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CANONICAL INTEGRALS OF ADMISSIBLE DIFFERENTIAL GEOMETRIC STRUCTURES ON SUBMANIFOLDS OF CODIMENSION TWO IN PSEUDOEUCLIDEAN SPACE $E_{n+1}^{2(n+1)}$

SAMVEL HAROUTUNIAN *

ABSTRACT. Some classes of n-tuple integrals depending on n parameters and differential geometric structures on $2n$ dimensional manifolds of integration's variables and parameters M are studying. These integrals (when no degenerate) induce the structure of the pseudoriemannian Rashevsky-Einstein space on M. Using the Cartan's method of exterior forms on manifolds the inverse problem of the discovery of above-mentioned integrals inducing the given admissible differential geometric structure on M is studying. Obtained results contain new kernels for integral transforms.

Introduction

Creation of the method of exterior forms by French academician Cartan (1933) in the beginning XX century and its further development formed an opportunity for Differential Geometry to enter in different fields of modern Mathematics, analyze some important concepts of these fields from a differential geometric viewpoint and solve some problems at the contact point of these fields and modern Differential Geometry. For the first time the concept of differential geometric research of mathematical objects was introduced by Cartan (1936). Studying the ordinary integral, he associated the corresponding subintegral form to the integral and studied it as geometric object. Such study was considering as differential geometric study of integral.

On the initial stage of the study of multiple integrals depending on parameters we consider the geometric description of main structures of the given theory and connections between them. The next step is the differential geometric analysis of these structures and identification of the corresponding most general characteristic (geometric) properties. Finally, on the last stage of research these properties or the part of them only become the foundation for generalizations and new problems in initial theory. New level research starts. This approach has some similarity with iterations and it gives definitive opportunity to keep always the studying objects in the center of differential geometric study penetrating in the more and more deep areas of investigation. Saying differential geometry of analytic objects, we have in mind above-mentioned research.

All this is true for the geometry of multiple integral depending on parameters. At the same time, the presence of parameters totally changes the circle of arising problems and corresponding results. Solution of the inverse problem of discovering canonical integrals of admissible differential geometric structures on the manifold of double fiber bundle of variables of integration and parameters M (Haroutunian 1998) and further analysis of obtained kernels comes to the necessity to study submanifolds of pseudoeuclidean space $E^{2(n+1)}_{n+1}$ and calculate canonical integrals on these submanifolds. Following the Whitney theorem (Whitney 1957) it is interesting to study submanifolds with enough big dimensions. The case of submanifolds of codimension two is studied completely. The present work is devoted to the calculation of canonical integrals of above-mentioned structures on some submanifolds of dimension $2n$ in pseudoeuclidean space $E^{2(n+1)}_{n+1}$. Moreover it is useful to bring together all obtained results and analyze them from the general viewpoint.

1. About geometry of multiple integral depending on parameters

As established by Haroutunian (1975) the study of a n-tuple integral depending on n parameters

$$I(y_1,\dots,y_n)=\int\dots\int K(x^1,\dots,x^n,y_1,\dots,y_n)\,dx^1\dots dx^n \tag{1.1}$$

is equivalent to the study of subintegral form $K(x^1,\dots,x^n,y_1,\dots,y_n)\,dx^1\dots dx^n$ or exterior form $K(x^1,\dots,x^n,y_1,\dots,y_n)\,dx^1\wedge\dots\wedge dx^n$ on $2n$ dimensional manifold M of integration's variables $x^1,\dots,x^n$ and parameters $y_1,\dots,y_n$. The manifold M has the structure of double fiber bundle: two smooth mappings $\pi_i : M\to M_i$ $(i=1,2)$, $\dim M=2n$, $\dim M_i=n$, where M_1, M_2 are smooth manifolds, are given and

$$T_p(\pi_1^{-1}(x))\cap T_p(\pi_2^{-1}(y))=p,\quad \pi_1(p)=x,\quad \pi_2(p)=y;\qquad x\in M_1,\ y\in M_2.$$

Therefore at each point of M the tangent space to M is decomposing into the direct sum of two n-dimensional subspaces. Also, note that generally the manifold M is a Cartesian product $M_1\times M_2$, we consider the double fiber bundle for the maximal generality. Moreover, we study those properties of this structure, which are invariant with respect to arbitrary smooth change of variables and parameters and integral's transformation of the type

$$I\to f(x^1,\dots,x^n)\,g(y_1,\dots,y_n)I.$$

Such transformations are typical for integral transforms theory and our differential geometric study must be coherent with philosophy of this theory. Subintegral form and special structure of the manifold M (double fiber bundle) induce a special type pseudoriemannian metrics of the signature (n,n) on M. It determines all invariant (with respect to above mentioned changes and transformations) properties of the integral I. For the first time this metrics was studied by Shirokov (1925) (elliptic case) and Rashevsky (1949) (hyperbolic case). Here we have a deal with hyperbolic case that is with Rashevsky metrics (Rashevsky space). Rashevsky studied a scalar field $U(x^1,\dots,x^n,y_1,\dots,y_n)$ (in our notations) in $2n$ dimensional fiber bundle space and introduced a special metrics tensor with nonzero components $g_i^j=\frac{\partial^2U}{\partial x^i\partial y_j}$. He established that this space was a pseudoriemannian space with metrics of signature (n,n) and studied some properties of this metrics and corresponding pseudoriemannian connection. Different geometers studied it as a pseudoriemannian space.

Let us note that the parabolic case was studied by Vishnevskii (1968) who constructed the corresponding classification of so-called *A*-spaces.

As indicated above the first remarkable result of the geometry of *n*-tuple integrals depending on n parameters is the following statement (Haroutunian 1975).

Theorem 1.1. *A n-tuple integral of the form* $K(x^1,\dots,x^n,y_1,\dots,y_n)\,dx^1\wedge\dots\wedge dx^n$ *depending on n parameters and satisfying condition* $\det\left(\frac{\partial^2 \ln K}{\partial x^i \partial y_j}\right)\neq 0$ *induces the differential geometric structure of* $2n$ *dimensional pseudoriemannian Rashevsky space on the manifold of integration's variables and parameters M.*

In the case of *n*-tuple integral depending on n parameters (1.1) the nonzero components of this metrics tensor are defined by formula $g_i^j = \frac{\partial^2 \ln K}{\partial x^i \partial y_j}$.

The differential geometric description of the integral's reversibility was established by Haroutunian (1975): if the subintegral forms $\Omega = K(x^1,\dots,x^n,y_1,\dots,y_n)\,dx^1\wedge\dots\wedge dx^n$ and $\Theta = L(x^1,\dots,x^n,y_1,\dots,y_n)\,dy_1\wedge\dots\wedge dy_n$ of bundles (M,π_1,M_1) and (M,π_2,M_2) respectively induce the same Rashevsky pseudoriemannian geometry with volume form $V=\Omega\wedge\Theta$ in the total space of double fiber bundle *M*, then Rashevsky space is a Einstein space (Ricci tensor is proportional to the metric tensor with constant coefficient).

It is natural here to search *n*-tuple integrals depending on *n* parameters inducing the given admissible differential geometric structure on corresponding manifold *M*, i.e. the double fiber bundle with pseudoriemannian Rashevsky-Einstein geometry. The simplest result in this direction is as follows (Haroutunian 1990a): any *n*-tuple integral depending on *n* parameters inducing the structure of $2n$ dimensional pseudoeuclidean Rashevsky space on the manifold of double fiber bundle of integration's variables and parameters *M* may be represented as integral of the form

$$\Omega = P(x^1,\dots,x^n)\,Q(y_1,\dots,y_n)\,e^{cx^i y_i} dx^1\wedge dx^2\dots\wedge dx^n, \qquad c = const. \tag{1.2}$$

If $c=i$ we obtain the classic Fourier integral kernel. It gives the interior geometric meaning of Fourier - Laplace integrals. Integrals inducing Rashevsky-Einstein metrics are natural generalizations of these classic integrals. Besides it means that the classic Fourier kernel is the most simple representative of enough rich family of integral kernels and the position of the Fourier kernel in this family is analogous to the position of $2n$ dimensional pseudoeuclidean Einstein space with the metrics of signature (n,n) in the family of $2n$ dimensional pseudoriemannian Rashevsky-Einstein spaces. It's natural to try to pick up *n*-tuple integrals depending on *n* parameters inducing the given structure of pseudoriemannian Rashevsky-Einstein space on the manifold of integration's variables and parameters *M*. On the other hand, the pseudoriemannian structure on *M* may be considered as a special affine connection on *M* and it interesting to discover which structure is the essential here. The study of $(n+s)$-tuple integrals depending on *n* parameters and realization of above mentioned program was effective in solving this problem.

