PROPERTIES OF $\gamma\mathcal{H}$-COMPACT SPACES
WITH HEREDITARY CLASSES

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ABSTRACT. By using $\gamma$-operations on an $m$-structure and a hereditary class $\mathcal{H}$, we define the notion of $\gamma$-compactness modulo hereditary classes (or ideals) called $\gamma\mathcal{H}$-compact. We obtain several properties of $\gamma\mathcal{H}$-compact spaces and $\gamma\mathcal{H}$-compact sets relative to $m$-structures.

1. Introduction

Let $(X, \tau)$ be a topological space and $\mathcal{P}(X)$ the power set of $X$. Ogata (1991) introduced the notions of $\gamma$-operations and $\gamma$-open sets and investigated the associated topology $\tau_\gamma$ and weak separation axioms $\gamma-T_i$ ($i = 0, 1/2, 1, 2$). More recently Noiri (2011) defined an operation on an $m$-structure with property $B$. The operation is defined as a function $m\gamma : m \to \mathcal{P}(X)$ such that $U \subseteq m\gamma(U)$ for each $U \in m$ and is called an operation $m\gamma$ on $m$. Then it turns out that the operation is an unified form of several operations (for example, semi-$\gamma$-operation: Sai Sundara Krishnan et al. 2007; pre-$\gamma$-operation: An et al. 2008) defined on the family of generalized open sets. Moreover, Noiri obtained some characterizations of $m\gamma$-compactness.

In this paper, by using hereditary classes (Császár 2007) and ideals (Janković and Hamlett 1990), we define the notion of $\gamma$-compactness modulo hereditary classes (or ideals) called $\gamma\mathcal{H}$-compact. In Section 3 we obtain several properties of $\gamma\mathcal{H}$-compact spaces and $\gamma\mathcal{H}$-compact sets. In Section 4 we deal with functions between $m$-spaces with operations and hereditary classes and obtain several properties of such functions and some preservation theorems of $\gamma\mathcal{H}$-compact sets. Recent papers have introduced some new classes of sets via hereditary classes (Al-Omari and Noiri 2016, 2019).

This paper is dedicated to Professor Filippo Cammaroto (University of Messina) on the occasion of his retirement.
2. Preliminaries

Definition 2.1. Let $X$ be a nonempty set and $\mathcal{P}(X)$ the power set of $X$. A subfamily $m$ of $\mathcal{P}(X)$ is called a minimal structure (briefly $m$-structure) on $X$ if $m$ satisfies the following conditions:

1. $\emptyset \in m$ and $X \in m$,
2. The union of any family of subsets belonging to $m$ belongs to $m$.

A set $X$ with an $m$-structure is called an $m$-space and is denoted by $(X, m)$. Each member of $m$ is said to be $m$-open and the complement of an $m$-open set is said to be $m$-closed.

Definition 2.2. Let $(X, m)$ be an $m$-space. For a subset $A$ of $X$, the $m$-closure of $A$ is defined by Maki et al. (1999) as follows:

$$ m\text{Cl}(A) = \bigcap \{F : A \subset F, X \setminus F \in m\}. $$

Lemma 2.3. (Maki et al. 1999). Let $(X, m)$ be an $m$-space. For the $m$-closure, the following properties hold, where $A$ and $B$ are subsets of $X$:

1. $A \subset m\text{Cl}(A)$,
2. $m\text{Cl}(\emptyset) = \emptyset$, $m\text{Cl}(X) = X$,
3. If $A \subset B$, then $m\text{Cl}(A) \subset m\text{Cl}(B)$,
4. $m\text{Cl}(m\text{Cl}(A)) = m\text{Cl}(A)$.

Lemma 2.4. (Popa and Noiri 2000). Let $(X, m)$ be an $m$-space and $A$ a subset of $X$. Then $x \in m\text{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in m$ containing $x$.

Lemma 2.5. (Popa and Noiri 2002). Let $(X, m)$ be an $m$-space and $A$ a subset of $X$. Then, the following properties hold:

1. $A$ is $m$-closed if and only if $m\text{Cl}(A) = A$,
2. $m\text{Cl}(A)$ is $m$-closed.

Remark 2.6. Lemmas 2.3 and 2.4 hold without the condition (2) (property $B$) in Definition 2.1.

Definition 2.7. (Noiri 2011). Let $(X, m)$ be an $m$-space. Let $m\gamma : m \rightarrow \mathcal{P}(X)$ be a function from $m$ into $\mathcal{P}(X)$ such that $U \subset m\gamma(U)$ for each $U \in m$. The function $m\gamma$ is called an $m\gamma$-operation on $m$. Hereafter, an $m\gamma$-operation is called a $\gamma$-operation and denoted by $\gamma : m \rightarrow \mathcal{P}(X)$.

Definition 2.8. (Noiri 2011). Let $(X, m)$ be an $m$-space and $\gamma$ an operation on $m$. A subset $A$ of $X$ is said to be $\gamma$-open if for each $x \in A$ there exists $U \in m$ such that $x \in U \subset \gamma(U) \subset A$. The complement of a $\gamma$-open set is said to be $\gamma$-closed. The family of all $\gamma$-open sets of $(X, m)$ is denoted by $\gamma(X)$.

Remark 2.9. We assume that the empty set $\emptyset$ is a $\gamma$-open set, that is, $\emptyset \in \gamma(X)$.

Lemma 2.10. Let $(X, m)$ be an $m$-space. For $\gamma(X)$, the following properties hold:

1. $\emptyset, X \in \gamma(X)$,
2. If $A_\alpha \in \gamma(X)$ for each $\alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} A_\alpha \in \gamma(X)$,
3. $\gamma(X) \subset m$. 

Lemma 2.13. (Noiri 2011). By (1) and (2) of Lemma 2.10, it turns out that $\gamma(X)$ is an $m$-structure. However, in general, $\gamma(X)$ is not a topology. It was shown by Ogata (1991, Example 2.8) that the intersection of two $\gamma$-open sets is not always $\gamma$-open.

