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L^p-THEORY OF VENTTSEL BVPS WITH DISCONTINUOUS DATA

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ABSTRACT. We provide $W^{2,p}$-*a priori* estimates for the strong solutions to Venttsel boundary value problems for linear elliptic operators with discontinuous coefficients.

1. Introduction

The history of the Venttsel boundary value problems starts with the notable article (Venttsel 1959) where, given a general second-order linear elliptic operator

$$\mathscr{E}u := \sum_{i,j=1}^{n} a^{ij}(x)D_iD_ju + \sum_{i=1}^{n} b^i(x)D_iu + c(x)u$$

over a bounded domain $\Omega \subset \mathbb{R}^n$, A.D. Venttsel found the most general admissible boundary conditions which restrict $\mathscr{E}$ to an infinitesimal generator of a Markov process in Ω. These conditions are given in terms of the second-order integro-differential operator

$$\begin{aligned}\mathscr{V}u := & \sum_{i,j=1}^{n} \alpha^{ij}(x)d_id_ju + \sum_{i=1}^{n} \beta^i(x)d_iu + \gamma(x)u + \beta_0(x)\partial_{\mathbf{n}}u + a(x)\mathscr{E}u \\ & + \int_{\partial\Omega} k_1(x,y)\Big(u(y) - k_2(x,y)\Big(u(x) + \sum_{j=1}^{n}(y_j - x_j)d_ju(x)\Big)\Big)dy \\ & + \int_{\Omega} k_3(x,y)\big(u(y) - u(x)\big)\,dy, \qquad x \in \partial\Omega, \quad y \in \Omega,\end{aligned}$$

where $d = (d_1, \ldots, d_n)$ stands for the tangential gradient to $\partial\Omega$ with components given by $d_i = D_i - \sum_{j=1}^{n} \mathbf{n}^i\mathbf{n}^jD_j$, $\partial_{\mathbf{n}}$ means the directional derivative along the outward normal $\mathbf{n}$ to $\partial\Omega$ and the integral kernels k_i verify appropriate hypotheses.

To Nino Maugeri with thanks for the long standing friendship and with all the best wishes on the occasion of his 75th anniversary

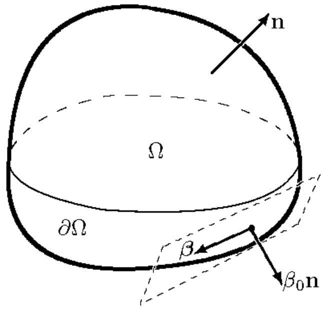
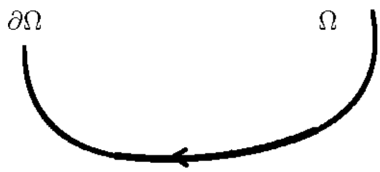
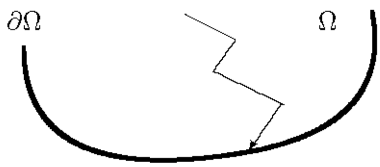


FIGURE 1. The domain Ω

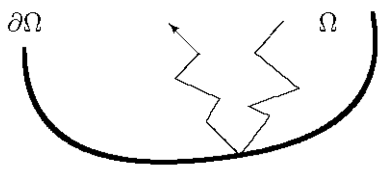
According to the theory of Markov processes, the first two terms above correspond to *diffusion* and *drift* of the process along the boundary, $\gamma(x)u$, $\beta_0(x)\partial_{\mathbf{n}}u$ and $a(x)\mathscr{E}u$ are related to *absorption, reflection* and *viscosity* phenomena, respectively, while the non-local integral terms represent *jumps* of the process along $\partial\Omega$ and *inward jumps* from $\partial\Omega$ into Ω (cf. Watanabe 1979; Ikeda and Watanabe 1989).



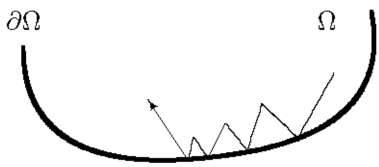
$\sum_{i,j=1}^{n} \alpha^{ij}(x)d_i d_j u + \sum_{i=1}^{n} \beta^i(x) d_i u$
diffusion and drift along the boundary



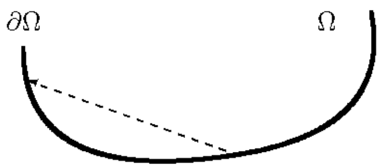
$\gamma(x)u$
absorption



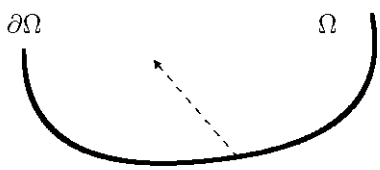
$\beta_0(x)\partial_{\mathbf{n}}u$
reflection



$a(x)\mathscr{E}u$
viscosity



$$\int_{\partial\Omega} k_1(x,y)\Big(u(y) - k_2(x,y)\big(u(x) + \textstyle\sum_{j=1}^{n}(y_j - x_j)d_j u(x)\big)\Big)dy$$
jumps along $\partial\Omega$



$$\int_{\Omega} k_3(x,y)\big(u(y) - u(x)\big)\,dy$$
inward jumps from $\partial\Omega$ into Ω

The Venttsel boundary conditions include as particular cases the Dirichlet, Neumann, oblique derivative and Robin boundary conditions, and Venttsel problems arise in various branches of science, technology and industry (see, *e.g.*, the references in Coclite *et al.* 2009; Apushkinskaya *et al.* 2019) for instance in models of fluid diffusion, elasticity, electromagnetic and phase-transition phenomena, hydraulic fracturing, different climate models and in various aspects of the financial mathematics.