Let us introduce a $(n+s)$-tuple integral

$$I(y_1,\dots,y_n) = \int\dots\int K(x^1,\dots,x^{n+s},y_1,\dots,y_n)\,dx^1\dots dx^{n+s} \tag{1.3}$$

depending on *n* parameters, fix a chart with local coordinates $x^1,\dots,x^{n+s},y_1,\dots,y_n$, the natural frame $\{P,(e_\alpha)_0,(e^i)_0,\alpha=1,2,\dots,n+s;\ i=1,2,\dots,n\}$ adapted to the structure of double fiber bundle of $(2n+s)$ dimensional manifold of integration's variables and parameters M and vectors

$$e_\alpha = \tilde{x}_\alpha^\beta (e_\beta)_0, \quad e^i = \tilde{y}_k^i (e^k)_0,$$
$$\operatorname{rank}(\tilde{x}_\alpha^\beta) = n+s, \quad \operatorname{rank}(\tilde{y}_k^i) = n,$$

where new variables $\tilde{x}_\alpha^\beta$ and $\tilde{y}_k^i$ compose non degenerate matrices, don't depend on previously introduced coordinates and each other. The naturality of the frame means that the differentials of local coordinates in the given chart are coordinates of vectors tangent to this manifold with respect to this frame. Fixing this natural frame generates a one-to-one correspondence between the Cartesian product $GL(n+s,R) \times GL(n,R)$ and the family of the first order tangent frames on M: the frame with basic vectors (e_α), (e^i) corresponds to each ordered pair of matrices $((\tilde{x}_\alpha^\beta),(\tilde{y}_k^i))$. As a result e_α, e^i may be considered as a smooth functions on the Cartesian product $GL(n+s,R) \times GL(n,R)$.

It is natural now to introduce the cobasis of linear differential forms

$$\omega^\alpha = x_\beta^\alpha \, dx^\beta, \quad \omega_i = y_i^k \, dy_k; \qquad i,k = 1,2,\ldots,n; \;\; \alpha,\beta = 1,2,\ldots,n+s,$$

where $\tilde{x}_\gamma^\alpha x_\beta^\gamma = \delta_\beta^\alpha$, $\tilde{y}_p^i y_k^p = \delta_k^i$. Therefore there is a basis on each tangent space of fibers $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ transforming by elements of groups $GL(n,R)$ and $GL(n+s,R)$ respectively. There is a moving frame on M such that at each point of M n vectors from this frame compose the basis of the tangent vector space to the fiber of the bundle $\pi_1 : M \to M_1$ at this point, the rest $n+s$ vectors compose the basis in the tangent vector space of the fiber of bundle $\pi_2 : M \to M_2$ at the same point and the system of linear differential forms $\omega^1, \omega^2, \ldots, \omega^{n+s}, \omega_1, \omega_2, \ldots, \omega_n$ on M which do not depend on the choice of the chart. The fibers of bundles $\pi_1 : M \to M_1$ and $\pi_2 : M \to M_2$ are integral manifolds of maximal dimensions for the systems of linear differential equations $\omega^\alpha = 0$, $\alpha = 1,\ldots,n+s$ and $\omega = 0, i = 1,\ldots,n$ respectively (Haroutunian 1990a).

Theorem 1.2. *If* $(n+s)$*-tuple integral* (1.3) *depending on n parameters induces the bilinear form* $d\varphi = \omega^i \wedge \omega_i$ *of maximal rank on* $(2n+s)$ *dimensional manifold M of integration's variables and parameters in an invariant way and* $s \leq \frac{1}{2}n(n+1)$*, then there exists a special type affine connection* γ *on M invariantly associated to M, determined by linear differential forms* $\omega^i, \omega^\xi, \omega_i, \omega_k^i, \omega_\eta^\xi$ $(i,k = 1,\ldots,n;\ \xi,\eta = n+1,\ldots,n+s)$ *defined on the manifold of the second order tangent frames* $M^{(2)}$ *and satisfying the following structure equations*

$$\begin{aligned}
d\omega^i &= \omega_k^i \wedge \omega^k + a_\xi^{ik} \omega_k \wedge \omega^\xi, \\
d\omega^\xi &= \omega_\eta^\xi \wedge \omega^\eta + A_i^{\xi k} \omega_k \wedge \omega^i, \\
d\omega_i &= -\omega_i^k \wedge \omega_k, \\
d\omega_k^i &= \omega_p^i \wedge \omega_k^p + R_{kp}^{it} \omega^p \wedge \omega_t + a_\xi^{ip} A_k^{\xi t} \omega_p \wedge \omega_t, \\
d\omega_\eta^\xi &= \omega_\mu^\xi \wedge \omega_\eta^\mu - R_{\eta p}^{\xi k} \omega^p \wedge \omega_k - R_{\eta\mu}^{\xi k} \omega^\mu \wedge \omega_k + A_t^{\xi k} a_\eta^{it} \omega_k \wedge \omega_p
\end{aligned} \tag{1.4}$$

besides the relation $a_{ik}^\xi a_\eta^{ikp} = 0$ *holds. The tensor of curvature and torsion of this affine connection has a special structure.*

Coefficients of the system of structure equations (1.4) form a system of relative invariants on M: vanishing these coefficients has an invariant geometric meaning. If $s=0$ we come to the case of pseudoriemannian metrics on M.

Let us introduce a concept of the semibasic form.

Definition 1.1. *Exterior differential form of the degree $p \leq n$, given on double fiber bundle M $(\dim M = 2n)$, is said to be a semibasic form of the first (second) bundle if it is equal to zero for all totalities of p vectors such that at least one of them has a non trivial decomposition by basic vectors of the tangent linear space at each point of fibers of the bundle (M, π_1, M_1) $((M, \pi_2, M_2))$.*

The form $\Omega = \lambda\,\omega^1 \wedge \omega^2 \ldots \wedge \omega^n$ determines a smooth measure (volume) on each fiber of the bundle (M, π_2, M_2), $\dim M = 2n$ and satisfies the second part of the definition for the semibasic form of the bundle (M, π_1, M_1). Besides, it shall be a form on M.

Reversibility of the integral comes to the following result (Haroutunian 1990a).

Theorem 1.3. *Assume $(n+s)$-tuple integral (1.3) depending on n parameters induces some differential geometric structure on the manifold of double fiber bundle of integration's variables and parameters M in an invariant way. There exists a "volume form" on M of the form $\Omega \wedge \Theta$, where Ω is an exterior $(n+s)$-form associated with subintegral form and Θ is the semibasic n-form of the bundle (M, π_2, M_2), both inducing the same differential geometric structure on M, if and only if the form $\omega_1^1 + \omega_2^2 + \ldots + \omega_n^n$ is closed.*

Taking into account that the forms ω^α, $\alpha = 1, 2, \ldots, n+s$ are linearly independent linear combinations of differentials dx^α, $\alpha = 1, 2, \ldots, n+s$, these differentials are linearly independent combinations of forms $\omega^1, \ldots, \omega^{n+s}$. It means that the exterior form $K(x^1, \ldots, x^n, y_1, \ldots, y_n)\, dx^1 \wedge \ldots \wedge dx^n$ is equivalent to the form $\lambda\,\omega^1 \wedge \omega^2 \ldots \wedge \omega^n$. The same is true for differential forms constructed on parameters of integration.