Definition 2.12. (Noiri 2011). An $m$-space $(X, m)$ is said to be $\gamma$-regular if for each $x \in X$ and each $U \in m$ containing $x$, there exists $V \in m$ such that $x \in V \subset \gamma(V) \subset U$.

Lemma 2.13. (Noiri 2011). For an $m$-space $(X, m)$, the following properties are equivalent:

1. $m = \gamma(X)$;
2. $(X, m)$ is $\gamma$-regular;
3. For each $x \in X$ and each $U \in m$ containing $x$, there exists $W \in \gamma(X)$ such that $x \in W \subset \gamma W \subset U$.

Definition 2.14. (Noiri 2011). Let $(X, m)$ be an $m$-space. For a subset $A$ of $X$, the $\gamma$-closure of $A$ ($\gamma Cl(A)$) and $\gamma$-interior of $A$ ($\gamma Int(A)$), are defined as follows:

1. $\gamma Cl(A) = \cap \{F : A \subset F, X \setminus F \in \gamma(X)\}$,
2. $\gamma Int(A) = \cup \{U : U \subset A, U \in \gamma(X)\}$.

Lemma 2.15. (Noiri 2011). Let $(X, m)$ be an $m$-space on $X$. For the $\gamma$-closure and the $\gamma$-interior, the following properties hold, where $A$ and $B$ are subsets of $X$:

1. $\gamma Int(A) \subset A \subset \gamma Cl(A)$,
2. $\gamma Cl(\emptyset) = \emptyset = \gamma Int(\emptyset)$, $\gamma Cl(X) = X = \gamma Int(X)$,
3. If $A \subset B$, then $\gamma Cl(A) \subset \gamma Cl(B)$ and $\gamma Int(A) \subset \gamma Int(B)$,
4. $\gamma Cl(\gamma Cl(A)) = \gamma Cl(A)$ and $\gamma Int(\gamma Int(A)) = \gamma Int(A)$,
5. $A$ is $\gamma$-closed if and only if $\gamma Cl(A) = A$ and $A$ is $\gamma$-open if and only if $\gamma Int(A) = A$,
6. $\gamma Cl(A)$ is $\gamma$-closed and $\gamma Int(A)$ is $\gamma$-open,
7. (i) $x \in \gamma Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every $\gamma$-open set $U$ containing $x$,
(ii) $x \in \gamma Int(A)$ if and only if for each $x \in A$ there exists a $\gamma$-open set $U$ containing $x$ such that $U \subset A$.

Proof. The proof follows easily from Lemmas 2.3, 2.4 and 2.5.

3. $\gamma\mathcal{H}$-compact spaces

First, we recall the definitions of a hereditary class and an ideal used in the sequel. A subfamily $\mathcal{H}$ of the power set $\mathcal{P}(X)$ is called a hereditary class on $X$ (Császár 2007) if it satisfies the following property: $A \in \mathcal{H}$ and $B \subset A$ implies $B \in \mathcal{H}$. A hereditary class $\mathcal{H}$ is called an ideal (Janković and Hamlett 1990) if it satisfies the additional condition: $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$. An $m$-space $(X, m)$ with a hereditary class $\mathcal{H}$ on $X$ is called a hereditary $m$-space and is denoted by $(X, m, \mathcal{H})$.

Definition 3.1. Let $(X, m, \mathcal{H})$ be a hereditary $m$-space and $\gamma$ an operation on $m$, where $\mathcal{H}$ a hereditary class on $X$. A subset $A$ of $X$ is said to be $\gamma\mathcal{H}$-compact (resp. $\mathcal{H}$-compact) relative to $m$ if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of $A$ by $m$-open sets of $X$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $A \setminus \cup \{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}$ (resp. $A \setminus \cup \{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$).

Definition 3.2. Let $(X, m, \mathcal{H})$ be a hereditary $m$-space and $\gamma$ an operation on $m$. The $m$-space $(X, m)$ is said to be $\gamma\mathcal{H}$-compact (resp. $\mathcal{H}$-compact) relative to $m$. The
Definition 3.3. Let \((X, m)\) be an \(m\)-space and \(\gamma\) an operation on \(m\). Then \((X, m)\) is said to be \(\gamma\)-compact (Noiri 2011) if for each cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(X\) by \(m\)-open sets of \(X\), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(\bigcup \{\gamma(U_\alpha) : \alpha \in \Delta_0\} = X\).

Definition 3.4. Let \((X, m)\) be an \(m\)-space. A subset \(A\) of \(X\) is said to be \(m\)-compact (Popa and Noiri 2002) (resp. \(m\)-closed, Popa and Noiri 2002) relative to \(m\) if for each cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(A\) by \(m\)-open sets of \(X\), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(A \subseteq \bigcup \{U_\alpha : \alpha \in \Delta_0\}\) (resp. \(A \subseteq \bigcup \{\text{Cl}(U_\alpha) : \alpha \in \Delta_0\}\)).

Definition 3.5. An \(m\)-space \((X, m)\) is said to be \(m\)-compact (Popa and Noiri 2002) (resp. \(m\)-closed, Popa and Noiri 2002) if \(X\) is \(m\)-compact (resp. \(m\)-closed) relative to \(m\).

Remark 3.6. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space and \(\gamma\) an operation on \(m\).

1. If \(\gamma\) is the identity (resp. \(m\)-closure) operation, then "\(\gamma\mathcal{H}\)-compact relative to \(m\)" coincides with "\(\mathcal{H}\)-compact (resp. \(m\)-closed) relative to \(m\)".

2. If \(\mathcal{H} = \{\emptyset\}\), then "\(\gamma\mathcal{H}\)-compact relative to \(m\)" coincides "\(\gamma\)-compact relative to \(m\)". Moreover if \(\gamma\) is the identity operation, then "\(\gamma\mathcal{H}\)-compact relative to \(m\)" coincides with "\(m\)-compact relative to \(m\)".