The systematic study of Venttsel BVPs has been initiated by B. Paneah in the mid '80-es (see Paneah 2000, and the references therein) who combined the theory of pseudo-differential operators with Hörmander's vector field approach to deal with Venttsel problems for linear elliptic operators with C^∞-coefficients. At the same time Taira (2014) studied mainly operators with constant coefficients, employing semigroup techniques. The Schauder $C^{2,\alpha}$-theory of Venttsel BVPs for linear elliptic operators with $C^{0,\alpha}$-smooth principal coefficients has been developed by Luo and Trudinger (1991), while the Venttsel L^p-theory of operators with merely uniformly continuous principal coefficients has been elaborated by Apushkinskaya and Nazarov (1995). The study of *quasilinear* problems with Venttsel boundary conditions was initiated by Luo (1991) and continued later in a series of publications by Apushkinskaya and Nazarov. A detailed survey on the results obtained up to 1999 can be found in the paper by Apushkinskaya and Nazarov (2000) (see also Apushkinskaya and Nazarov 2001, for results regarding the two-phase Venttsel problems). All these results regard equations and boundary conditions with principal coefficients that depend at least *continuously* on the variable x.

Our general aim here is to present a very recent results, concerning the L^p-theory of linear Venttsel problems for elliptic operators with *discontinuous* coefficients. These were obtained by Apushkinskaya *et al.* (2019) and announced by Apushkinskaya *et al.* (2020), and regard *a priori* estimates in the framework of the Sobolev spaces and strong solvability theory of linear and quasilinear boundary-value problems with Venttsel boundary conditions. The discontinuity of the principal coefficients of the elliptic operators considered is measured in terms of their belonging to the Sarason class *VMO* of functions with vanishing mean oscillation, while the lower-order terms are taken in suitable Lebesgue or Orlicz spaces. Actually, the results presented by Apushkinskaya *et al.* (2019) concern solutions which belong to the Sobolev space $W^{2,p}$ inside the underlying domain Ω, and their trace on the boundary $\partial\Omega$ belongs to $W^{2,q}$ and satisfies the Venttsel boundary condition. To fix the ideas and for the sake of simplicity, we restrict ourselves here to a situation when the lower-order terms are avoided and $p = q$, referring the interested reader to Apushkinskaya *et al.* (2019) for the general case. We provide an *a priori* estimate in the Sobolev space $W^{2,p}(\Omega)$ for any strong solution of the discontinuous Venttsel problem considered. The result is preceded by a brief overview regarding the Dirichlet and the oblique derivative problems for linear elliptic operators with *VMO* principal coefficients and their relations to the Venttsel problems studied.

2. A brief overview on the Dirichlet and oblique derivative problems with discontinuous data

In what follows, we will consider elliptic equations over a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with $C^{1,1}$-smooth boundary $\partial\Omega$. We denote by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, $p \in [1,\infty)$,

$k \in \mathbb{N}$, the standard Lebesgue and Sobolev spaces with their respective norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, while $W_0^{k,p}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{k,p}(\Omega)}$.

We will consider differential operators with discontinuous principal coefficients that belong to the Sarason class of functions with mean oscillation that vanishes over shrinking balls. The space *BMO* of functions with *bounded mean oscillation* has been introduced by John and Nirenberg (1961). Later, Sarason (1975) attracted the attention to a natural subspace of *BMO* consisting of the functions with *vanishing mean oscillation (VMO)*. Let us recall the definitions of these spaces.

Definition 2.1. *A locally integrable function $f\colon\ \mathbb{R}^n \to \mathbb{R}$ belongs to BMO if its mean integral oscillation is bounded,*

$$\|f\|_* := \sup_B ⨍_B \big|f(x) - f_B\big|\, dx < \infty.$$

Here B varies in the class of all *balls in $\mathbb{R}^n$ and f_B and the dashed integral stand for the integral average $|B|^{-1}\int_B f(x)\,dx$. Modulo constant functions, the quantity $\|\cdot\|_*$ defines a norm under which BMO becomes a Banach space.*

For a function $f \in BMO$ define

$$\omega_f(r) = \sup_{\rho \le r} ⨍_{B_\rho} \big|f(x) - f_{B_\rho}\big|\, dx,$$

where B_ρ varies now in the class of all *balls of radius ρ. Then $f \in VMO$ if*

$$\lim_{r \to 0} \omega_f(r) = 0$$

and we refer to $\omega_f(r)$ as VMO-modulus of f.

For a bounded domain $\Omega \subset \mathbb{R}^n$, the localized spaces $BMO(\Omega)$ and $VMO(\Omega)$ are defined in the same manner, replacing B and B_ρ above by the respective intersections with Ω. Similarly, if $\partial\Omega$ is smooth, $BMO(\partial\Omega)$ and $VMO(\partial\Omega)$ are defined in a natural way by considering surface integral oscillations over $B \cap \partial\Omega$ and $B_\rho \cap \partial\Omega$ with balls centered at points of $\partial\Omega$.

