} \ge \beta\, e_{kl} e_{kl}, \qquad \beta > 0,\ e_{kl} = e_{lk}\ . \\ &\bullet\ \dot{\mathbb{G}}_{klmn} e_{kl} e_{mn} \le 0 \\ &\bullet\ \ddot{\mathbb{G}}_{klmn} e_{kl} e_{mn} \ge 0 \end{aligned} \tag{12}$$

Note that also these sign conditions are crucial to prove Lemma 1, the *a priori* estimate on which the solution existence result is based.

3. Singular memory kernel problem

This Section is concerned about a singular viscoelasticity problem which represents a generalization of the regular one presented in the previous Section. Specifically, a 3-dimensional singular viscoelasticity problem is proved to admit solution generalizing the result obtained by Carillo *et al.* (2013), where the 1-dimensional case is studied. An analogous result for details can be stated also in 3-dimensional rigid thermodynamics (Carillo *et al.* 2014).

To take into account a wider class of materials, as specified for instance by Carillo (2015) and Carillo and Giorgi (2016), the regularity assumptions on $\mathbb{G}$ are relaxed. In particular the request $(5)_2$ is removed, that is, we do not require the integrability of $\dot{\mathbb{G}}$ in $\mathbb{R}^+$ but only the integrability of $\mathbb{G}$. Hence, the tensor $\mathbb{G}$ is unbounded at the origin and, therefore, the integro-differential problem can not be written under the form (8) where $\mathbb{G}(0)$, not defined, appears. To overcome this difficulty, following Carillo *et al.* (2013), observe that condition (10) guarantees that the *integrated relaxation tensor* $\mathbb{K}$ can be defined via

$$\mathbb{K}(\xi) := \int_0^\xi \mathbb{G}(\tau)d\tau \quad \text{which implies} \quad \mathbb{K}(0) = 0\ . \tag{13}$$

Then, the following integral problem

$$\mathrm{P}: \ \mathbf{u}(t) = \int_0^t \mathbb{K}(t-\tau)\Delta\mathbf{u}(\tau)d\tau + \mathbf{u}^1 t + \mathbf{u}^0 + \int_0^t d\tau \int_0^\tau \mathbf{f}(\xi)d\xi \tag{14}$$

where, respectively, $\mathbf{u}^0, \mathbf{u}^1, \mathbf{f}$ denote the initial data and the external force, which, once again, includes the past history, is well defined and represents an equivalent formulation of the i.b.v.p. (9). Now, let the *translated relaxation tensor* be introduced via

$$\mathbb{G}^\varepsilon(\cdot) := \mathbb{G}(\varepsilon + \cdot) \ , \ \varepsilon > 0 \, , \tag{15}$$

which, recalling condition (10), is well defined $\forall \varepsilon > 0$. Hence, if the initial time $t_0 = \varepsilon$ is considered, the following integro-differential problem P^ε_D can be defined

$$P^\varepsilon_D : \ \mathbf{u}^\varepsilon_{tt} = \mathbb{G}^\varepsilon(0)\,\Delta\mathbf{u}^\varepsilon + \int_0^t \dot{\mathbb{G}}^\varepsilon(t-\tau)\,\Delta\mathbf{u}^\varepsilon(\tau)\,d\tau + \mathbf{f} \ . \tag{16}$$

Imposing on the latter the initial and boundary conditions

$$\mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}^0(x), \quad \mathbf{u}^\varepsilon_t|_{t=0} = \mathbf{u}_1(x), \quad \mathbf{u}^\varepsilon_x|_{\partial\Omega\times(0,T)} = 0 \ , \ t < T \tag{17}$$

a regular problem, which can be regarded as an approximation of the singular problem (14), is obtained. The arbitrariness of ε, allows to look for a solution of the singular problem of interest according to the following steps.

- let $\{\varepsilon_h\}_{h\in\mathbb{N}}$ denote an infinitesimal sequence, as $h \to \infty$, then construct a suitable sequence $\{P^{\varepsilon_h}\}_{h\in\mathbb{N}}$ of *approximated regular problems* P^{ε_h};
- find the admitted corresponding approximated solutions to the regular problems $P^{\varepsilon_h}_D$ (16)-(17), denoted as $\mathbf{u}^{\varepsilon_h}$;
- show that the sequence of solutions admits a limit, *i.e.*, $\exists\, \mathbf{u} := \lim_{\varepsilon\to 0} \mathbf{u}^\varepsilon$, that is $\mathbf{u} := \lim_{h\to\infty} \mathbf{u}^{\varepsilon_h}$.