Theorem 3.7. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space and \(\gamma\) an operation on \(m\). Then the following properties are equivalent:

1. \((X, m)\) is \(\mathcal{H}\)-compact;
2. For every family \(\{F_\alpha : \alpha \in \Delta\}\) of \(m\)-closed sets satisfying \(\bigcap \{F_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}\) for every finite subfamily \(\Delta_0\) of \(\Delta\), \(\bigcap \{F_\alpha : \alpha \in \Delta_0\} \neq \emptyset\).

Proof. (1) \(\Rightarrow\) (2): Let \((X, m)\) be \(\mathcal{H}\)-compact. Suppose that \(\bigcap \{F_\alpha : \alpha \in \Delta\} = \emptyset\). Then \(X \setminus F_\alpha\) is \(m\)-open for each \(\alpha \in \Delta\) and \(\bigcup_{\alpha \in \Delta} X \setminus F_\alpha = X \setminus \bigcap_{\alpha \in \Delta} F_\alpha = X\). By (1), there exists a finite subfamily \(\Delta_0\) of \(\Delta\) such that \(X \setminus \bigcup_{\alpha \in \Delta_0} (X \setminus F_\alpha) = \bigcap \{F_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}\). This is a contradiction.

(2) \(\Rightarrow\) (1): Suppose that \((X, m)\) is not \(\mathcal{H}\)-compact. There exists a cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(X\) by \(m\)-open sets of \(X\) such that \(X \setminus \bigcup \{U_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}\) for every finite subset \(\Delta_0\) of \(\Delta\). Since \(X \setminus U_\alpha\) is \(m\)-closed for each \(\alpha \in \Delta\) and \(\bigcap \{(X \setminus U_\alpha) : \alpha \in \Delta_0\} \notin \mathcal{H}\), for every finite subset \(\Delta_0\) of \(\Delta\). By (2), we have \(\bigcap \{(X \setminus U_\alpha) : \alpha \in \Delta\} \neq \emptyset\). Therefore, \(X \setminus \bigcup \{U_\alpha : \alpha \in \Delta\} \neq \emptyset\). This is contrary that \(\{U_\alpha : \alpha \in \Delta\}\) is an \(m\)-open cover of \(X\).

Theorem 3.8. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space, \(\gamma\) an operation on \(m\) and \(A\) a subset of \(X\). The implications (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) \(\Rightarrow\) (4) hold. If \((X, m)\) is \(\gamma\)-regular, then the following properties are equivalent:

1. \(A\) is \(\mathcal{H}\)-compact relative to \(m\);
2. \(A\) is \(\gamma\mathcal{H}\)-compact relative to \(m\);
3. \(A\) is \(\mathcal{H}\)-compact relative to \(\gamma(X)\);
4. \(A\) is \(\gamma\mathcal{H}\)-compact relative to \(\gamma(X)\).

Proof. (1) \(\Rightarrow\) (2): For any cover \(\{U_\alpha : \alpha \in \Delta\}\) of \(A\) by \(m\)-open sets of \(X\), there exists a finite subset \(\Delta_0\) of \(\Delta\) such that \(A \setminus \bigcup \{U_\alpha : \alpha \in \Delta_0\} \notin \mathcal{H}\); hence \(A \setminus \bigcup \{\gamma(U_\alpha) : \alpha \in \Delta_0\} \in \mathcal{H}\). Therefore, \(A\) is \(\gamma\mathcal{H}\)-compact relative to \(m\).

(2) \(\Rightarrow\) (3): Let \(A\) be \(\gamma\mathcal{H}\)-compact relative to \(m\) and \(\{U_\alpha : \alpha \in \Delta\}\) a cover of \(A\) by \(\gamma\)-open sets of \(X\). For each \(x \in A\) there exists \(\alpha(x) \in \Delta\) such that \(x \in U_{\alpha(x)}\). Since \(U_{\alpha(x)}\) is \(\gamma\)-open,
there exists $V_{α(x)} ∈ m$ such that $x ∈ V_{α(x)} ⊆ γ(V_{α(x)}) ⊆ U_{α(x)}$. Since the family $\{V_{α(x)} : x ∈ A\}$ is an $m$-open cover of $A$ and $A$ is $γH$-compact relative to $m$, there exists a finite subset $A_0$ of $A$ such that $A \setminus \bigcup \{γ(V_{α(x)}) : x ∈ A_0\} ∈ H$ and hence $A \setminus \bigcup \{U_{α(x)} : x ∈ A_0\} ∈ H$. This shows that $A$ is $H$-compact relative to $γ(X)$.

(3) $⇒$ (4): By Lemma 2.10, $γ(X)$ is an $m$-structure and it follows from the same argument as (1) $⇒$ (2) that $A$ is $γH$-compact relative to $γ(X)$.

(4) $⇒$ (1): Suppose that $(X,m)$ is $γ$-regular. Let $A$ be $γH$-compact relative to $γ(X)$.

By Lemma 2.13, $m = γ(X)$ and $A$ is $γH$-compact relative to $γ(X)$. Let $\{U_α : α ∈ Δ\}$ be any cover of $A$ by $m$-open sets of $X$. For each $x ∈ A$, there exists $α(x) ∈ Δ$ such that $x ∈ U_{α(x)}$. Since $(X,m)$ is $γ$-regular, there exists $V_{α(x)} ∈ γ(X)$ such that $x ∈ V_{α(x)} ⊆ γ(V_{α(x)}) ⊆ U_{α(x)}$. Since $\{V_{α(x)} : x ∈ A\}$ is a cover of $A$ by $γ$-open sets of $X$ and $A$ is $γH$-compact relative to $γ(X)$, there exists a finite subset $A_0$ of $A$ such that $A \setminus \bigcup \{γ(V_{α(x)}) : x ∈ A_0\} ∈ H$ and hence $A \setminus \bigcup \{U_{α(x)} : x ∈ A_0\} ∈ H$. This shows that $A$ is $H$-compact relative to $m$.