Refering the reader to Maugeri *et al.* (2000, Section 2.1) for more details about the spaces *BMO* and *VMO*, we will restrict ourselves here only to mention some properties which will be used in the sequel. If $f \in VMO(\Omega)$ is defined on a Lipschitz domain, then it is possible to extend it to the whole $\mathbb{R}^n$ by preserving the corresponding *VMO*-modulus. The *BMO*-functions are not necessarily bounded, but $L^\infty \subset BMO$ and the inclusion is proper as shows the function $\log|x|$. The space of the bounded and uniformly continuous functions belongs to *VMO* with the modulus of continuity taken as *VMO*-modulus. However, *VMO* contains discontinuous functions as shows the embedding $W^{1,n} \subset VMO$ that is a simple consequence of the Poincaré inequality. Also this inclusion is proper and this can be seen by considering the functions $f_\alpha(x) = |\log|x||^\alpha$ with $\alpha > 0$. In particular, $f_1 = \log|x| \in BMO \setminus VMO$, $f_\alpha \in VMO$ for each $\alpha \in (0,1)$, but $f_\alpha \in W^{1,n}$ *only* if $\alpha \in (0, 1 - 1/n)$.

An alternative description of *VMO* has been given by Sarason (1975), who proved that $f \in VMO$ *if and only if* f belongs to the *BMO*-closure of the space of bounded and uniformly continuous functions. Moreover, $\lim_{y \to 0} \|f(\cdot - y) - f(\cdot)\|_* = 0$ which guarantees the good behavior of the mollifiers of *VMO* functions and this is crucial in the study of PDEs with *VMO* principal coefficients.

We will consider hereafter linear, second-order differential operator in non-divergence form

$$\mathscr{L}u := a^{ij}(x)D_{ij}u,$$

with generally measurable coefficients $a^{ij}\colon\ \Omega \to \mathbb{R}$, where $D_{ij} = \partial^2/\partial x_i \partial x_j$ and the usual summation convention on the repeated indices is adopted. The operator $\mathscr{L}$ will be supposed to be *uniformly elliptic,* that is, there exists a constant $\lambda > 0$ such that

$$\begin{cases} \lambda^{-1}|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 & \text{for almost all } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n, \\ a^{ij}(x) = a^{ji}(x) & \text{for almost all } x \in \Omega. \end{cases} \tag{1}$$

Indeed, as it follows from (1), the coefficients a^{ij} are essential bounded, $a^{ij} \in L^\infty(\Omega)$, but this is not enough to develop a relevant regularity theory for the operator $\mathscr{L}$ (cf. Maugeri *et al.* 2000, Chapter 1). That is why, we impose the additional assumption that a^{ij}'s are functions of *vanishing mean oscillation,*

$$a^{ij} \in VMO(\Omega). \tag{2}$$

The Dirichlet problem. Let $f \in L^p(\Omega)$ with $p \in (1,\infty)$, and consider the Dirichlet problem for the operator $\mathscr{L}$

$$\begin{cases} \mathscr{L}u := a^{ij}(x)D_{ij}u = f(x) & \text{for a.a. } x \in \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

A *strong solution* to (3) is a twice weakly differentiable function $u \in W^{2,p}(\Omega)$ that satisfies the equation in (3) *almost everywhere* in Ω and assumes zero boundary values in the sense of $W_0^{1,p}(\Omega)$.

Regarding the problem (3), we dispose of the *Schauder theory* which provides regularity and solvability results in the framework of the Hölder spaces. Namely, if $\partial\Omega$ is $C^{2,\alpha}$-smooth and $u \in C^2(\overline{\Omega})$ is a *classical solution* of (3) with Hölder continuous coefficients and right-hand side ($a^{ij}, f \in C^{0,\alpha}(\overline{\Omega})$, $\alpha \in (0,1)$) then $u \in C^{2,\alpha}(\overline{\Omega})$. Moreover, (3) is *uniquely solvable* in $C^{2,\alpha}(\overline{\Omega})$ for each $f \in C^{0,\alpha}(\overline{\Omega})$ (see Gilbarg and Trudinger 2001, Chapters 4, 6).

If a^{ij} are *merely continuous,* the Schauder theory is no more valid, and relevant regularity theory has been developed by *Calderón and Zygmund* in the settings of the L^p-spaces. The essence of that theory asserts that if $\partial\Omega \in C^{1,1}$ and u is a strong solution to (3) with $a^{ij} \in C^0(\overline{\Omega})$ and $f \in L^p(\Omega)$, $p \in (1,\infty)$, then $u \in W^{2,p}(\Omega)$ (cf. Gilbarg and Trudinger 2001, Chapter 9).

In a series of seminal papers Chiarenza *et al.* (1991, 1993) succeeded to extend the Calderón–Zygmund results to the problem (3) with *discontinuous* coefficients a^{ij}. Actually, the discontinuity cannot be arbitrary (cf. Maugeri *et al.* 2000, Chapter 1) and the one allowed by Chiarenza *et al.* (1991, 1993) is measured exactly in terms of *VMO*. The main result of Chiarenza *et al.* (1991, 1993) is the following *a priori* estimate for the strong solutions to (3).

Theorem 2.2. *Let* $\partial\Omega \in C^{1,1}$, $p \in (1,\infty)$ *and assume* (1) *and* (2). *Suppose further that* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *is a strong solution of* (3) *with* $f \in L^p(\Omega)$.