Definition 1.2. *A $(n+s)$-tuple integral depending on n parameters is said to be a canonical integral of the given differential geometric structure on manifold M of integration's variables and parameters if it induces this structure on M and besides the total number of variables and parameters is equal to dimension of M.*

Outline of admissible differential geometric structures on M is presented by Haroutunian (2015).

Theorem 1.4. *If $(n+s)$-tuple integral depending on n parameters induces the differential geometric structure*

$$d\omega^i = a_\xi^{ik}\omega_k \wedge \omega^\xi, \quad d\omega^\xi = 0,$$
$$d\omega_i = 0, \quad da_\xi^{ik} = a_\xi^{ikp}\omega_p,$$

on the manifold of double fiber bundle of integration's variables and parameters M then it can be reduced to the integral of the form

$$\Omega = P(x)Q(y)\exp(x^i y_i + a_\xi^{ik} x^\xi y_k y_i)\, dx^1 \wedge \ldots \wedge dx^{n+s}, \tag{1.5}$$

where $P(x^1, \ldots, x^{n+s})$ and $Q(y_1, \ldots, y_n)$ are exponents of some smooth functions.

As we see the main structure on the manifold of integration's variables and parameters induced by integral is an affine connection. Let us note that the canonical $(n+s)$-tuple integral depending on $n+s$ parameters inducing the structure of $2(n+s)$ dimensional pseudoeuclidean Rashevsky space is reducing to the form

$$\Omega = P(x)Q(y)\exp(x^i y_i + x^\xi y_\xi)\,dx^1 \wedge \ldots \wedge dx^{n+s}, \tag{1.6}$$

where $i = 1,\ldots,n$; $\xi = n+1,\ldots,n+s$. Comparing formulas (1.6) and (1.5) we see that the term $a^{ik}_\xi x^\xi y_k y_i$ in exponential presented in (1.5) is a realization of the addend $x^\xi y_\xi$. It means the necessity to study submanifolds of pseudoeuclidean Rashevsky space and calculate the canonical integrals of admissible differential geometric structures on them. In general case this problem is complicate enough, so it is more reasonable to start from submanifolds of codimension two, i.e. $2n$ dimensional submanifolds in pseudoeuclidean space $E^{2(n+1)}_{n+1}$.

2. About geometry of $2n$ dimensional submanifolds in $E^{2(n+1)}_{n+1}$

The structure equations of pseudoeuclidean $2(n+1)$ dimensional space with metrics of half index may be represented in the form (Haroutunian 1975)

$$\begin{aligned} d\omega^I &= \omega^I_K \wedge \omega^K, \\ d\omega_I &= -\omega^K_I \wedge \omega_K \qquad I,K,P = 1,2,\ldots,n+1, \\ d\omega^I_K &= \omega^I_P \wedge \omega^P_K. \end{aligned} \tag{2.1}$$

Generally speaking the essential classes of $2n$ dimensional submanifolds in $E^{2(n+1)}_{n+1}$ are determined by following relations

$$\begin{aligned} \omega^{n+1} &= a^i \omega_i + a_i \omega^i, \\ \omega_{n+1} &= b^i \omega_i + b_i \omega^i, \\ i &= 1,\ldots,n. \end{aligned} \tag{2.2}$$

The metric form $d\varphi = \omega^I \wedge \omega_I$ of the pseudoeuclidean Rashevsky space $E^{2(n+1)}_{n+1}$ induces the bilinear form

$$d\varphi^* = (\delta^i_k - a^i b_k)\omega^k \wedge \omega_i, \tag{2.3}$$

on the submanifold M and the condition of the rank's maximality of this form is equivalent to inequality

$$\det(\delta^i_k - a^i b_k) \neq 0.$$

The following statement holds.

Theorem 2.1. *The metrical connection of the pseudoeuclidean space $E^{2(n+1)}_{n+1}$ in imbedding $M \to E^{n+1}_{2(n+1)}$ defined by relations* (2.2)*, under condition of the maximality of the rank of bilinear form* (2.3)*, induces an affine connection with torsion defined by forms ω^i, ω_i, ω^i_k,*

$i,k=1,\dots,n$ *satisfying structure equations*

$$\begin{aligned} d\omega^i &= \omega^i_k \wedge \omega^k + a^k b^i_p \omega^p \wedge \omega_k, \\ d\omega_i &= -\omega^k_i \wedge \omega_k - b_k a^p_i \omega_p \wedge \omega^k, \\ d\omega^i_k &= \omega^i_p \wedge \omega^p_k + (a^{ip}_{n+1} b^{n+1}_{kt} - b^i_t a^p_k)\omega^t \wedge \omega_p + b^i_p b^{n+1}_{kt} \omega^p \wedge \omega^t - a^{ip}_{n+1} a^t_k \omega_p \wedge \omega_t, \end{aligned} \tag{2.4}$$

on $2n$ *dimensional submanifold of the double fiber bundle* $M \subset E^{2(n+1)}_{n+1}$, *where*

$$\begin{aligned} da^i &= a^k \omega^i_k + a^i \omega^{n+1}_{n+1} + a^i_k \omega^k + a^{ik} \omega_k, \\ db_i &= -b_k \omega^k_i - b_i \omega^{n+1}_{n+1} - b_{ik} \omega^k + b^k_i \omega_k, \\ \omega^i_{n+1} &= b^i_k \omega^k + a^{ik}_{n+1} \omega_k, \\ \omega^{n+1}_i &= -a^k_i \omega_k + b^{n+1}_{ik} \omega^k. \end{aligned}$$

The following statement is true (Haroutunian 1991).

Theorem 2.2. *The essential classes of imbedding* $M \to E^{n+1}_{2(n+1)}$, $\dim = 2n$ *are reducing to following cases:*

$$\omega^{n+1} = \omega_n, \quad \omega_{n+1} = -\omega^n; \tag{2.5}$$

$$\omega^{n+1} = \omega_n, \quad \omega_{n+1} = \omega^1; \tag{2.6}$$

$$\omega^{n+1} = \omega_n, \quad \omega_{n+1} = 0; \tag{2.7}$$

$$\omega^{n+1} = 0, \quad \omega_{n+1} = 0. \tag{2.8}$$

Let us note that in cases (2.5) and (2.6) the rank of a matrix composed by coefficients of decompositions (2.2) is maximal.

3. Canonical integrals

As is well known the first problems of Calculus were in close relation with geometric problems (tangent of a curve, area of a curvilinear trapezium etc) and the first researches in this field always had correct geometric interpretations. Later, especially in XVIII century, thanks a very active development of this new field of Mathematics, the geometric aspect was forgotten little by little. Creation of such field of Mathematics like the theory of integral transforms in XIX century was not supported by geometric interpretations. It is not strange because the technical opportunities of Differential Geometry were strongly limited at that time. Introduction of the method of exterior forms by Elie Cartan and especially further development of this method change significantly the technical opportunities of modern Differential Geometry. This section is devoted to calculation of some integrals depending on parameters, corresponding to imbeddings (2.5)–(2.8). It means that we are going to solve the inverse problem of finding canonical integrals of differential geometric structures induced on the double fiber manifold of integration's variables and parameters M, determined by relations (2.5)–(2.8), bring together obtained kernels of integrals and analyze them.