**Corollary 3.9.** For any $γ$-regular $m$-space $(X,m)$, the following properties are equivalent. The implications (1) $⇒$ (2) $⇒$ (3) $⇒$ (4) hold without the assumption $γ$-regular on $(X,m)$.

1. $(X,m)$ is $H$-compact;
2. $(X,m)$ is $γH$-compact;
3. $(X,γ(X))$ is $H$-compact;
4. $(X,γ(X))$ is $γH$-compact.

**Remark 3.10.** In Corollary 3.9, if we put $H = \{0\}$, then we obtain Theorem 5.1 of Noiri (2011).

**Theorem 3.11.** Let $(X,m,H)$ be a hereditary $m$-space, $γ$ an operation on $m$ and $A,B$ be subsets of $X$. If $A$ is $γH$-compact relative to $m$ and $B$ is $γ$-closed, then $A \cap B$ is $γH$-compact relative to $m$.

**Proof.** Let $\{U_α : α ∈ Δ\}$ be a cover of $A \cap B$ by $m$-open subsets of $X$. Then $A \setminus B ⊆ X \setminus B$ and $A \setminus \bigcup \{U_α : α ∈ Δ\}$ is $γ$-open. For each $x ∈ A \setminus B$, there exists an $m$-open set $U_α$ containing $x$ such that $x ∈ U_α ⊆ γ(U_α) ⊆ X \setminus B$. Then $\{U_α : α ∈ Δ\} \cup \{U_α : x ∈ A \setminus B\}$ is a cover of $A$ by $m$-open sets of $X$. Since $A$ is $γH$-compact relative to $m$, there exist finite subsets $A_0$ of $Δ$ and $A_0$ of $A$ such that $A \subseteq \bigcup \{γ(U_α) : α ∈ Δ_0\} \cup \bigcup \{γ(U_α) : x ∈ A_0\} \cup H_0$, where $H_0 ∈ H$. Then we have

\[
A \cap B \subseteq \bigcup \{γ(U_α) \cap B : α ∈ Δ_0\} \cup \bigcup \{γ(U_α) \cap B : x ∈ A_0\} \cup (H_0 \cap B)
\]

\[
\subseteq \bigcup \{γ(U_α) : α ∈ Δ_0\} \cup H_0.
\]

Therefore, $(A \cap B) \setminus \bigcup \{γ(U_α) : α ∈ Δ_0\} ⊆ H_0 ∈ H$. This shows that $A \cap B$ is $γH$-compact relative to $m$.

**Corollary 3.12.** Let $(X,m,H)$ be a hereditary $m$-space and $γ$ an operation.

1. If $(X,m,H)$ is $γH$-compact and $B$ is $γ$-closed, then $B$ is $γH$-compact relative to $m$.
2. If $(X,m)$ is $γ$-compact and $B$ is $γ$-closed, then $B$ is $γ$-compact relative to $m$.

**Definition 3.13.** Let $(X,m,H)$ be a hereditary $m$-space. A subset $A$ of $X$ is said to be...
(1) $m\mathcal{H}$-g-closed (Al-Omari and Noiri 2020) if $mcl(A) \subseteq U$ whenever $A \setminus U \in \mathcal{H}$ and $U \in m$, 
(2) mg-closed (Noiri 2007) if $mcl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in m$.

Remark 3.14. We note that:
(1) If $\mathcal{H} = \{\emptyset\}$, then $m\{\emptyset\}$-g-closed and mg-closed coincide.
(2) If $A$ is $m\mathcal{H}$-g-closed, then $A$ is mg-closed. The converse is not always true as shown by the following example due to Qahis et al. (2021).

Example 3.15. Let $X = \mathbb{R}$, $m = \{\emptyset, \mathbb{R}\} \cup \{(r, \infty) : r \in \mathbb{R}\}$ and $\mathcal{H} = \{H : H \subseteq \mathbb{Q} \cap [0, \infty) \text{ or } H \subseteq \mathbb{Q} \cap (-\infty, 0]\}$. If $A = \mathbb{Q}$, then

(1) $A$ is mg-closed because if $A \subseteq U$ and $U \in m$, then $U = \mathbb{R}$ and $mcl(A) = \mathbb{R} \subseteq U$.
(2) $A$ is not $m\mathcal{H}$-g-closed because if $U = (0, \infty)$, then $U \in m$ and $A \setminus U = \mathbb{Q} \setminus (0, \infty) = \mathbb{Q} \cap (-\infty, 0]$ \in $\mathcal{H}$ but $mcl(A) = \mathbb{R} \not\subseteq (0, \infty)$.

Theorem 3.16. Let $(X, m, \mathcal{H})$ be an $\mathcal{H}$-compact space. If $A$ is mg-closed, then $A$ is $\mathcal{H}$-compact relative to $m$.

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of $A$ by $m$-open sets of $X$. Then $mcl(A) \subseteq \bigcup\{U_\alpha : \alpha \in \Delta\}$. Then we have $(X \setminus mcl(A)) \cup \bigcup\{U_\alpha : \alpha \in \Delta\}$ is an $m$-open cover of $X$. Since $X \in \mathcal{H}$-compact, there exists a finite subset $\Delta_0$ of $\Delta$ such that $X \setminus [(X \setminus mcl(A)) \cup \bigcup\{U_\alpha : \alpha \in \Delta_0\}] \in \mathcal{H}$. Then

$$X \setminus [(X \setminus mcl(A)) \cup \bigcup\{U_\alpha : \alpha \in \Delta_0\}] = mcl(A) \cap (X \setminus (\bigcup\{U_\alpha : \alpha \in \Delta_0\}))$$
$$\supseteq A \cap (X \setminus (\bigcup\{U_\alpha : \alpha \in \Delta_0\}))$$
$$= A \cap \bigcup\{U_\alpha : \alpha \in \Delta_0\}.$$

Therefore, $A \cap \bigcup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$ and $A$ is $\mathcal{H}$-compact relative to $m$.