Then there exists a constant C, *depending only on* n, p, λ, $\operatorname{diam}\Omega$, $\partial\Omega$ *and the VMO-moduli of the coefficients* a^{ij}, *such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\Big(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\Big). \tag{4}$$

The proof of Theorem **2.2** relies on explicit local (interior and boundary) representation formulae for the second-order derivatives $D_{ij}u$ of *any* strong solution to (3) via Calderón–Zygmund singular integrals $\mathscr{K}f$ and their commutators $\mathscr{C}[a^{ij}, D_{ij}u] = a^{ij}\mathscr{K}D_{ij}u - \mathscr{K}(a^{ij}D_{ij}u)$. This leads to

$$\|D_{ij}u\|_{L^p(B_\rho)} \leq C\Big(\|\mathscr{K}f\|_{L^p(B_\rho)} + \|\mathscr{C}[a^{ij}, D_{ij}u]\|_{L^p(B_\rho)}\Big),$$

where B_ρ is a ball of radius ρ contained in Ω when dealing with the interior estimate, and B_ρ stands for a half-ball lying in the locally flatten Ω in the case of the boundary estimate. It follows from the Calderón–Zygmund theory of singular integral operators that $\|\mathscr{K}f\|_{L^p(B_\rho)} \leq C\|f\|_{L^p(B_\rho)}$, while

$$\|\mathscr{C}[a^{ij}, D_{ij}u]\|_{L^p(B_\rho)} \leq C\|a^{ij}\|_* \|D_{ij}u\|_{L^p(B_\rho)}.$$

At this point the vanishing property of the *VMO*-modulus of a^{ij}'s plays a crucial role that permits to make the commutator norm small enough if ρ is small, and thus to get a local version of (4). The global estimate (4) then follows by a standard procedure consisting in local flattering of $\partial\Omega$ and finite covering of Ω with small enough balls.

Furthermore, combining Theorem **2.2** with fixed-point arguments and the Aleksandrov–Bakel'man maximum principle, Chiarenza, Frasca and Longo succeeded to get also *regularization property* of the operator $\mathscr{L}$ in $W^{2,p}$-scales and *unique strong solvability* of (3) for each $f \in L^p(\Omega)$. Namely, the following result holds true.

Theorem 2.3. *Assume* $\partial\Omega \in C^{1,1}$, (1) *and* (2)*, and let* $p, q \in (1,\infty)$ *with* $q \leq p$. *Suppose further that* $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ *is such that* $\mathscr{L}u \in L^p(\Omega)$ *almost everywhere in* Ω.

Then $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

Moreover, for each $p \in (1,\infty)$ *and each* $f \in L^p(\Omega)$, *the Dirichlet problem* (3) *admits a unique solution* $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, *satisfying the bound*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

with a constant C *independent of* u.

The L^p-theory of linear elliptic *systems* with *VMO* principal coefficients has been developed by Chiarenza *et al.* (1994) (see also Palagachev and Softova 2006, for fine regularity results).

The results of Theorems **2.2** and **2.3** have been combined by Palagachev (1995) with the Leray–Schauder fixed point principle in order to prove strong solvability in $W^{2,n}(\Omega)$ of the *quasilinear* Dirichlet problem

$$\begin{cases} a^{ij}(x,u)D_{ij}u + b(x,u,Du) = 0 & \text{for a.a. } x \in \Omega, \\ u = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

with Carathéodory nonlinear terms, $\varphi \in W^{2,n}(\Omega)$ and where the principal coefficients a^{ij} are *VMO*-functions with respect to x, while $b(x,u,Du)$ grows at most quadratically with respect to the gradient.

The corresponding linear theory of *parabolic* operators with *VMO* principal coefficients has been developed by Bramanti and Cerutti (1993), while Softova (2003) provides strong solvability results of quasilinear Cauchy–Dirichlet problems.

The oblique derivative problem. Recalling that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $C^{1,1}$-smooth boundary, we set $\mathbf{n}(x)$ for the unit outward normal to $\partial\Omega$ at the point x, and let $\boldsymbol{\ell}(x)$ be a unit and Lipschitz continuous vector field defined on $\partial\Omega$ which is *strictly exterior* to Ω, that is,

$$\beta_0(x) := \mathbf{n}(x)\cdot\boldsymbol{\ell}(x) = \sum_{i=1}^{n} \mathbf{n}^i(x)\boldsymbol{\ell}^i(x) > 0 \quad \text{for all } x \in \partial\Omega. \tag{5}$$

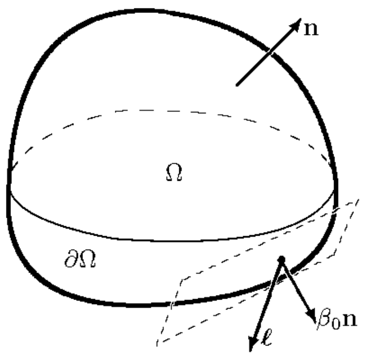


FIGURE 2. The domain Ω and the vector field $\boldsymbol{\ell}$ on $\partial\Omega$

Given $f \in L^p(\Omega)$ with $p \in (1,\infty)$ and a function $g\colon\ \partial\Omega \to \mathbb{R}$ belonging to the fractional Sobolev space $W^{1-1/p,p}(\partial\Omega)$, we consider now the *oblique derivative problem*

$$\begin{cases} \mathscr{L}u := a^{ij}(x)D_{ij}u = f(x) & \text{for almost all } x \in \Omega,\\ \mathscr{B}u := \partial_{\boldsymbol{\ell}}u + \gamma(x)u = g(x) & \text{in the sense of trace on } \partial\Omega, \end{cases} \tag{6}$$

where $\partial_{\boldsymbol{\ell}}$ stands for the directional derivative along the field $\boldsymbol{\ell}$.