3.1. Approximation strategy. The generic Approximated Problem P^ε_D (16)-(17), fixed $\varepsilon > 0$, reads

$$P^\varepsilon_D : \begin{cases} \mathbf{u}^\varepsilon_{tt} = \mathbb{G}(\varepsilon)\,\Delta\mathbf{u}^\varepsilon + \displaystyle\int_0^t \dot{\mathbb{G}}(\varepsilon + t - \tau)\,\Delta\mathbf{u}^\varepsilon(\tau)\,d\tau + \mathbf{f} \\ \mathbf{u}^\varepsilon(0) = \mathbf{u}^0, \quad \mathbf{u}^\varepsilon{}_t(0) = \mathbf{u}^1 \ \text{in} \ \Omega; \ \ \mathbf{u}^\varepsilon = 0 \quad \text{on} \ \Sigma = \partial\Omega\times(0,T) \ . \end{cases} \tag{18}$$

The latter, according to Dafermos (1970), admits a unique solution, here denoted as $\mathbf{u}^\varepsilon$: it is such that the following Lemma holds.

Lemma 1 *Let* $\mathbf{u}^\varepsilon$ *be, according to* Dafermos (1970)*, the solution to* (18)*, then the following equality holds*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{G}(t)|\nabla\mathbf{u}^\varepsilon|^2 d\mathbf{x} + \frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}^\varepsilon_t|^2 d\mathbf{x} - \frac{1}{2}\frac{d}{dt}\int_0^t ds \int_\Omega \dot{\mathbb{G}}(s)\,|\nabla\mathbf{u}^\varepsilon(t) - \nabla\mathbf{u}^\varepsilon(t-s)|^2\,d\mathbf{x} =$$

$$= \int_\Omega \mathbf{f}\cdot\mathbf{u}^\varepsilon_t\,d\mathbf{x} + \frac{1}{2}\int_\Omega \dot{\mathbb{G}}(t)\nabla - \frac{1}{2}\int_0^t ds \int_\Omega \ddot{\mathbb{G}}(s)\,|\nabla\mathbf{u}^\varepsilon(t) - \nabla\mathbf{u}^\varepsilon(t-s)|^2\,d\mathbf{x} \ . \tag{19}$$

Proof. Consider the integro-differential P^ε_D, subject to the assigned initial and boundary conditions, equation $(18)_1$ can be written in the equivalent form (8) as

$$\mathbf{u}^\varepsilon_{tt} - \operatorname{div}\left(\mathbb{G}(\varepsilon)\nabla\mathbf{u}^\varepsilon_t \int_0^t \dot{\mathbb{G}}(\varepsilon + s)\,[\nabla\mathbf{u}^\varepsilon_t(t) - \nabla\mathbf{u}^\varepsilon_t(t-s)]\,ds\right) = \mathbf{f} \, . \tag{20}$$

The latter on multiplication by $\mathbf{u}_t^\varepsilon$, followed by integration over Ω, gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}_t|^2\,d\mathbf{x}+\frac{1}{2}\int_\Omega \mathbb{G}(t)\nabla\mathbf{u}\cdot\nabla\mathbf{u}_t\,d\mathbf{x}+\\&+\int_\Omega \mathbf{u}_t(t)\,d\mathbf{x}\int_0^t \operatorname{div}\left[\dot{\mathbb{G}}(s)\left(\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)\right)\right]ds=\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,d\mathbf{x}\;,\end{aligned}\tag{21}$$

wherein all the indices ε are omitted to simplify the notation. Then, re-writing the second and the third terms in a more convenient form, (21) reads

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \mathbb{G}(t)|\nabla\mathbf{u}|^2\,d\mathbf{x}-\frac{1}{2}\int_\Omega \dot{\mathbb{G}}(t)|\nabla\mathbf{u}|^2\,d\mathbf{x}+\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}_t|^2 d\mathbf{x}-\\&-\int_\Omega\int_0^t \dot{\mathbb{G}}(s)\,\nabla\mathbf{u}_t(t)\cdot[\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)]\,d\mathbf{x}ds=\int_\Omega \mathbf{f}\cdot\mathbf{u}_t\,d\mathbf{x}\;.\end{aligned}\tag{22}$$