Corollary 3.17. If $(X, m)$ is $m$-compact and $A$ is mg-closed, then $A$ is $m$-compact relative to $m$.

Let $(X, \tau)$ be a topological space. A subset $A$ of $X$ is said to be g-closed (Levine 1970) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U \in \tau$.

Corollary 3.18. (Levine 1970) If $(X, \tau)$ is a compact topological space and $A$ is g-closed, then $A$ is compact.

Theorem 3.19. Let $(X, m, \mathcal{H})$ be a hereditary $m$-space. An $m\mathcal{H}$ g-closed subset of $X$ and $A \subseteq B \subseteq mcl(A)$, then the following properties are equivalent:

(1) $A$ is $\mathcal{H}$-compact relative to $m$,
(2) $B$ is $\mathcal{H}$-compact relative to $m$.

Proof. (1) $\Rightarrow$ (2): Suppose that $A$ is $\mathcal{H}$-compact relative to $m$. Let $\{U_\alpha : \alpha \in \Delta\}$ be any cover of $B$ by $m$-open sets of $X$. Then $\{U_\alpha : \alpha \in \Delta\}$ is a cover of $A$ by $m$-open sets of $X$. Since $A$ is $\mathcal{H}$-compact relative to $m$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $A \setminus \bigcup\{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H}$. Since $A$ is $m\mathcal{H}$-g-closed, $mcl(A) \subseteq \bigcup\{U_\alpha : \alpha \in \Delta_0\}$. Since $B \subseteq mcl(A)$, we have $B \setminus \bigcup\{U_\alpha : \alpha \in \Delta_0\} \subseteq mcl(A) \setminus \bigcup\{U_\alpha : \alpha \in \Delta_0\} = \emptyset \in \mathcal{H}$. Therefore, $B$ is $\mathcal{H}$-compact relative to $m$. 

(2) ⇒ (1): Suppose that \( B \) is \( \mathcal{H} \)-compact relative to \( m \). Let \( \{U_\alpha : \alpha \in \Delta\} \) is any cover of \( A \) by \( m \)-open sets of \( X \). Since \( A \) is \( m \mathcal{H} \)-g-closed, \( A \) is \( mg \)-closed and hence we have \( B \subseteq m\text{cl}(A) \subseteq \cup \{U_\alpha : \alpha \in \Delta\} \). Since \( B \) is \( \mathcal{H} \)-compact relative to \( m \), there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( B \setminus \cup \{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H} \). Since \( A \subseteq B \), \( A \setminus \cup \{U_\alpha : \alpha \in \Delta_0\} \in \mathcal{H} \). Therefore, \( A \) is \( \mathcal{H} \)-compact relative to \( m \).

**Theorem 3.20.** Let \((X, m)\) be an \( m \)-space and \( \mathcal{H} \) an ideal. If subsets \( A \) and \( B \) of \( X \) are \( \gamma \mathcal{H} \)-compact relative to \( m \), then \( A \cup B \) is \( \gamma \mathcal{H} \)-compact relative to \( m \).

**Proof.** Let \( \mathcal{U} = \{U_\alpha : \alpha \in \Delta\} \) be a cover of \( A \cup B \) by \( m \)-open sets of \( X \). Then \( \mathcal{U} \) is a cover of \( A \) and \( B \) by \( m \)-open sets of \( X \). Since \( A \) and \( B \) are \( \gamma \mathcal{H} \)-compact relative to \( m \), there exist finite subsets \( \Delta_A \) and \( \Delta_B \) of \( \Delta \) and subsets \( H_A \) and \( H_B \) of \( \mathcal{H} \) such that \( A \subseteq \cup \{\gamma(U_\alpha) : \alpha \in \Delta_A\} \cup H_A \) and \( B \subseteq \cup \{\gamma(U_\alpha) : \alpha \in \Delta_B\} \cup H_B \). Hence we have \( A \cup B \subseteq \cup \{\gamma(U_\alpha) : \alpha \in \Delta_A \cup \Delta_B\} \cup (H_A \cup H_B) \). Since \( \mathcal{H} \) is an ideal, we have \( (A \cup B) \setminus \cup \{\gamma(U_\alpha) : \alpha \in \Delta_A \cup \Delta_B\} \in \mathcal{H} \). This shows that \( A \cup B \) is \( \gamma \mathcal{H} \)-compact relative to \( m \).

The following example is due to Qahis et al. (2021).

**Example 3.21.** Let \( X = \mathbb{R} \) be the real number, \( m = \tau \) the usual topology, \( \gamma : m \rightarrow \mathcal{P}(X) \) such that \( \gamma(U) = \text{cl}(U) \) for every \( U \subseteq m \) and \( \mathcal{H} = \{H \subseteq \mathbb{R} : H \subseteq (0, 1) \text{ or } H \subseteq (1, 2)\} \). If \( A = (0, 1) \) and \( B = (1, 2) \), then

1. \( A \) and \( B \) are \( \gamma \mathcal{H} \)-compact relative to \( m \).
2. \( A \cup B \) is not \( \gamma \mathcal{H} \)-compact relative to \( m \).

**Proof.** (1) The proof is obvious.

(2) The family \( \{(\frac{1}{n}, 2 - \frac{1}{n}) : n \in \mathbb{Z}^+\} \), where \( \mathbb{Z}^+ \) is the family of positive integers, is a cover of \( A \cup B \) by \( m \)-open sets of \( X \). For any finite subsets \( \{n_1, n_2, ..., n_k\} \) of \( \mathbb{Z}^+ \), put \( N = \max\{n_1, n_2, ..., n_k\} \). Then we have
\[
(A \cup B) \setminus \cup \{\gamma(\frac{1}{n}, 2 - \frac{1}{n}) : 1 \leq i \leq k\} = (A \cup B) \setminus \{[\frac{1}{n}, 2 - \frac{1}{n}] : 1 \leq i \leq k\} = (A \cup B) \setminus [\frac{1}{N}, 2 - \frac{1}{N}] = (0, \frac{1}{N}) \cup (2 - \frac{1}{N}, 2) \notin \mathcal{H}. 
\]
Therefore, \( A \cup B \) is not \( \gamma \mathcal{H} \)-compact relative to \( m \).