The L^p-theory of oblique derivative problems (6) for elliptic operators with *VMO*-principal coefficients has been developed by Di Fazio and Palagachev (1996a) by proving the following global *a priori* estimate for the strong solutions to (3).

Theorem 2.4. *Let* $\partial\Omega \in C^{1,1}$, $\boldsymbol{\ell}$, $\gamma \in C^{0,1}(\partial\Omega)$, $p \in (1,\infty)$ *and assume* (1), (2) *and* (5). *Let* $u \in W^{2,p}(\Omega)$ *be a strong solution of the problem* (6) *with* $f \in L^p(\Omega)$ *and* $g \in W^{1-1/p,p}(\partial\Omega)$.

Then there exists a constant C, *depending only on* n, p, λ, $\operatorname{diam}\Omega$, $\partial\Omega$, $\boldsymbol{\ell}$, γ *and the VMO-moduli of the coefficients* a^{ij}, *such that*

$$\|u\|_{W^{2,p}(\Omega)} \le C\Big(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\partial\Omega)}\Big). \tag{7}$$

The interior local version of (7) follows from Chiarenza *et al.* (1991), while the local boundary estimate is derived on the base of an explicit representation formula for the second-order derivatives $D_{ij}u$ of the strong solution to (6). That formula contains the same singular integrals of Calderón–Zygmund type and their commutators as in the case of Dirichlet problem, plus additional, *non-singular* terms, due to the first-order boundary condition in (6). A crucial role in getting that representation formula is played by the strict obliqueness condition (5) that ensures non-degeneracy of the problem considered. That condition guarantees also the required L^p-estimate of the non-singular terms in the representation formula. For what concerns the singular ingredient, these are estimated in the same manner as in the papers by Chiarenza *et al.* (1991, 1993), leading this way to the desired bound (7).

Under the additional sign-condition on the coefficient γ,

$$\gamma(x) > 0 \quad \text{for all } x \in \partial\Omega, \tag{8}$$

regularizing property of the couple $(\mathscr{L}, \mathscr{B})$ and *Fredholmness* of (6) have been also proved by Di Fazio and Palagachev (1996a) on the base of the Aleksandrov–Bakel'man maximum principle.

Theorem 2.5. *Let $\partial\Omega \in C^{1,1}$, $\boldsymbol{\ell}$, $\gamma \in C^{0,1}(\partial\Omega)$ and assume* (1)*,* (2)*,* (5) *and* (8)*.*
Let $p,q \in (1,\infty)$ with $q \le p$, and suppose that $u \in W^{2,q}(\Omega)$ satisfies $\mathscr{L}u \in L^p(\Omega)$ almost everywhere in Ω and $\mathscr{B}u \in W^{1-1/p,p}(\partial\Omega)$ in the sense of trace on $\partial\Omega$. Then $u \in W^{2,p}(\Omega)$. Moreover, for each $p \in (1,\infty)$ and all $f \in L^p(\Omega)$, $g \in W^{1-1/p,p}(\partial\Omega)$ the oblique derivative problem (6) *possesses a unique solution $u \in W^{2,p}(\Omega)$ that satisfies the bound*

$$\|u\|_{W^{2,p}(\Omega)} \le C\Big(\|f\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}(\partial\Omega)}\Big)$$

with a constant C independent of u.

The results of Di Fazio and Palagachev (1996a) have been generalized by Maugeri and Palagachev (1998) to oblique derivative problem for general elliptic operators $a^{ij}(x)D_{ij} + b^i(x)D_i + c(x)$ with $a^{ij} \in VMO$ and $b^i, c \in L^r(\Omega)$ with $r > n$ if $p \le n$, $r = p$ when $p > n$, while oblique derivative problem for quasilinear elliptic equations $a^{ij}(x,u)D_{ij}u + b(x,u,Du) = 0$ with *VMO* principal coefficients has been studied by Di Fazio and Palagachev (1996b). We refer the reader to Softova (2000, 2011, 2013) for what concerns the regularity and solvability theory of linear and quasilinear *parabolic* operators with *VMO* coefficients.

In all these studied, the *strict obliqueness* (5) is crucial and it ensures ellipticity of the oblique derivative problem considered. If (5) fails then there exists a *non-empty* set $\mathscr{S} \subset \partial\Omega$ where the vector field $\boldsymbol{\ell}$ becomes *tangent* to $\partial\Omega$. What happens now is that (6) is no more a regular boundary value problem (see Popivanov and Palagachev 1997; Paneah 2000), and its properties depend essentially on the behaviour of the field $\boldsymbol{\ell}$ near the set $\mathscr{S}$ of tangency. In particular, new effects appear such as loss of smoothness of the solution, loss of Fredholmness, etc. Some partial results regarding tangential oblique derivative problems for elliptic operators with discontinuous coefficients have been obtained by Maugeri *et al.* (1998, 2001), and also by Palagachev (2005, 2006, 2008a,b), while the parabolic case has been considered by Softova (2004).