As noted by Haroutunian (1975), the calculation of canonical integral of the differential form $K(x^1,\dots,x^n,y_1,\dots,y_n)\,dx^1\dots dx^n$ or (what is the same) $\lambda\,\omega^1 \wedge \omega^2 \dots \wedge \omega^n$ comes to the solution of the following system of differential equations

$$d\ln\lambda = \lambda_u\omega^u + \lambda_\xi\omega^\xi + \lambda^u\omega_u + \lambda^\xi\omega_\xi, \tag{3.1}$$
$$d(\lambda^u\omega_u + \lambda^\xi\omega_\xi) = \omega^u\wedge\omega_u + \omega^\xi\wedge\omega_\xi. \tag{3.2}$$

We will study imbeddings (2.5)–(2.8) separately. These relations are identities on M. Differentiating both sides of identities (2.5) by exterior way, using structure equations of the pseudoeuclidean space $E^{2(n+1)}_{n+1}$ and then Cartan's lemma (Cartan 1933) we conclude that the secondary forms ω_i^{n+1}, $\omega_{n+1}^{n+1}+\omega_n^n$, ω_n^i, ω_{n+1}^i, ω_i^n have decompositions by principal forms:

$$\begin{aligned}
\omega_i^{n+1} &= a_{ik}^{n+1}\omega^k + a_{ni}^k\omega_k + a_{in}^n\omega_n,\\
\omega_{n+1}^{n+1}+\omega_n^n &= a_{in}^n\omega^i + a_n^{in}\omega_i,\\
\omega_n^i &= a_{nk}^i\omega^k + a_n^{ik}\omega_k + a_n^{in}\omega_n,\\
\omega_{n+1}^i &= a_{n+1}^{ik}\omega_k - a_k^{ni}\omega^k - a_n^{in}\omega^n,\\
\omega_i^n &= a_i^{nk}\omega_k + a_{ik}^n\omega^k + a_{in}^n\omega^n
\end{aligned} \tag{3.3}$$

with conditions of symmetry in coefficients, corresponding Cartan's lemma. Substitution of these decompositions in the corresponding structure equations comes to the general structure equations of the submanifold M. On the other hand, differentiation of these relations (being identities on M), further substitution of obtained structure equations of M and application of Cartan's lemma comes to the differential equations for coefficients of these decompositions, moreover the following finite relations hold

$$\begin{gathered}
a_{in}^n a_n^{in} = 0,\ a_n^{ik}a_{kn}^n = 0,\ a_k^{ni}a_n^{kn} = 0,\ 2a_{in}^n a_n^{ik} = a_{ni}^k a_n^{in},\ a_{nk}^i a_n^{kn} = 0,\ a_{ik}^n a_n^{in} = 0,\\
2a_{ip}^n a_{nk}^i + a_{ip}^{n+1}a_k^{ni} + a_p^{ni}a_{ik}^{n+1} = 2a_{ik}^n a_{np}^i + a_{ik}^{n+1}a_p^{ni} + a_k^{ni}a_{ip}^{n+1}.
\end{gathered} \tag{3.4}$$

The bilinear form $d\varphi^* = \omega^i\wedge\omega_i + 2\omega^n\wedge\omega_n$, $i=1,\dots,n-1$ being the restriction of the form $d\varphi$ on M is closed and nondegenerate on M.

These relations give an opportunity to bring the study of the geometry of submanifold M to consideration of particular cases.

a) Analyzing conditions (3.4) let us consider the case when

$$a_i^{nk} = 0, \quad a_{nk}^i = 0, \quad a_n^{in} = 0, \quad a_{in}^n = 0.$$

The system of relations (3.4) becomes simpler and the system of general structure equations of the submanifold M comes to the following form

$$
\begin{aligned}
d\omega^i &= \omega^i_k \wedge \omega^k + a^{ik}_n \omega_k \wedge \omega^n + a^{ik}_{n+1} \omega_k \wedge \omega_n,\\
d\omega_i &= -\omega^k_i \wedge \omega_k - a^n_{ik} \omega^k \wedge \omega_n + a^{n+1}_{ik} \omega^k \wedge \omega^n,\\
d\omega^n &= -\omega^n_n \wedge \omega^n + \omega^n_{n+1} \wedge \omega_n,\\
d\omega_n &= -\omega^n_n \wedge \omega_n + \omega^{n+1}_n \wedge \omega^n,\\
d\omega^i_k &= \omega^i_p \wedge \omega^p_k - (a^{it}_n a^n_{kp} + a^{it}_{n+1} a^{n+1}_{kp}) \omega^p \wedge \omega_t,\\
d\omega^n_n &= \omega^n_{n+1} \wedge \omega^{n+1}_n + a^n_{ip} a^{ik}_n \omega^p \wedge \omega_k,\\
d\omega^n_{n+1} &= 2\omega^n_n \wedge \omega^n_{n+1} + a^n_{ik} a^{ip}_{n+1} \omega^k \wedge \omega_p,\\
d\omega^{n+1}_n &= -2\omega^n_n \wedge \omega^{n+1}_n + a^{n+1}_{ik} a^{ip}_n \omega^k \wedge \omega_p,
\end{aligned}
\tag{3.5}
$$

where

$$a^{n+1}_{ik} a^{kp}_n = 0, \quad a^{ik}_{n+1} a^n_{kp} = 0, \quad a^{ik}_n a^n_{kp} = 0, \quad a^{n+1}_{ik} a^{kp}_{n+1} = 0.$$

Let us consider the case when

$$\operatorname{rank}(a^{ik}_n) = \operatorname{rank}(a^{ik}_{n+1}) = A, \quad \operatorname{rank}(a^n_{ik}) = \operatorname{rank}(a^{n+1}_{ik}) = B,$$
$$A + B = n - 1.$$

Then without any loss of generality we can assume

$$(a^n_{ik}) = \begin{pmatrix} a^n_{uv} & 0 \\ 0 & 0 \end{pmatrix}, \ \left(a^{ik}_n\right) = \begin{pmatrix} 0 & 0 \\ 0 & a^{\xi\eta}_n \end{pmatrix}, \ (a^{n+1}_{ik}) = \begin{pmatrix} a^{n+1}_{uv} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\left(a^{ik}_{n+1}\right) = \begin{pmatrix} 0 & 0 \\ 0 & a^{\xi\eta}_{n+1} \end{pmatrix},$$

$$u, v = 1, \dots, A < n - 1; \qquad \xi, \eta = A + 1, \dots, n - 1.$$

Breaking down indices ($u, v = 1, \dots, A$; $\xi, \eta = A + 1, \dots, n - 1$) by the corresponding way it is not difficult to check that the systems of Pfaff's equations $\omega^u_v = 0$, $\omega^u_\xi = 0$, $\omega^\xi_\eta = 0$; $u, v = 1, \dots, A$; $\xi, \eta = A + 1, \dots, n - 1$ are totally integrable. Therefore we can narrow the study by consideration of the subbundle of moving frames on which the system of structure equations of M may be presented in the form

$$
\begin{aligned}
d\omega^u &= 0, & d\omega^\xi &= \omega^\xi_u \wedge \omega^u + a^{\xi\eta}_n \omega_\eta \wedge \omega^n + a^{\xi\eta}_{n+1} \omega_\eta \wedge \omega_n,\\
d\omega^n &= \omega^n_n \wedge \omega^n + \omega^n_{n+1} \wedge \omega_n, & d\omega_u &= -\omega^\xi_u \wedge \omega_\xi - a^n_{uv} \omega^v \wedge \omega_n + a^{n+1}_{uv} \omega^v \wedge \omega^n,\\
d\omega_\xi &= 0, & d\omega_n &= -\omega^n_n \wedge \omega_n + \omega^{n+1}_n \wedge \omega^n,\\
d\omega^\xi_u &= (a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1} a^{n+1}_{uv}) \omega_\eta \wedge \omega^v, & d\omega^n_n &= \omega^n_{n+1} \wedge \omega^{n+1}_n,\\
d\omega^n_{n+1} &= 2\omega^n_n \wedge \omega^n_{n+1}, & d\omega^{n+1}_n &= -2\omega^n_n \wedge \omega^{n+1}_n
\end{aligned}
\tag{3.6}
$$

and coefficients of this system satisfy the following differential equations:

$$\begin{aligned} da^n_{uv} &= a^n_{uv}\omega^n_n + a^{n+1}_{uv}\omega^n_{n+1} + a^n_{uvw}\omega^w, \\ da^{\xi\eta}_n &= -a^{\xi\eta}_n\omega^n_n - a^{\xi\eta}_{n+1}\omega^{n+1}_n + a^{\xi\eta v}_n\omega_v, \\ da^{n+1}_{uv} &= -a^{n+1}_{uv}\omega^n_n + a^n_{uv}\omega^{n+1}_n + a^{n+1}_{uvw}\omega^w, \\ da^{\xi\eta}_{n+1} &= a^{\xi\eta}_{n+1}\omega^n_n - a^{\xi\eta}_n\omega^n_{n+1} + a^{\xi\eta v}_{n+1}\omega_v. \end{aligned} \tag{3.7}$$