**Definition 3.22.** (Noiri and Popa 2018) Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space and \( A \) a subset of \( X \).

1. The minimal local function \( A^*_m(H)(\mathcal{H}, m) \) of \( A \) is defined as follows: \( A^*_m(H)(\mathcal{H}, m) = \{x \in X : U \cap A \notin \mathcal{H} \text{ for every } U \subseteq m(x)\} \), where \( m(x) = \{U : x \in U \subseteq m\} \). Hereafter, \( A^*_m(H)(\mathcal{H}, m) \) is denoted by \( A^*_m(H) \).
2. The minimal \( \gamma \)-closure \( m\text{Cl}_\gamma(H)(A) \) of \( A \) is defined as \( m\text{Cl}_\gamma(H)(A) = A \cup A^*_m(H) \). The \( m\text{Cl}_\gamma(H) \)-structure is defined as follows: \( m\text{Cl}_\gamma(H)(A) = \{U \subseteq X : m\text{Cl}_\gamma(H)(A \setminus U) = X \setminus U\} \). Each member of \( m\text{Cl}_\gamma(H)(A) \) is said to be \( m\text{Cl}_\gamma(H) \)-open and the complement of an \( m\text{Cl}_\gamma(H) \)-open set is said to be \( m\text{Cl}_\gamma(H) \)-closed.

**Remark 3.23.** (Noiri and Popa 2018) Let \((X, m, \mathcal{H})\) be a hereditary \( m \)-space and \( A \) a subset of \( X \). If \( \mathcal{H} = \{\emptyset\} \) (resp. \( \mathcal{P}(X) \)), then \( A^*_m(H) = m\text{Cl}(A) \) (resp. \( A^*_m(H) = \emptyset \)).

**Lemma 3.24.** (Noiri and Popa 2018) For a hereditary \( m \)-space \((X, m, \mathcal{H})\), the following properties hold:

1. \( m\text{Cl}_\gamma(H)(A) \) is an \( m \)-structure on \( X \) such that \( m\text{Cl}_\gamma(H)(A) \) has property \( \mathcal{B} \) and \( m \subseteq m\text{Cl}_\gamma(H)(A) \).
2. \( \mathcal{B}(m, \mathcal{H}) = \{U \subseteq m : U \subseteq m, H \subseteq \mathcal{H}\} \) is a basis for \( m\text{Cl}_\gamma(H)(A) \) such that \( m \subseteq \mathcal{B}(m, \mathcal{H}) \).
Theorem 3.25. Let \((X, m, \mathcal{H})\) be a hereditary \(m\)-space. If a subset \(A\) of \(X\) is \(\mathcal{H}\)-compact relative to \(m^*_H\), then \(A\) is \(\mathcal{H}\)-compact relative to \(m\). The converse is true if \(\mathcal{H}\) is an ideal.

Proof. Let \(A\) be \(\mathcal{H}\)-compact relative to \(m^*_H\). By Lemma 3.24, \(m \subseteq m^*_H\) and hence \(A\) is \(\mathcal{H}\)-compact relative to \(m\).

Conversely, suppose that \(\mathcal{H}\) is an ideal and \(A\) is \(\mathcal{H}\)-compact relative to \(m\). Let \(\{V_\alpha : \alpha \in \Delta\}\) be any cover of \(A\) by \(m^*_H\)-open sets of \(X\). For each \(x \in A\) there exists \(\alpha(x) \in \Delta\) such that \(x \in V_{\alpha(x)} \subseteq m^*_H\). By Lemma 3.24, there exists \(U_{\alpha(x)} \subseteq m\) and \(H_{\alpha(x)} \subseteq \mathcal{H}\) such that \(x \in U_{\alpha(x)} \setminus H_{\alpha(x)} \subseteq V_{\alpha(x)}\); hence \(x \in U_{\alpha(x)} \subseteq V_{\alpha(x)} \cup H_{\alpha(x)}\), where \(H_{\alpha(x)} \subseteq \mathcal{H}\). Then \(\{U_{\alpha(x)} : x \in A\}\) is a cover of \(A\) by \(m\)-open sets of \(X\). Since \(A\) of \(X\) is \(\mathcal{H}\)-compact relative to \(m\), there exists a finite subset \(A_0\) of \(A\) such that \(A \setminus \bigcup\{U_{\alpha(x)} : x \in A_0\} \subseteq \mathcal{H}\). Hence we have

\[
A \subseteq \bigcup\{U_{\alpha(x)} : x \in A_0\} \cup H_0, \quad \text{where} \quad H_0 \in \mathcal{H}
\]

\[
\subseteq \bigcup\{V_{\alpha(x)} \cup H_{\alpha(x)} : x \in A_0\} \cup H_0
\]

\[
= \bigcup\{V_{\alpha(x)} : x \in A_0\} \cup \left(\bigcup\{H_{\alpha(x)} : x \in A_0\}\right) \cup H_0.
\]

Therefore, we obtain \(A \setminus \bigcup\{V_{\alpha(x)} : x \in A_0\} \subseteq \left(\bigcup\{H_{\alpha(x)} : x \in A_0\}\right) \cup H_0 \in \mathcal{H}\). This shows that \(A\) is \(\mathcal{H}\)-compact relative to \(m^*_H\).

Corollary 3.26. Let \((X, m, \mathcal{H})\) be an ideal \(m\)-space, then for a subset \(A\) of \(X\) the following properties are equivalent:

1. \(A\) is \(\mathcal{H}\)-compact relative to \(m\),
2. \(A\) is \(\mathcal{H}\)-compact relative to \(m^*_H\).