3. The Venttsel problem

Recall that $\partial\Omega \in C^{1,1}$ and let $f\colon\ \Omega \to \mathbb{R}$ and $g\colon\ \partial\Omega \to \mathbb{R}$ be two arbitrary functions belonging respectively to $L^p(\Omega)$ and $L^p(\partial\Omega)$ with $p > 1$. Consider the *Venttsel problem*

$$\begin{cases} \mathscr{L}u := \qquad a^{ij}(x)D_{ij}u = f(x) & \text{for almost all } x \in \Omega, \\ \mathscr{V}u := \alpha^{ij}(x)d_{ij} + \beta^i(x)d_i + \beta_0(x)\partial_{\mathbf{n}}u = g(x) & \text{for almost all } x \in \partial\Omega, \end{cases} \tag{9}$$

where $\partial_{\mathbf{n}}$ stands for the directional derivative along the outward normal $\mathbf{n}$ to $\partial\Omega$, $d_{ij} := d_i d_j$ and d_i are the components of the tangential gradient $d = (d_1, \ldots, d_n)$ to $\partial\Omega$, given by

$$d_i = D_i - \sum_{j=1}^{n} \mathbf{n}^i\mathbf{n}^j D_j, \quad i = 1, \ldots, n.$$

The coefficients of the uniformly elliptic operator $\mathscr{L}$ are supposed to be $VMO(\Omega)$-functions, that is, these satisfy (1) and (2), while α^{ij}, β^i and β_0 are measurable functions defined on the boundary $\partial\Omega$. We will assume that the Venttsel operator $\mathscr{V}$ is *uniformly elliptic* one with *VMO* principal coefficients,

$$\begin{cases} \lambda^{-1}|\xi'|^2 \le \alpha^{ij}(x)\xi'_i\xi'_j \le \lambda|\xi'|^2 & \text{for a.a. } x \in \partial\Omega,\ \forall \xi' \in \mathbb{R}^n,\ \xi' \perp \mathbf{n}(x), \\ \alpha^{ij}(x) = \alpha^{ji}(x) & \text{for a.a. } x \in \Omega, \\ \alpha^{ij} \in VMO(\partial\Omega). \end{cases} \tag{10}$$

For what concerns the lower-order coefficients of the operator $\mathscr{V}$, we set
$\boldsymbol{\beta}(x) := \big(\beta^1(x), \ldots, \beta^n(x)\big)$ and suppose

$$\begin{cases} |\boldsymbol{\beta}| \in L^{\max\{p,n-1\}}(\partial\Omega) & \text{if} \quad p \neq n-1, \\ |\boldsymbol{\beta}|\big(\log(1+|\boldsymbol{\beta}|)\big)^{1-1/(n-1)} \in L^{n-1}(\partial\Omega) & \text{if} \quad p = n-1, \end{cases} \tag{11}$$

together with

$$\begin{cases} \beta_0 \in L^p(\partial\Omega) & \text{if} \quad p > n, \\ \beta_0\big(\log(1+|\beta_0|)\big)^{1-1/n} \in L^n(\partial\Omega) & \text{if} \quad p = n, \\ \beta_0 \in L^{p(n-1)/(p-1)}(\partial\Omega) & \text{if} \quad p < n. \end{cases} \tag{12}$$

The strong solutions of (9) will be taken in the space $V^{2,p}(\Omega)$ consisting of all functions $u \in W^{2,p}(\Omega)$ with boundary traces in $W^{2,p}(\partial\Omega)$, and the norm in $V^{2,p}(\Omega)$ is naturally given by

$$\|u\|_{V^{2,p}(\Omega)} := \|u\|_{W^{2,p}(\Omega)} + \|u\|_{W^{2,p}(\partial\Omega)}.$$

Our main result provides an *a priori* estimate for any strong solution to the Venttsel problem (9) in terms of the data of the problem.

Theorem 3.1. *Let $p > 1$ and $\partial\Omega \in C^{1,1}$. Assume* (1), (2), (10), (11) *and* (12)*, and let $u \in V^{2,p}(\Omega)$ be a strong solution of the problem* (9) *with $f \in L^p(\Omega)$ and $g \in L^p(\partial\Omega)$.*

Then there exists a constant C, depending only on n, p, λ, $\operatorname{diam}\Omega$, $\partial\Omega$, on the VMO-moduli of the coefficients a^{ij} and α^{ij}, and on the moduli of continuity of the functions $|\boldsymbol{\beta}|$ and β_0 in the functional spaces corresponding to (11) *and* (12)*, such that*

$$\|u\|_{V^{2,p}(\Omega)} \le C\Big(\|u\|_{L^p(\Omega)} + \|u\|_{L^p(\partial\Omega)} + \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\partial\Omega)}\Big). \tag{13}$$

Remark 3.2. We refer the reader to Apushkinskaya *et al.* (2019) where a general *a priori* estimate is proved for the strong solutions $u \in W^{2,p}(\Omega)$ to (9) with traces in $W^{2,q}(\partial\Omega)$ and in the case when the operators $\mathscr{L}$ and $\mathscr{V}$ contain also lower-order terms. Moreover, on the base of Aleksandrov–Bakel'man type maximum principle strong solvability and regularity theories are developed, and the results are applied to the *quasilinear* Venttsel problem.

The main ingredients of the **proof** of Theorem **3.1** are the localized version of Theorem **2.2** and the following extension result which is a modification of Theorem 6.1 discussed by Apushkinskaya and Nazarov (1995), that allows to extend Sobolev functions defined on $\partial\Omega$ to Sobolev functions in the whole Ω.

Lemma 3.3. *Let $p > 1$ and $\partial\Omega \in C^{1,1}$.*
Then there exists an extension operator

$$\mathrm{E}\colon\ W^{2,p}(\partial\Omega) \to W^{2,p}(\Omega)$$

such that

$$\|\mathrm{E}u\|_{W^{2,p}(\Omega)} \leq C\|u\|_{W^{2,p}(\partial\Omega)}, \tag{14}$$

with a constant C depending only on n, p and the regularity $\partial\Omega$.