In particular we obtain from here the equation for components of the curvature tensor:

$$d\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right) = \left(a^{\xi\eta v}_n a^n_{uv} + a^{\xi\eta v}_{n+1}a^{n+1}_{uv}\right)\omega_v + \left(a^{\xi\eta}_n a^n_{uvw} + a^{\xi\eta}_{n+1}a^{n+1}_{uvw}\right)\omega^w.$$

In contrast to the previous case we do not put here the condition of the total integrability of the systems of forms $\omega^1,\dots,\omega^n$; $\omega_1,\dots,\omega_n$. In the system of forms and functions satisfying these equations the subsystem of forms ω^n_n, ω^n_{n+1}, ω^{n+1}_n is closed: it defines the structure of the Lie Group $SL_2(R)$ (Lang 1975).

It is easy to check that the system of forms ω^u, ω^ξ, ω^n, ω_u, ω_ξ, ω_n, ω^ξ_u, ω^n_n, ω^n_{n+1}, ω^{n+1}_n and functions $a^{\xi\eta}_n$, a^n_{uv}, $a^{\xi\eta}_{n+1}$, a^{n+1}_{uv}, satisfying equations (3.6) – (3.7) is closed and in virtue of Laptev's theorem (Laptev 1953) we can conclude that the metrical connection of the pseudoeuclidean space $E^{2(n+1)}_{n+1}$ induces a special affine connection defined by these forms satisfying the structure equations (3.6). This affine connection is metrical: the bilinear form $d\varphi^*$ is closed, no degenerate on M and determines some metrics on M. On the other hand the submanifold M is not Rashevsky space (the torsion tensor has nonzero components). Analyzing the structure equations (3.6) it is easy to see that the system of Pfaff's equations

$$\{\omega^n = 0,\ \omega_n = 0\}$$

is totally integrable and determines a $2(n-1)$ dimensional submanifold $M_1 \subset M$ with structure equations

$$\begin{aligned} &d\omega^u = 0, \quad d\omega^\xi = \omega^\xi_u \wedge \omega^u, \\ &d\omega_u = -\omega^\xi_u \wedge \omega_\xi, \quad d\omega_\xi = 0, \\ &d\omega^\xi_u = \left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)\omega_\eta \wedge \omega^v, \end{aligned} \tag{3.6'}$$

It is easy to see that the submanifold M_1 is a pseudoriemannian Rashevsky space and it is determined by imbedding $\{\omega^{n+1} = 0, \omega^n = 0, \omega_{n+1} = 0, \omega_n = 0\}$.

Let us calculate the canonical integral of differential geometric structure $(3.6')$. Using structure equations $(3.6')$ we can define principal and secondary forms in a following way

$$\begin{aligned} &\omega^u = dx^u, \quad \omega^\xi = dx^\xi - \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)x^v y_\eta dx^u, \\ &\omega_u = dy_u - \frac{1}{2}\left(a^{\xi\eta}_u a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)x^v y_\eta dy_\xi, \quad \omega_\xi = dy_\xi, \\ &\omega^\xi_u = \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)\left(y_\eta dx^v - x^v dy_\eta\right). \end{aligned}$$

Speaking more correctly such way defined principal and secondary forms ω^u, ω^ξ, ω_u, ω_ξ, ω^ξ_u; $u = 1,\dots,\alpha$; $\xi = \alpha+1,\dots,n-1$ satisfy structure equations $(3.6')$. Let us introduce now the following notations for coefficients of decompositions for differentials of coefficients λ^u, λ^ξ, λ_u, λ_ξ:

$$d\lambda^u = a^u_v\omega^v + a^u_\xi\omega^\xi + a^{uv}\omega_v + a^{u\xi}\omega_\xi, \quad d\lambda^\xi = a^\xi_u\omega^u + a^\xi_\eta\omega^\eta + a^{\xi u}\omega_u + a^{\xi\eta}\omega_\eta,$$
$$d\lambda_u = a_{uv}\omega^v + a_{u\xi}\omega^\xi + b^v_u\omega_v + b^\xi_u\omega_\xi, \quad d\lambda_\xi = a_{\xi u}\omega^u + a_{\xi\eta}\omega^\eta + b^u_\xi\omega_u + b^\eta_\xi\omega_\eta.$$

Substitution of these decompositions in the result of exterior differentiation of relations (3.2) comes to the following relations:

$$b^u_v = a^u_v = \delta^u_v,\ b^\xi_\eta = a^\xi_\eta = \delta^\xi_\eta,\ b^u_\xi = a^u_\xi = 0,$$
$$a^\xi_u = \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)\lambda^v y_\eta,$$
$$b^\xi_u = \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)x^v\lambda_\eta,$$
$$a^{u\xi} = a^{\xi u},\ a_{u\xi} = a_{\xi u},\ a^{\xi\eta} = a^{\eta\xi},\ a_{uv} = a_{vu}.$$

Inverse substitution of these conditions comes to the system of differential equations for coefficients of the system (3.2):

$$d\lambda^u = \omega^u + a^{uv}\omega_v + a^{u\xi}\omega_\xi,$$
$$d\lambda^\xi = \omega^\xi + \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)\lambda^v y_\eta\omega^u + a^{\xi u}\omega_u + a^{\xi\eta}\omega_\eta,$$
$$d\lambda_u = \omega_u + \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)x^v\lambda_\eta\omega_\xi + a_{uv}\omega^v + a_{u\xi}\omega^\xi,$$
$$d\lambda_\xi = \omega_\xi + a_{\xi u}\omega^u + a_{\xi\eta}\omega^\eta.$$

Solving this system, we obtain the following functions:

$$\lambda^u = x^u + a^u(y),\ \lambda^\xi = x^\xi + \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)a^v(y)y_\eta x^u + a^\xi(y),$$
$$\lambda_u = y_u + \frac{1}{2}\left(a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}\right)x^v a_\eta(x) + a_u(x),\ \lambda_\xi = y_\xi + a_\xi(x).$$

Substituting these expressions in the equation (3.1), we obtain after some simplest transformations

$$d\ln\lambda = d\left(x^u y_u + x^\xi y_\xi - \frac{1}{4}R^{\xi\eta}_{uv}x^u x^v y_\xi y_\eta\right) + d\varphi(x) + d\psi(y),$$

where

$$R^{\xi\eta}_{uv} = a^{\xi\eta}_n a^n_{uv} + a^{\xi\eta}_{n+1}a^{n+1}_{uv}$$

are nonzero components of the curvature tensor.

We come to the following conclusion. One can see that the system of differential forms ω^u, ω^ξ, ω_u, ω_ξ, ω^ξ_u; $u = 1,\ldots,\alpha$; $\xi = \alpha+1,\ldots,n-1$ and functions $a^{\xi\eta}_n$, a^n_{uv}, $a^{\xi\eta}_{n+1}$, a^{n+1}_{uv}, satisfying equations $(3.6')$ – (3.7) (under conditions $\omega^n = 0$, $\omega_n = 0$) is closed, therefore following Laptev's theorem (Laptev 1953) the following statement is true.