4. \((\gamma, \delta)\)-continuous functions

In this section, let \((X, m)\) and \((Y, n)\) be minimal spaces and \(\gamma\) (resp. \(\delta\)) be an operation on \(m\) (resp. \(n\)).

Definition 4.1. A function \(f : (X, m) \to (Y, n)\) is said to be

1. \((\gamma, \delta)\)-continuous if for each \(x \in X\) and each \(V \in n\) containing \(f(x)\), there exists \(U \in m\) containing \(x\) such that \(f(\gamma(U)) \subseteq \delta(V)\).
2. weakly \((\gamma, \delta)\)-continuous if \(f^{-1}(V)\) is \(\gamma\)-open in \(X\) for every \(\delta\)-open set \(V\) of \(Y\).

Theorem 4.2. If \(f : (X, m) \to (Y, n)\) is \((\gamma, \delta)\)-continuous, then it is weakly \((\gamma, \delta)\)-continuous.

Proof. Let \(V\) be any \(\delta\)-open set of \(Y\). For each \(x \in f^{-1}(V)\), \(f(x) \in V\) and there exists \(V_0 \subset n\) such that \(f(x) \in V_0 \subseteq \delta(V_0) \subseteq V\). Since \(f\) is \((\gamma, \delta)\)-continuous, there exists \(U \in m\) containing \(x\) such that \(f(\gamma(U)) \subseteq \delta(V_0)\). Therefore, we have \(x \in U \subseteq \gamma(U) \subseteq f^{-1}(f(\gamma(U))) \subseteq f^{-1}(\delta(V_0)) \subseteq f^{-1}(V)\). This shows that \(f\) is weakly \((\gamma, \delta)\)-continuous.

The converse of Theorem 4.2 is not always true as shown by the following example.

Example 4.3. Let \(X = Y = \{a, b, c\}\), \(m = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\), \(n = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}\). Then \(m\) (resp. \(n\)) is a topology on \(X\) (resp. \(Y\)). Let \(\gamma(A) = m\text{Cl}(A) = \text{Cl}(A)\) for every subset \(A\) of \(X\) and \(\delta(B) = n\text{Cl}(B) = \text{Cl}(B)\) for every subset \(B\) of \(Y\). Then the identity function \(f : (X, m) \to (Y, n)\) is weakly \((\gamma, \delta)\)-continuous because \(\gamma(X) = \emptyset, X\) and \(\delta(Y) = \emptyset, Y\).

For \(a \in X\) and \(f(a) = a \in \{a\} = V \in n\), \(\delta(V) = \text{Cl}(\{a\}) = \{a, b\}\) and \(f(\gamma(U))\) is not contained in \(\delta(V)\) for every \(U \in m\) containing \(a\). Therefore, \(f\) is not \((\gamma, \delta)\)-continuous.
Theorem 4.4. Let \((Y, n)\) be \(\delta\)-regular. Then for a function \(f : (X, m) \to (Y, n)\) the following properties are equivalent:

1. \(f\) is \((\gamma, \delta)\)-continuous;
2. \(f\) is weakly \((\gamma, \delta)\)-continuous.

Proof. (1) \(\Rightarrow\) (2): The proof follows from Theorem 4.2.
(2) \(\Rightarrow\) (1): For any \(x \in X\) and \(V \in n\) containing \(f(x)\), by Lemma 2.13 there exists \(W \subseteq \delta(Y)\) such that \(f(x) \in W \subseteq \delta(W) \subseteq V\). By (2), \(f^{-1}(W)\) is a \(\gamma\)-open set containing \(x\) and hence there exists \(U \subseteq m\) such that \(x \in U \subseteq \gamma(U) \subseteq f^{-1}(W)\). Therefore, \(f(\gamma(U)) \subseteq W \subseteq \delta(W) \subseteq V \subseteq \delta(V)\). This shows that \(f\) is \((\gamma, \delta)\)-continuous.

Lemma 4.5. (Al-Omari and Noiri 2020) Let \(f : X \to Y\) be a function.

1. If \(\mathcal{H}\) is a hereditary class on \(X\), then \(f(\mathcal{H})\) is a hereditary class on \(Y\).
2. If \(\mathcal{H}\) is a hereditary class on \(Y\), then \(f^{-1}(\mathcal{H})\) is a hereditary class on \(X\).

Theorem 4.6. If \(f : (X, m, \mathcal{H}) \to (Y, n, f(\mathcal{H}))\) is \((\gamma, \delta)\)-continuous and \(A\) is \(\mathcal{H}\)-compact relative to \(m\), then \(f(A)\) is \(\delta f(\mathcal{H})\)-compact relative to \(n\).

Proof. Let \(\{V_a : a \in \Delta\}\) be any cover of \(f(A)\) by \(n\)-open sets of \(Y\). For each \(x \in A\), there exists \(\alpha(x) \in \Delta\) such that \(f(x) \in V_{\alpha(x)}\). Since \(f\) is \((\gamma, \delta)\)-continuous, there exists \(U_{\alpha(x)} \subseteq m\) containing \(x\) such that and \(f(\gamma(U_{\alpha(x)}) \subseteq \delta(V_{\alpha(x)})\). Since \(\{U_{\alpha(x)} : x \in A\}\) is a cover of \(A\) by \(m\)-open sets of \(X\), there exists a finite subset \(A_0\) of \(A\) such that \(A \subseteq \bigcup \{\gamma(U_{\alpha(x)}): x \in A_0\} \subseteq H_0\), where \(H_0 \in \mathcal{H}\). Therefore, \(f(A) \subseteq \bigcup \{f(\gamma(U_{\alpha(x)})): x \in A_0\} \cup f(H_0) \subseteq \bigcup \{\delta(V_{\alpha(x)}): x \in A_0\} \cup f(H_0)\). Hence we have \(f(A) \subseteq \bigcup \{\delta(V_{\alpha(x)}): x \in A_0\} \subseteq f(\mathcal{H})\) and hence \(f(A)\) is \(\delta f(\mathcal{H})\)-compact relative to \(n\).