Now, the last term on the left hand side can be written as follows:

$$\begin{aligned}&-\int_\Omega\int_0^t \dot{\mathbb{G}}(s)\,\nabla\mathbf{u}_t(t)\cdot[\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)]\,d\mathbf{x}ds=\\&=-\frac{1}{2}\frac{d}{dt}\int_0^t ds\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2+\frac{1}{2}\int_\Omega\int_0^t \ddot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2 d\mathbf{x}ds\;,\end{aligned}\tag{23}$$

which, substituted into (22), proves Lemma 1. Here, for sake of completeness, detailed computations which prove the identity (23) are reported:

$$\begin{aligned}&-\int_\Omega\int_0^t \dot{\mathbb{G}}(s)\,\nabla\mathbf{u}_t(t)\cdot[\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)]\,d\mathbf{x}ds=-\frac{1}{2}\frac{d}{dt}\int_0^t ds\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2 d\mathbf{x}+\\&+\frac{1}{2}\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2 d\mathbf{x}-\int_\Omega\int_0^t \dot{\mathbb{G}}(s)\nabla\mathbf{u}_t(t-s)\cdot[\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)]\,d\mathbf{x}ds=\\&=-\frac{1}{2}\frac{d}{dt}\int_0^t ds\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2+\frac{1}{2}\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(0)|^2 d\mathbf{x}+\\&+\int_\Omega\int_0^t \dot{\mathbb{G}}(s)\,|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2\,d\mathbf{x}ds=-\frac{1}{2}\frac{d}{dt}\int_0^t ds\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2 d\mathbf{x}+\\&+\frac{1}{2}\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(0)|^2-\int_\Omega\int_0^t \dot{\mathbb{G}}(s)[\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)]\cdot\frac{d}{ds}[\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)]\,d\mathbf{x}ds=\\&=-\frac{1}{2}\frac{d}{dt}\int_0^t ds\int_\Omega \dot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2 d\mathbf{x}+\frac{1}{2}\int_\Omega\int_0^t \ddot{\mathbb{G}}(s)|\nabla\mathbf{u}(t)-\nabla\mathbf{u}(t-s)|^2\,d\mathbf{x}ds\;.\end{aligned}$$

This completes the proof of (23) and, hence, of Lemma 1. □

Then, the further estimate can be proved.

Lemma 2 *Let* $\mathbf{u}^\varepsilon$ *denote a solution to* (18)*, then the following estimate holds*

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,d\mathbf{x} + \frac{1}{2}\int_\Omega |u_t|^2\,d\mathbf{x} \le \gamma\, e^T C(\mathbf{f}) \tag{24}$$

wherein $\gamma = \max\{(|\mathbb{G}(T+1)|)^{-1}, 1\}$.

Proof.

Consider (19) and integrate it w.r.t. time, over the time interval $(0,t)$, since $\mathbf{u}^\varepsilon$ is a solution of the integro-differential problem P_D^ε (18), it satisfies also the assigned initial and boundary conditions; hence

$$\frac{1}{2}\int_\Omega \mathbb{G}(t)|\nabla\mathbf{u}^\varepsilon|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega |\mathbf{u}_t^\varepsilon|^2 d\mathbf{x} \le$$

$$\le \int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t^\varepsilon\,d\mathbf{x} + \frac{1}{2}\int_0^t\int_\Omega \dot{\mathbb{G}}(t)|\nabla\mathbf{u}^\varepsilon(0)|^2\nabla d\mathbf{x} + \frac{1}{2}\int_0^t\int_\Omega |\mathbf{u}_t(\mathbf{x},0)|^2\,d\mathbf{x} \tag{25}$$

$$\le \int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t^\varepsilon\,d\mathbf{x} + \frac{1}{2}\int_0^t\int_\Omega |\mathbf{u}_1|^2\,d\mathbf{x}.$$

If, in addition, homogeneous initial conditions are imposed, then, the last inequality reduces to

$$\frac{1}{2}\int_\Omega \mathbb{G}(t)|\nabla\mathbf{u}^\varepsilon|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega |\mathbf{u}_t^\varepsilon|^2 d\mathbf{x} \le \int_0^t\int_\Omega \mathbf{f}\cdot\mathbf{u}_t^\varepsilon\,d\mathbf{x}. \tag{26}$$