Theorem 3.1. *Differential geometric structure $(3.6')$ on $2(n-1)$ dimensional submanifold of the pseudoeuclidean Rashevsky space $M_1 \subset E^{2(n+1)}_{n+1}$ is induced by the integral of the form*

$$\Omega = P(x)Q(y)\exp\left(x^u y_u + x^\xi y_\xi - \frac{1}{4}R^{\xi\eta}_{uv}x^u x^v y_\xi y_\eta\right)dx^1\wedge\ldots\wedge dx^{n-1}, \qquad (3.8)$$

where

$$P(x) = P(x^1,\ldots,x^{n-1}) = \exp[\varphi(x^1,\ldots,x^{n-1})],\ Q(y) = Q(y_1,\ldots,y_{n-1}) = \exp[\psi(y_1,\ldots,y_{n-1})]$$

are some smooth positive functions on M_1, $(R^{\xi\eta}_{uv})$ *is a curvature tensor of submanifold* M.

b) Analyzing conditions (3.4) let us consider a special case when the curvature tensor is vanishing:

$$a^i_{np}a^{nt}_k - a^{ni}_p a^t_{nk} = 0.$$

This condition generates the relations $a^i_{nk} = 0$, $\omega^{n+1}_n = 0$. The structure equations of submanifold M come to the form

$$\begin{aligned} &d\omega^i = a^{ni}_k \omega_n \wedge \omega^k. && d\omega^n = \omega^n_{n+1} \wedge \omega_n + a^{ni}_k \omega_i \wedge \omega^k, \\ &d\omega_i = a^{nk}_i \omega_n \wedge \omega_k, && d\omega_n = 0, \\ &d\omega^n_{n+1} = a^{np}_i a^{ni}_k \omega^k \wedge \omega_p, && da^{ni}_k = -a^{np}_k a^{ni}_p \omega_n. \end{aligned} \tag{3.9}$$

The submanifold M has some critical points (Haroutunian 1997).

Theorem 3.2. $2n$ *dimensional submanifold* $M \subset E^{2(n+1)}_{n+1}$ *defined by structure equations* (3.9) (*under condition* $a^i_{nk} = 0$) may be defined by two parametric equations

$$X^{n+1} = Y_n, \quad Y_{n+1} = -X^n - \sum_{i=1}^{n-1} \frac{X^i Y_i}{Y_n - c_i}.$$

It is easy to see from here that the submanifold M is located in the hyperplane of the pseudoeuclidean Rashevsky space $E^{2(n+1)}_{n+1}$.

Let us find now the canonical integral of differential geometric structure determined by equations (3.9). Since the submanifold $M \subset E^{2(n+1)}_{n+1}$ has critical points, the calculation of the integral inducing this differential geometric structure on the manifold M of variables and parameters is too much complicate. Let us calculate it on the submanifold $N \subset M$, $\dim N = 2n-1$ defined by equations

$$\omega^{n+1} = 0, \quad \omega_n = 0, \quad \omega_{n+1} = 0, \tag{3.10}$$

with structure equations

$$d\omega^i = 0, \quad d\omega^n = a^{ni}_k \omega_i \wedge \omega^k, \quad d\omega_i = 0, \quad d\omega^n_{n+1} = a^{np}_i a^{ni}_k \omega^k \wedge \omega_p. \tag{3.11}$$

Applying the standard procedure (see for example the proof of the theorem 3.1) we can obtain the following result (Haroutunian 1997).

Theorem 3.3. *The differential geometric structure* (3.11) *on* $(2n-1)$ *dimensional submanifold* $N \subset E^{2(n+1)}_{n+1}$ *defined by equations* (2.3) *is induced by the integral of the form*

$$\Omega = P(x)\,Q(y)\,\exp(x^i y_i)\,dx^1 \wedge \ldots \wedge dx^n$$

where $P(x) = P(x^1,\ldots,x^n)$ *and* $Q(y) = Q(y_1,\ldots,y_{n-1})$ *are some smooth positive functions on fibers of the bundle* N.

It is interesting to note that the integral of this form is not canonical integral of this differential geometric structure. Indeed the integral of this form is the canonical integral for $2(n-1)$ dimensional pseudoeuclidean Rashevsky space $E_{n-1}^{2(n-1)}$ but here $\dim N = 2n-1$.

Let us study now the imbedding $M \to E_{n+1}^{2(n+1)}$, defined by relations (2.6). These equations determine the most general class in the family of smooth submanifolds of codimension two in $E_{n+1}^{2(n+1)}$. In spite of this, the considered submanifolds have interesting geometric properties. The bilinear form $d\varphi$ comes to the more complicate form

$$d\varphi^* = \omega^1 \wedge \omega_1 + \omega^i \wedge \omega_i + \omega^n \wedge \omega_n + \omega_n \wedge \omega^1$$

($i = 2, \ldots, n-1$) on M and exterior differentiation shows that it is closed. Exterior differentiation of relations (2.6), further substitution of structure equations of $E_{2(n+1)}^{n+1}$ and application of Cartan's lemma shows that the secondary forms ω_n^{n+1}, $\omega_{n+1}^{n+1} + \omega_n^n$, $\omega_1^1 - \omega_n^n$, ω_i^{n+1}, ω_n^i, ω_n^1, ω_1^{n+1}, ω_{n+1}^n, ω_i^1, ω_{n+1}^i, ω_{n+1}^1 are principal forms, i.e. are linear combinations of principal basic forms $\omega^1, \ldots, \omega^n, \omega_1, \ldots, \omega_n$. The further simplification of these forms expressions can be reached using the transformation of following forms: ω_n^n, ω_1^1, ω_1^i, ω_1^n, ω_i^n. We obtain the following system of decompositions

$$\begin{aligned}
&\omega_n^{n+1} = b_{nn}^1 \omega_1 + b_{nn}^t \omega_t + b_n^{n+1\,n} \omega_n,\\
&\omega_{n+1}^{n+1} + \omega_n^n = b_t^{n+1\,n} \omega^t + b_n^{n+1\,n} \omega^1 + b_n^{n+1\,n} \omega^n + b_n^{tn} \omega_t + b_{n+1}^{n+1\,n} \omega_n,\\
&\omega_1^1 - \omega_n^n = \left(b_{tn}^1 - b_t^{n+1\,n}\right) \omega^t + \left(b_{nn}^1 - b_n^{n+1\,n}\right) \omega^n + b_1^{11} \omega_1 - b_n^{tn} \omega_t + b_1^{1n} \omega_n,\\
&\omega_i^{n+1} = b_{ni}^1 \omega_1 + b_{ni}^k \omega_k + b_i^{n+1\,n} \omega_n,\\
&\omega_n^i = b_{nk}^i \omega^k + b_{nn}^i \omega^1 + b_{nn}^i \omega^n + b_n^{i1} \omega_1 + b_n^{ik} \omega_k + b_n^{in} \omega_n,\\
&\omega_n^1 = b_{ni}^1 \omega^i + b_{nn}^1 \omega^1 + b_{nn}^1 \omega^n + b_n^{1n} \omega_1 + b_n^{k1} \omega_k + b_n^{1n} \omega_n,\\
&\omega_1^{n+1} = b_{nn}^{n+1} \omega^1,\\
&\omega_{n+1}^n = b_{n+1}^{nn} \omega_n,\\
&\omega_i^1 = b_{ik}^1 \omega^k + b_{i1}^1 \omega^1 + b_{ni}^1 \omega^n + b_i^{11} \omega_1 + b_i^{1k} \omega_k + b_i^{11} \omega_n,\\
&\omega_{n+1}^i = b_k^{1i} \omega^k + b_n^{i1} \omega^n,\\
&\omega_{n+1}^1 = b_k^{11} \omega^k + \left(b_1^{11} + b_n^{1n}\right) \omega^1 + b_n^{1n} \omega^n,
\end{aligned} \tag{3.12}$$

and besides the following relation holds

$$b_1^{1n} + b_{n+1}^{n+1\,n} - b_1^{11} - b_n^{1n} = 0.$$

Substitution of decompositions (3.12) in structure equations of pseudoeuclidean Rashevsky space comes to the complicate preliminary structure equations of submanifold M. Exterior differentiation of structure equations of M comes to identities, which generate the finite relations

$$\begin{aligned}
b_{nn}^1 b_{nn}^{n+1} = b_{ni}^1 b_{nn}^{n+1} = b_{ik}^1 b_{nn}^{n+1} = 0,\\
b_n^{1n} b_{n+1}^{nn} = b_n^{i1} b_{n+1}^{nn} = b_n^{ik} b_{n+1}^{nn} = 0.
\end{aligned} \tag{3.13}$$

These relations are foundations for classification of possible cases. Three essentially different cases are possible:

c) $b^{nn}_{n+1} \neq 0 \neq b^{n+1}_{nn}$;

d) $b^{nn}_{n+1} = 0 \neq b^{n+1}_{nn}$; d') $b^{nn}_{n+1} \neq 0 = b^{n+1}_{nn}$;

e) $b^{nn}_{n+1} = 0 = b^{n+1}_{nn}$.