Theorem 4.7. If \(f : (X, m, \mathcal{H}) \to (Y, n, f(\mathcal{H}))\) is weakly \((\gamma, \delta)\)-continuous and \(A\) is \(\mathcal{H}\)-compact relative to \(\gamma(X)\), then \(f(A)\) is \(f(\mathcal{H})\)-compact relative to \(\delta(Y)\).

Proof. Let \(\{V_a : a \in \Delta\}\) be any cover of \(f(A)\) by \(\delta\)-open sets of \(Y\). For each \(x \in A\), there exists \(\alpha(x) \in \Delta\) such that \(f(x) \in V_{\alpha(x)}\). Since \(f\) is weakly \((\gamma, \delta)\)-continuous, \(x \in f^{-1}(V_{\alpha(x)}) \subseteq \gamma(X)\) and \(\{f^{-1}(V_{\alpha(x)}): x \in A\}\) is a cover of \(A\) by \(\gamma\)-open sets of \(X\). Since \(A\) is \(\mathcal{H}\)-compact relative to \(\gamma(X)\), there exists a finite subset \(A_0\) of \(A\) and \(H_0 \in \mathcal{H}\) such that \(A \subseteq \bigcup \{f^{-1}(V_{\alpha(x)}): x \in A_0\} \cup H_0\); hence \(f(A) \subseteq \bigcup \{V_{\alpha(x)}: x \in A_0\} \cup f(H_0)\). Therefore, \(f(A)\) is \(f(\mathcal{H})\)-compact relative to \(\delta(Y)\).

Corollary 4.8. Let \(f : (X, m) \to (Y, n)\) be a surjective function.

1. If \(f\) is \((\gamma, \delta)\)-continuous and \((X, m)\) is \(\mathcal{H}\)-compact, then \((Y, n)\) is \(\delta f(\mathcal{H})\)-compact.
2. If \(f\) is weakly \((\gamma, \delta)\)-continuous and \((X, \gamma(X))\) is \(\mathcal{H}\)-compact, then \((Y, \delta(Y))\) is \(f(\mathcal{H})\)-compact.

Definition 4.9. (Noiri and Popa 2006) A function \(f : (X, m) \to (Y, n)\) is said to be \(M\)-closed if \(f(B)\) is \(n\)-closed in \(Y\) for every \(m\)-closed subset \(B\) of \(X\).

Lemma 4.10. (Noiri and Popa 2006) For a function \(f : (X, m) \to (Y, n)\), the following properties are equivalent:

1. \(f\) is \(M\)-closed;
2. for each \(y \in Y\) and each \(U \in m\) containing \(f^{-1}(y)\), there exists \(V \in n\) containing \(y\) such that \(f^{-1}(V) \subseteq U\).
Theorem 4.11. Let \( f : (X, m) \to (Y, n, \mathcal{H}) \) be an \( M \)-closed surjective function. If \( f^{-1}(y) \) is \( m \)-compact relative to \( m \) for each \( y \in Y \) and \( B \) is \( \mathcal{H} \)-compact relative to \( n \), then \( f^{-1}(B) \) is \( f^{-1}(\mathcal{H}) \)-compact relative to \( m \).

Proof. Let \( \{ U_{\alpha} : \alpha \in \Delta \} \) be any cover of \( f^{-1}(B) \) by \( m \)-open sets of \( X \). Then for each \( y \in B \), since \( f^{-1}(y) \) is \( m \)-compact relative to \( m \), there exists a finite subset \( \Delta(y) \) of \( \Delta \) such that \( f^{-1}(y) \subseteq \bigcup \{ U_{\alpha} : \alpha \in \Delta(y) \} = U_y \). Since \( U_y \) is an \( m \)-open set of \( X \) containing \( f^{-1}(y) \) and \( f \) is \( M \)-closed, by Lemma 4.10 there exists an \( n \)-open set \( V_y \) containing \( y \) such that \( f^{-1}(V_y) \subseteq U_y \). Since \( \{ V_y : y \in B \} \) is a cover of \( B \) by \( n \)-open sets of \( Y \) and \( B \) is \( \mathcal{H} \)-compact relative to \( n \), there exists a finite subset \( B_0 \) of \( B \) such that \( B \setminus \bigcup \{ V_y : y \in B_0 \} \in \mathcal{H} \). Therefore, \( B \subseteq \bigcup \{ V_y : y \in B_0 \} \cup H_0 \), where \( H_0 \in \mathcal{H} \). Hence we have

\[
\begin{align*}
f^{-1}(B) & \subseteq \bigcup \{ f^{-1}(V_y) : y \in B_0 \} \cup f^{-1}(H_0) \\
& \subseteq \bigcup \{ U_y : y \in B_0 \} \cup f^{-1}(H_0) \\
& \subseteq \bigcup \{ U_{\alpha} : \alpha \in \Delta(y), y \in B_0 \} \cup f^{-1}(H_0) .
\end{align*}
\]

We obtain \( f^{-1}(B) \setminus \bigcup \{ U_{\alpha} : \alpha \in \Delta(y), y \in B_0 \} \in f^{-1}(\mathcal{H}) \). This shows that \( f^{-1}(B) \) is \( f^{-1}(\mathcal{H}) \)-compact relative to \( m \).

Corollary 4.12. Let \( f : (X, m) \to (Y, n, \mathcal{H}) \) be a \( M \)-closed surjective function. If \( f^{-1}(y) \) is \( m \)-compact relative to \( m \) for each \( y \in Y \) and \( Y \) is \( \mathcal{H} \)-compact, then \( X \) is \( f^{-1}(\mathcal{H}) \)-compact.

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References