Then, it follows

$$\frac{1}{2}\int_\Omega \mathbb{G}(t)|\nabla\mathbf{u}^\varepsilon|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2\,d\mathbf{x} - \int_0^t\int_\Omega |\mathbf{u}_t|^2\,d\mathbf{x}ds \le C(\mathbf{f})\ , \tag{27}$$

when $\mathbf{u}_1 = 0$, otherwise, in the latter and also in the following estimates, $C(\mathbf{f})$ should be replaced by $C(\mathbf{f},\mathbf{u}_1)$. Gronwall's lemma applied to (27) gives

$$\frac{1}{2}\int_\Omega \mathbb{G}(t)|\nabla\mathbf{u}^\varepsilon|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega |\mathbf{u}_t|^2\,d\mathbf{x} \le e^T C(\mathbf{f})\ , \tag{28}$$

which, when the sign conditions (11) and (12) are recalled, $|\mathbb{G}(t+\varepsilon)| \ge |\mathbb{G}(T+1)|$, and therefore

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,d\mathbf{x} + \frac{1}{2}\int_\Omega |u_t|^2\,d\mathbf{x} \le \gamma e^T C(\mathbf{f})\ \ ,\ \ \gamma := \max\{(|\mathbb{G}(T+1)|)^{-1}, 1\}. \tag{29}$$

□

4. Solution existence

This Section is concerned about the existence result: the main one obtained in this note. Specifically, the existence of a weak solution is stated in Theorem 1, together with a suitable weak formulation of the problem; then, after the proof's sketch, a Lemma is stated and proved. The Section closes with the proof of the Theorem.

Let $\left\{\mathrm{P}_D^{\varepsilon_h}\right\}_{h\in\mathbb{N}}$ denote a sequence of approximated problems $\mathrm{P}_D^{\varepsilon}$, (16), consider the corresponding sequence of admitted solutions $\{\mathbf{u}^{\varepsilon_h}\}_{h\in\mathbb{N}}$, each of which satisfies also the assigned initial and boundary conditions (17). Theorem 1 shows that given a solution sequence, it admits a limit which represents a solution to the singular problem under investigation. Specifically, the following Theorem can be stated.

Theorem 1 *The integral problem (14)*

$$\mathrm{P}:\ \mathbf{u}(t)=\int_0^t \mathbb{K}(t-\tau)\Delta\mathbf{u}(\tau)d\tau+\mathbf{u}^1 t+\mathbf{u}^0+\int_0^t d\tau\int_0^\tau \mathbf{f}(\xi)d\xi$$

admits a solution $\mathbf{u}$ *given by the limit of the solution sequence* $\{\mathbf{u}^\varepsilon\}$, *where* $\mathbf{u}^\varepsilon$ *denotes the solution to the integral problem* P_I^ε

$$\mathrm{P}_I^\varepsilon:\mathbf{u}^\varepsilon(t)=\int_0^t \mathbb{K}^\varepsilon(t-\tau)\Delta\mathbf{u}^\varepsilon(\tau)d\tau+\mathbf{u}_1 t+\mathbf{u}_0+\int_0^t d\tau\int_0^\tau \mathbf{f}(\xi)d\xi\,, \tag{30}$$

that is

$$\exists\,\mathbf{u}(t)=\lim_{\varepsilon\to 0}\mathbf{u}^\varepsilon(t)\ \ in\ \ \mathbf{L}^2(\mathscr{Q})\ ,\ \mathscr{Q}=\Omega\times(0,T). \tag{31}$$

The outline of the proof, based on the equivalence between the integral formulation (30) and the integro-differential one (16)-(17), is based on the following steps:

- consider the weak formulation of the problem;
- introduce an infinitesimal sequence $\{\varepsilon_h\}_{h\in\mathbb{N}}$, and denote as $\{\mathbf{u}^{\varepsilon_h}\}_{h\in\mathbb{N}}$ the corresponding sequence of approximate solutions;
- consider separately the terms which do not depend on ε_h;
- consider the terms where $\mathbf{u}^{\varepsilon_h}$ & $\mathbb{K}^{\varepsilon_h}$ appear;
- apply the convergence result in Lemma 2.

Previous to the proof of the Theorem, the weak formulation of the problem and a Lemma are needed to prove the existence theorem. First of all, on introduction of the following test functions

$$\mathbf{v}\equiv(v_1,v_2,v_3),\ \ \mathbf{v}\in H_0^1(\mathscr{Q})\ \ ,\ \ \mathscr{Q}=\Omega\times(0,T),\ \ \text{s.t.}\ \ \mathbf{v}=\mathbf{0}\ \ \text{on}\ \ \partial\Omega\,, \tag{32}$$

the weak formulation of the problem can be constructed asserting that, given $\mathbf{u}^{\varepsilon_h}$ it represents a *weak* solution to (30) whenever