In the case c) the values b^1_{nn}, b^1_{ni}, b^1_{ik}, b^{1n}_n, b^{i1}_n, b^{ik}_n are equal to zero and besides

$$b^1_{i1} b^{in}_n + b^{11}_1 b^{n+1\,n}_n = 0,$$

which follows from condition $b^{1n}_n = 0$. Exterior differentiation of structure equations for forms ω^n and ω_1 gives $b^i_{nn} = 0$ and $b^{11}_i = 0$. Exterior differentiation of the expressions of forms ω^1_n, ω^{n+1}_n, ω^{n+1}_i, ω^1_{n+1}, ω^i_{n+1} comes to the following algebraic relations

$$\begin{aligned} &b^i_{nk} b^1_{i1} = b^{1i}_k b^k_{np} = b^{1i}_k b^{kn}_n = 0, \\ &b^i_{nk} b^{n+1}_n = b^i_{nk} b^k_{np}, \quad b^{1i}_k b^{11}_1 = b^{1i}_p b^{1p}_k, \end{aligned} \tag{3.14}$$

that in its turn comes to new simplifications. For example, it is easy to prove that matrices (b^i_{nk}) and (b^{1i}_k) cannot be nondegenerate. Indeed suppose for example that $\det(b^i_{nk}) \neq 0$. In virtue of the last system of algebraic relations we come to the following conditions

$$b^1_{i1} = 0, \quad b^{1i}_k = 0, \quad b^i_{nk} = \delta^i_k b^{n+1\,n}_n.$$

It follows from here that $b^{n+1\,n}_n \neq 0$. Exterior differentiation of the structure equation of the basic linear differential form ω^n under condition $b^{1i}_k = 0$ comes to the identity

$$b^i_{nk} b^{nn}_{n+1} \omega_i \wedge \omega_n \wedge \omega^k - b^{11}_1 \omega^n_1 \wedge \omega_n \wedge \omega^1 = 0$$

and it is easy to see from here that $b^i_{nk} b^{nn}_{n+1} = 0$ and therefore $b^i_{nk} = 0$ because $b^{n+1\,n}_n \neq 0$. This is in contradiction with the assumption $\det(b^i_{nk}) \neq 0$. Therefore the matrix (b^i_{nk}) cannot be nondegenerate. The condition $b^{11}_1 = 0$ may be obtained from the previously obtained relation

$$b^1_{i1} b^{in}_n + b^{11}_1 b^{n+1\,n}_n = 0.$$

Similarly if we assume the matrix (b^{1i}_k) nondegenerate, then we obtain

$$b^{in}_n = 0, \quad b^{1i}_k = \delta^i_k b^{11}_1, \quad b^{11}_1 \neq 0.$$

Using some modifications, the structure equations of the submanifold M come to the form

$$\begin{aligned} &d\omega^1 = b^1_{t1} \omega^1 \wedge \omega^t + b^{11}_1 \omega^1 \wedge \omega_n, && d\omega^t = 0, \\ &d\omega^n = \omega^n_t \wedge \omega^t + a^{nn}_1 \omega_n \wedge \omega^1, && d\omega_1 = -b^{11}_1 \omega_n \wedge \omega_1, \\ &d\omega_t = -\omega^n_t \wedge \omega_n - b^1_{t1} \omega^1 \wedge \omega_1 - b^1_{t1} \omega_n \wedge \omega^1, && d\omega_n = 0, \\ &d\omega^n_t = b^1_{t1} a^{nn}_1 \omega_n \wedge \omega^1, \end{aligned} \tag{3.15}$$

where coefficients b^1_{t1}, b^{11}_1, a^{nn}_1 satisfy differential equations

$$\begin{aligned} db^1_{t1} &= -b^1_{r1} \omega^r_t + b^1_{t1} b^1_{r1} \omega^r, \\ db^{11}_1 &= b^{11}_1 \omega^n_n + \left(b^{11}_1\right)^2 \omega_n, \\ da^{nn}_1 &= -b^{11}_1 a^{nn}_1 \omega_1 - b^{nn}_{n+1} b^{n+1}_{11} \omega^1, \end{aligned} \tag{3.16}$$

The system of forms ω^1, ω^t, ω^n, ω_1, ω_t, ω_n, ω^n_t and functions b^1_{t1}, b^{11}_1, a^{nn}_1 satisfying equations (3.15)-(3.16) is closed and in virtue of Laptev's theorem (Laptev 1953) the following statement holds (Haroutunian 1989) (under condition $b^{nn}_{n+1} \neq 0 \neq b^{n+1}_{11}$).

Theorem 3.4. *The metrical connection of* $2(n+1)$ *dimensional pseudoeuclidean Rashevsky space generates under conditions c) the affine connection* γ *defined by forms* ω^1, ω^t, ω^n, ω_1, ω_t, ω_n, ω_t^n *and functions* b_{t1}^1, b_1^{11}, a_1^{nn} $(t,r=2,\dots,n-1)$ *satisfying structure equations* (3.15) *on* $2n$ *dimensional submanifold* M *defined by equations* (2.6).

Note that this affine connection has a nontrivial curvature tensor.

Let us calculate the canonical integral of this differential geometric structure in the simple case when $b_{t1}^1 = 0$. Then it is easy to conclude from the system of general structure equations of submanifold M that the system of linear differential equations $\omega_1^1 = 0$, $\omega_1^t = 0$, $\omega_t^n = 0$, $\omega_r^t = 0$; $t,r = 1,2,\dots,n-1$ is totally integrable. The system of structure equations comes to the form

$$\begin{aligned} &d\omega^1 = 0, \quad d\omega^t = 0, \quad d\omega^n = \omega_1^n \wedge \omega^1, \\ &d\omega_1 = -\omega_1^n \wedge \omega_n, \quad d\omega_t = 0, \quad d\omega_n = 0, \quad d\omega_1^n = b_{n+1}^{nn} b_{nn}^{n+1} \omega_n \wedge \omega^1, \end{aligned} \tag{3.17}$$

In this case $b_{n+1}^{nn} = b_{n+1}^{nn}(y_n)$, $b_{11}^{n+1} = b_{11}^{n+1}(x^1)$ and the following statement holds.