Proof. Since $\partial\Omega \in C^{1,1}$, at each $x_0 \in \partial\Omega$ there exists a Cartesian coordinate system centered at x_0 such that $\partial\Omega$ is tangent to the hyperplane $\{x_n = 0\}$ at x_0, and the part of $\partial\Omega$ lying in the neighborhood $U_R = \left\{x = (x', x_n) \in \mathbb{R}^n\colon\ |x'| < R, |x_n| < R\right\}$ can be given by the equation $x_n = \omega(x')$ with $\omega \in C^{1,1}(B'_R)$ where B'_R is the $(n-1)$-dimensional ball $\left\{x' \in \mathbb{R}^{n-1}\colon\ |x'| < R\right\}$. Moreover, the radius R can be chosen one and the same for all points $x_0 \in \partial\Omega$. This way, the change of variables $y' = x'$, $y_n = x_n - \omega(x')$ maps $\partial\Omega \cap U_R$ into the ball B'_R lying in the hyperplane $\{y_n = 0\}$. Actually, that change of variables induces the "shifting" operator $u(x) \to u(y)$ that acts continuously from $W^{2,p}(\partial\Omega \cap U_R)$ into $W^{2,p}(B'_R)$ and, without loss of generality, we may suppose that $u \in W_0^{2,p}(B'_R)$.

The next step is to construct an extension operator from a flat boundary surface to a boundary strip acting continuously from $W_0^{2,p}(B'_R)$ into the space $W^{2,p}(B'_R \times (0,R))$. Precisely, Triebel (1978, formula 2.8.1/18) yields the embedding $W^{2,p}(\mathbb{R}^{n-1}) \to \mathbf{B}_{p,p}^{2-1/p}(\mathbb{R}^{n-1})$, while Triebel (1978, Theorem 2.9.3 (a)) provides the extension $\mathbf{B}_{p,p}^{2-1/p}(\mathbb{R}^{n-1}) \to W^{2,p}(\mathbb{R}^n)$ with the Besov space $\mathbf{B}_{p,p}^{2-1/p}(\mathbb{R}^{n-1})$. Indeed, the norm of the extension operator is bounded in terms only of n, p and R, and, multiplying by a suitable cut-off function, we may ensure that

i) the extended function equals 0 for $|x_n| > R/2$;
ii) if the initial function equals 0 for $|x'| > R/2$, then the extended one is 0 for $|x'| > 3R/4$.

These arguments permit to construct local extension operators that map $W^{2,p}(\partial\Omega)$-functions with small enough support into $W^{2,p}(\mathbb{R}^n)$-functions that vanish when $|x_n| > R/2$ and $|x'| > 3R/4$. Finally, the desired operator E is built from the local operators via appropriate partition of unity. □

We start the proof of Theorem **3.1** with considering the simplest case $\beta_0 \equiv 0$ on $\partial\Omega$. Then the boundary equation in (9) becomes an autonomous one, that is,

$$\alpha^{ij}(x)d_{ij}u = g(x) - \beta^i(x)d_iu \quad \text{a.e. on } \partial\Omega.$$

We invoke now the standard procedure of finite covering of $\partial\Omega$ by balls, local flattening of $\partial\Omega$ and employing there the *a priori* estimate from Theorem **2.2**. After that, putting these estimates together with the aid of partition of unity, we get

$$\|u\|_{W^{2,p}(\partial\Omega)} \leq C_1 \Big(\|u\|_{L^p(\partial\Omega)} + \|\beta^i(x) d_i u\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)} \Big) \tag{15}$$

with C_1 depending on n, p, λ, the regularity of $\partial\Omega$ and by the *VMO*-moduli of the coefficients α^{ij}. In order to estimate the term $\|\beta^i(x)d_iu\|_{L^p(\partial\Omega)}$, we consider three possible cases.

Case 1: $p > n-1$. The Sobolev space $W^{2,p}(\partial\Omega)$ is compactly embedded into $C^1(\partial\Omega)$ whence we have the estimate

$$\begin{aligned} \|\beta^i d_i u\|_{L^p(\partial\Omega)} &\leq \|\boldsymbol{\beta}\|_{L^p(\partial\Omega)} \|du\|_{L^\infty(\partial\Omega)} \\ &\leq \varepsilon \|\boldsymbol{\beta}\|_{L^p(\partial\Omega)} \|u\|_{W^{2,p}(\partial\Omega)} + C_2(\varepsilon) \|\boldsymbol{\beta}\|_{L^p(\partial\Omega)} \|u\|_{L^p(\partial\Omega)}, \end{aligned}$$

for an arbitrary $\varepsilon > 0$ and where $C_2(\varepsilon)$ depends also on n, p, $\operatorname{diam}\Omega$ and the regularity of $\partial\Omega$.