$$\int_{\mathscr{Q}}\mathbf{v}\cdot\mathbf{u}^{\varepsilon_h}(t)dtd\mathbf{x}=\int_{\mathscr{Q}}\mathbf{v}\cdot\left[\int_0^t \mathbb{K}^{\varepsilon_h}(t-\tau)\Delta\mathbf{u}^{\varepsilon_h}(\tau)d\tau+\mathbf{u}_1 t+\mathbf{u}_0+\int_0^t d\tau\int_0^\tau \mathbf{f}(\xi)d\xi\right]dtd\mathbf{x}, \tag{33}$$

for all test functions $\mathbf{v}$ in (32).
Consider, now, the following Lemma, crucial to prove the existence theorem.

Lemma 2 *Given* $\mathbb{K}(\varepsilon)$*, which, according to* (13)*, is*

$$\mathbb{K}^{\varepsilon_h}(\xi) = \int_0^{\xi} \mathbb{G}(\varepsilon_h + \tau) d\tau \quad , \quad \mathbb{K}^{\varepsilon_h}(0) = 0 \quad , \forall \varepsilon_h \; , \tag{34}$$

and any test function $\mathbf{v}$*, defined in* (32)*, then it follows that*

$$\lim_{\varepsilon_h \to 0} \int_0^T \int_\Omega \Delta \mathbf{v} \cdot \int_0^t \left[\mathbb{K}^{\varepsilon_h}(s) - \mathbb{K}(s) \right] \mathbf{u}^{\varepsilon_h}(t-s) ds d\mathbf{x} dt = 0 \; . \tag{35}$$

Proof Note that, $\forall (\mathbf{x}, t) \in \Omega \times (0, T)$

$$|\mathbf{u}| \leq C|\Omega|, \qquad |\Delta \mathbf{v}| \leq M, \tag{36}$$

furthermore

$$|\mathbb{K}^{\varepsilon_h}(s) - \mathbb{K}(s)| = |\mathbb{K}(\varepsilon_h + s) - \mathbb{K}(s)| = \int_s^{\varepsilon_h + s} \mathbb{G}(\tau) d\tau \tag{37}$$

hence, since $\mathbb{G} \in L^1(0, T)$, Lebesgue's Theorem implies the limit convergence.

□

Now, the Theorem can be easily proved.

Proof of Theorem 1 To start with, consider the approximated problems in their formulation (18), the estimate (29) guarantees there is a subsequence $\{\varepsilon_h\}, h \in \mathbb{N}$ such that there exists a convergent subsequence of solutions $\{\mathbf{u}^{\varepsilon_h}\}$

$$\mathbf{u}^{\varepsilon_h} \longrightarrow \mathbf{u} \;\; \text{weakly} \;\; \text{in } \mathbf{H}^1(0, T, H_0^1(\Omega)) \;\; \text{as } \varepsilon_h \to 0; \tag{38}$$

$$\mathbf{u}^{\varepsilon_h} \longrightarrow \mathbf{u} \;\; \text{strongly} \;\; \text{in } \mathbf{L}^2(\Omega \times (0, T)) \;\; \text{as } \varepsilon_h \to 0; \tag{39}$$

hence

$$\exists \, \mathbf{u}(t) = \lim_{\varepsilon_h \to 0} \mathbf{u}^{\varepsilon_h}(t) \;\; \text{in} \;\; \mathbf{L}^2(\Omega \times (0, T)), \tag{40}$$

where $\mathbf{u}^{\varepsilon_h}$ is solution to the problem (16)-(17). This convergence result allows to prove Theorem 1. Consider the weak formulation of the integral problems $\mathrm{P}_I^{\varepsilon_h}$ expressed in (30). Scalar multiplication of (30) by $\mathbf{v}$ followed by integration over $\Omega \times (0, T)$, delivers (33). The following integral

$$\int_0^T \int_\Omega \mathbf{v} \cdot \left\{ \mathbf{u}_1 t + \mathbf{u}_0 + \int_0^t d\tau \int_0^\tau \mathbf{f}(\xi) d\xi \right\} d\mathbf{x} dt \tag{41}$$

collects all the terms which appear in (33), and do not dependent on ε_h. Such terms, conversely, depend on the history of the material as well as on the initial conditions, all of them assumed to be regular. Hence, since the integration domain $\mathscr{Q} = \Omega \times (0, T)$ is bounded, it follows that also the integral over $\Omega \times (0, T)$, in (41), is bounded. Furthermore, since such integral does not depend on ε_h, then it is unchanged in the limit $\varepsilon_h \to 0$. As a consequence, the only term which remains to consider, in the limit, is

$$\int_0^T \int_\Omega \mathbf{v} \cdot \left[\int_0^t \mathbb{K}^{\varepsilon_h}(t - \tau) \Delta u^{\varepsilon_h}(\tau) d\tau \right] d\mathbf{x} dt \; . \tag{42}$$