Theorem 3.5. *Differential geometric structure* (3.17) *on* $2n$ *dimensional submanifold* $M \subset E_{n+1}^{2(n+1)}$ *defined by equations* (2.6) *is induced by the integral of the form*

$$\Omega = P(x)\,Q(y)\,\exp(x^1 y_1 + x^t y_t + x^n y_n - x^1 y_n - b^{n+1} b_{n+1})\omega^1 \wedge \ldots \wedge \omega^n \tag{3.18}$$

where $P(x^1,\dots,x^n)$ *and* $Q(y_1,\dots,y_n)$ *are exponents of some smooth functions on fibers of fiber bundle* M *and* $b^{n+1} = b^{n+1}(x^1)$, $b_{n+1} = b_{n+1}(y_n)$ *are some smooth functions whose second derivatives coincide with* b_{11}^{n+1} *and* b_{n+1}^{nn} *respectively:*

$$b_{11}^{n+1} = \frac{\partial^2 b^{n+1}(x^1)}{\partial x^1 \partial x^1}, \quad b_{n+1}^{nn} = \frac{\partial^2 b_{n+1}(y_n)}{\partial y_n \partial y_n}.$$

In the case d) $b_{n+1}^{nn} = 0 \neq b_{nn}^{n+1}$ the calculation of canonical integral gives a complicate formulas. We will discuss here an interesting particular case of the system of decompositions (3.12) when $\omega_{n+1}^{n+1} = 0$, $\omega_1^1 = 0$, $b_n^{tr} = 0$. Then the system of structure equations corresponding to these decompositions comes to the form

$$\begin{aligned} &d\omega^1 = 0, \quad d\omega^t = b_{nn}^t \omega^1 \wedge \omega^n, \quad d\omega^n = 0, \\ &d\omega_1 = 0, \quad d\omega_t = 0, \qquad\qquad\quad d\omega_n = -b_{nn}^t \omega^n \wedge \omega_t, \end{aligned} \tag{3.19}$$

where coefficient b_{nn}^t is a constant. Using the standard procedure (see for example the proof of the Theorem 3.1 we come to the following statement (Haroutunian 1990b).

Theorem 3.6. *The differential geometric structure* (3.19) *is induced on* $2n$ *dimensional submanifold* $M \subset E_{n+1}^{2(n+1)}$ *by the integral of the form*

$$\Omega = P(x)\,Q(y)\,\exp\left[x^1 y_1 + x^t y_t + x^n y_n - x^1 y_n + b_{nn}^t x^1 x^n y_t - \frac{1}{2} b_{nn}^t (x^n)^2 y_t\right] dx^1 \wedge \ldots \wedge dx^n, \tag{3.20}$$

where $P(x) = P(x^1,\dots,x^n)$ *and* $Q(y) = Q(y_1,\dots,y_n)$ *are some positive smooth functions on fibers of the bundle* M.

The last part of this section is devoted to the study of the most interesting case e) $b^{nn}_{n+1} = 0 = b^{n+1}_{nn}$. The system of structure equations comes to the form

$$\begin{aligned} d\omega^1 &= -b^{n+1\,n}_n \omega^1 \wedge \omega_n, \\ d\omega^t &= b^{n+1\,n}_n \omega^t \wedge \omega^n + b^t_{nn} \omega^1 \wedge \omega^n, \\ d\omega^n &= 0, \\ d\omega_1 &= b^{n+1\,n}_n \omega^n \wedge \omega_1, \\ d\omega_t &= -b^{n+1\,n}_n \omega_t \wedge \omega^1, \\ d\omega_n &= -b^{n+1\,n}_n \omega^t \wedge \omega_t - b^t_{nn} \omega^n \wedge \omega_t - b^{tn}_n \omega_n \wedge \omega_t - b^{n+1\,n}_n \omega_n \wedge \omega^1, \end{aligned} \tag{3.21}$$

where functions b^s_{nn}, $b^{n+1\,n}_n$ satisfy differential equations

$$\begin{aligned} db^t_{nn} &= b^t_{nn} b^{n+1\,n}_r \omega^r + b^t_{nn} b^{n+1\,n}_n \omega^n - b^t_{nr} b^r_{nn} \omega^1, \\ db^{n+1\,n}_n &= \left(b^{n+1\,n}_n\right)^2 \omega^n. \end{aligned} \tag{3.22}$$

For the sake of simplicity we will study the case $b^{tr}_n = 0$. Then we can define the basic principal forms in the following way:

$$\begin{aligned} &\omega^1 = dx^1, \quad \omega^t = dx^t + b^t_{nr} x^r dx^n, \quad \omega^n = dx^n, \\ &\omega_1 = dy_1, \quad \omega_t = dy_t - b^r_{nt} y dx^1, \quad \omega_n = dy_n + b^t_{nr} y_t dx^r, \end{aligned}$$

where coefficients b^t_{nr} and b^t_{nn} are connected by relation $b^t_{nn} = -b^t_{ns} b^s_{nr} x^r$. Using the standard procedure used in the proof of the Theorem 3.1 we obtain the following statement (Haroutunian 1990b).

Theorem 3.7. *The differential geometric structure* (3.21)-(3.22) *is induced on* $2n$ *dimensional submanifold* $M \subset E^{2(n+1)}_{n+1}$ (under condition $b^{tr}_n = 0$) by the integral of the form

$$\Theta = P(x)\, Q(y)\, \exp\Big[-x^1 y_1 - x^t y_t - x^n y_n + x^1 y_n - b^t_{nr}(x^r x^n y_t + x^r y_t y_1 - x^r x^1 y_t + x^r y_t y_n) + \\ + \frac{1}{4}\, b^t_{ns}\, b^p_{nr}\, x^r x^s y_p\, y_t\Big]\, \omega_1 \wedge \ldots \wedge \omega_n. \tag{3.23}$$

The similar results are obtained for imbeddings (2.7), (2.8).

Let us start now the analysis of obtained results. Following the general result n-tuple integral depending on n parameters under some natural conditions induces the structure of pseudoriemannian Rashevsky space on the manifold of double fiber bundle of variables of integration and parameters M. This integral is reversible if and only if the Rashevsky space is the Einstein space. In the geometry of pseudoriemannian space the curvature tensor is playing the main role and the kernels of integrals for the forms (3.8), (3.18) are natural generalizations of the classic Fourier kernel. It is interesting to note that components of the curvature tensor are coefficients of the monomials of the fourth degree in canonical integral. On the other hand, the metrical connection of a pseudoriemannian space is a particular case of an affine connection on M. Following the result of the Theorem 1.2 the essential differential geometric structure associated to the $(n+s)$-tuple integral depending on n parameters by the invariant way for all values $s \leq \frac{1}{2}\, n(n+1)$ is the affine connection with structure equations (1.4). In general case the components of the torsion tensor appear in canonical integral's subintegral form as coefficients of the third degree monomials. This situation is seen in case (3.20). Besides the components of curvature and torsion tensors are present in the kernel of the form (3.23). The Theorem 3.3 shows that not only canonical integral of the given admissible differential geometric structure can induce this structure on manifold M. It means that the second requirement for canonical integral about equality of the total sum of variables and parameters quantity and dimension of manifold M is substantial. So we can see that canonical integral of the Rashevsky space, more exactly Einstein space with the metrics of half index, has the kernel

$$\exp[ix^k y_k + \alpha R^{ik}_{pq} x^p x^q y_i y_k], \qquad \alpha = const,$$

where R^{ik}_{pq} are non zero components of the curvature tensor of this space. Taking into account that the geometry of the Rashevsky space is in essence the geometry of its curvature tensor and canonical kernel contains all components of this tensor, then the theory of corresponding integral transforms in its essential part is in the field of the pseudoriemannian geometry. This fact confirms once more the idea that purely geometric problems are in the foundation of Calculus. On the other hand, these results show extraordinary potential opportunities of modern Differential Geometry.

Let us note that in some cases the kernel of canonical integral contains also the components of the torsion tensor of the space of affine connection induced on submanifold by given imbedding (see for example Theorem 3.7). Pseudoriemannian Rashevsky space has a vanishing torsion tensor, therefore such kernels are generalizations of (3.8). It is important to study this problem in the case of submanifolds of arbitrary dimensions and obtain more information about presence of components of the torsion tensor in kernel of canonical integral.

It remains to note that these kernels are new and are not studied by specialists of integral transforms, not yet.

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* Armenian State Pedagogical University
5, Khandjan str., 375010, Yerevan, Republic of Armenia

Email: sharoutunian2017@gmail.com

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