Let $s = t - \tau$, then

$$\int_0^t \mathbb{K}^{\varepsilon_h}(t - \tau) \Delta u^{\varepsilon_h}(\tau) d\tau = \int_0^t \mathbb{K}^{\varepsilon_h}(s) \Delta u^{\varepsilon_h}(t - s) ds \; . \tag{43}$$

On use of the homogeneous boundary conditions imposed on the test functions $\mathbf{v}$, on integration over Ω, two times, we obtain

$$\int_0^T\int_\Omega \left[\mathbf{v}\cdot\int_0^t \mathbb{K}^{\varepsilon_h}(t-\tau)\Delta u^{\varepsilon_h}(\tau)d\tau\right]d\mathbf{x}dt = \int_0^T\int_\Omega \left[\Delta\mathbf{v}\cdot\int_0^t \mathbb{K}^{\varepsilon_h}(t-\tau)u^{\varepsilon_h}(\tau)d\tau\right]d\mathbf{x}dt\ . \tag{44}$$

Then, recalling (43), the r.h.s. of the latter can be equivalently expressed as

$$\int_0^T\int_\Omega \Delta\mathbf{v}\cdot\int_0^t \mathbb{K}^{\varepsilon_h}(t-\tau)u^{\varepsilon_h}(\tau)d\tau d\mathbf{x}dt =$$

$$= \int_0^T\int_\Omega \Delta\mathbf{v}\cdot\int_0^t \left[\mathbb{K}^{\varepsilon_h}(s)-\mathbb{K}(s)\right]u^{\varepsilon_h}(t-s)dsd\mathbf{x}dt + \int_0^T\int_\Omega \Delta\mathbf{v}\cdot\int_0^t \mathbb{K}(s)u^{\varepsilon_h}(t-s)dsd\mathbf{x}dt\ , \tag{45}$$

where the term

$$\int_0^T\int_\Omega \Delta\mathbf{v}\cdot\int_0^t \mathbb{K}(s)u^{\varepsilon_h}(t-s)dsd\mathbf{x}dt$$

is added and subtracted. Then, the Theorem is proved via combination of (38) – (39) with Lemma 2.

□

5. Conclusions and perspectives

The result presented shows that solution existence holds also when the requirements on the regularity of the relaxation modulus $\mathbb{G}$ are weaker than the classical ones. Namely, $\mathbb{G}\in L^1$ but $\dot{\mathbb{G}}\notin L^1$. The achieved result generalizes the previous ones ([1]). In particular by Carillo *et al.* (2013) the existence and uniqueness of an initial boundary value problem in the case of a $1-$dimensional viscoelastic body with a singular kernel was treated. Notably, as a consequence of the mathematical analogy between the models which describe isothermal viscoelasticity, on one side, and rigid thermodynamics with memory, on the other one, singular kernel problems in the two different kind of materials can be compared (Carillo 2015). Thus, an existence result (Carillo *et al.* 2014) can be proved in the case of a singular kernel problem in rigid thermodynamics. Furthermore, new materials are also obtained inserting nano-particles which are magnetically active, in a viscoelastic gel; in this way, materials which can be termed magneto-viscoelastic are obtained. Also magneto-viscoelasticity problems are considered (Carillo *et al.* 2011, 2012; Carillo 2017; Carillo *et al.* 2017), where the coupling of the two different effects: viscoelastic response of the material and magnetization is considered. In particular, an existence result concerning a singular kernel problem in magneto-viscoelastic materials is obtained by Carillo *et al.* (2017). New perspective investigations are concerned about the possibility to model various different concomitant effects such as introduce also thermal effects in the case of the viscoelastic body, or, instead of magnetic effects, consider more general electro-magnetic ones, to mention only those effects which seems suitable not only to be modelled, but also to be analytically studied in a near future.

[1]Further generalizations, including also wedge continuous kernels, are considered in Carillo *et al.* (2018)

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* Università di Roma, “La Sapienza”
Dipartimento di Scienze di Base e Applicate per l’Ingegneria
Via A. Scarpa, 16, 00161 Rome, Italy

Email: sandra.carillo@uniroma1.it

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