Case 2: $p < n-1$. Now $|\boldsymbol{\beta}| \in L^{n-1}(\partial\Omega)$ by (11) and we use the well-known idea (see, for instance, Ladyzhenskaya and Ural'tseva 1968, Chapter III, § 8, Remark 8.2)) to decompose $|\boldsymbol{\beta}|$ into the sum

$$|\boldsymbol{\beta}(x)| = \varphi_1(x) + \varphi_2(x),$$

where $\varphi_1 \in L^{n-1}(\partial\Omega)$ and $\|\varphi_1\|_{n-1,\partial\Omega} \leq \delta$ with a small $\delta > 0$ to be chosen later, while $\varphi_2 \in L^\infty(\partial\Omega)$. It is worth noting that $\|\varphi_2\|_{L^\infty(\partial\Omega)}$ is also determined by δ and by the modulus of continuity of $|\boldsymbol{\beta}|$ in $L^{n-1}(\partial\Omega)$. At that point the Hölder inequality gives

$$\|\beta^i d_i u\|_{L^p(\partial\Omega)} \leq \|\varphi_1\|_{L^{n-1}(\partial\Omega)} \|du\|_{L^{p^*}(\partial\Omega)} + \|\varphi_2\|_{L^\infty(\partial\Omega)} \|du\|_{L^p(\partial\Omega)},$$

where $p^* = p(n-1)/(n-1-p)$. The first term above is estimated with the help of the Sobolev inequality on $\partial\Omega$, while the upper bound for the second term follows from the compact embedding of $W^{2,p}(\partial\Omega)$ into $W^{1,p}(\partial\Omega)$. Therefore,

$$\begin{aligned} \|du\|_{L^{p^*}(\partial\Omega)} &\leq C \|u\|_{W^{2,p}(\partial\Omega)}, \\ \|du\|_{L^p(\partial\Omega)} &\leq \varepsilon \|u\|_{W^{2,p}(\partial\Omega)} + C_3(\varepsilon) \|u\|_{L^p(\partial\Omega)} \end{aligned}$$

for each $\varepsilon > 0$, and we choose $\delta > 0$ small enough in order to get

$$\|\beta^i d_i u\|_{L^p(\partial\Omega)} \leq \varepsilon \big(1 + \|\varphi_2\|_{L^\infty(\partial\Omega)}\big) \|u\|_{W^{2,p}(\partial\Omega)} + C_3(\varepsilon) \|\varphi_2\|_{L^\infty(\partial\Omega)} \|u\|_{L^p(\partial\Omega)}, \tag{16}$$

with $C_3(\varepsilon)$ depending on the same parameters as $C_2(\varepsilon)$.

Case 3: $p = n-1$. The procedure is similar to that in the previous case, with the difference that the Yudovich–Pohozhaev embedding theorem into the Orlicz space must be used now,

$$W^{1,n-1}(\partial\Omega) \hookrightarrow L^{\psi}(\partial\Omega) \quad \text{with} \quad \psi(t) = \mathrm{e}^{|t|^{(n-1)/(n-2)}} - 1$$

(see, *e.g.*, Besov *et al.* 1978, Sections 10.5-10.6). Thus,

$$|du|^{n-1} \in L^{\Psi}(\partial\Omega) \quad \text{with} \quad \Psi(t) \sim \mathrm{e}^{|t|^{1/(n-2)}} \text{ as } |t| \to \infty$$

and we observe that in the considered case the assumption (11) ensures that $|\boldsymbol{\beta}|^{n-1}$ belongs to the Orlicz space $L^{\Psi^*}(\partial\Omega)$ dual to $L^{\Psi}(\partial\Omega)$ (see Krasnosel'skiĭ and Rutickiĭ 1961, Section 14). As a result we get again the estimate (16), but now $\|\varphi_2\|_{L^\infty(\partial\Omega)}$ is determined by the modulus of continuity of $|\boldsymbol{\beta}|$ in the Orlicz space related to (11).

Summarizing, choosing suitably $\varepsilon > 0$, we have

$$\|\beta^i d_i u\|_{L^p(\partial\Omega)} \leq \frac{1}{2C_1}\|u\|_{W^{2,p}(\partial\Omega)} + C_4\|u\|_{L^p(\partial\Omega)} \tag{17}$$

in the all three cases, where C_1 is the constant from (15), while C_4 is determined by n, p, $\operatorname{diam}\Omega$, the regularity of $\partial\Omega$ and on the moduli of continuity of $|\boldsymbol{\beta}|$ in corresponding functional spaces defined by conditions (11).

Substituting (17) into (15) yields

$$\|u\|_{W^{2,p}(\partial\Omega)} \leq C\Big(\|u\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}\Big).$$

We use now this bound and invoke Lemma **3.3** that ensures existence of a function $\mathrm{E}\big(u|_{\partial\Omega}\big) \in W^{2,p}(\Omega)$ such that

$$\big\|\mathrm{E}\big(u|_{\partial\Omega}\big)\big\|_{W^{2,p}(\Omega)} \leq C\|u\|_{W^{2,p}(\partial\Omega)} \leq C\Big(\|u\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}\Big). \tag{18}$$

The function $u - \mathrm{E}\big(u|_{\partial\Omega}\big) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ solves the equation

$$a^{ij}(x)D_{ij}\Big(u - \mathrm{E}\big(u|_{\partial\Omega}\big)\Big) = f(x) - a^{ij}(x)D_{ij}\mathrm{E}\big(u|_{\partial\Omega}\big) \quad \text{a.e. in } \Omega$$

and, according to Theorem **2.2**,

$$\big\|u - \mathrm{E}\big(u|_{\partial\Omega}\big)\big\|_{W^{2,p}(\Omega)} \leq C\Big(\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + \big\|a^{ij}(x)D_{ij}\mathrm{E}\big(u|_{\partial\Omega}\big)\big\|_{L^p(\Omega)}\Big)$$

which, together with (18) and (15), gives the claim (13) in the case when $\beta_0 \equiv 0$ on $\partial\Omega$.

In the general case $\beta_0 \not\equiv 0$, the boundary condition in (9) rewrites into

$$\alpha^{ij}(x)d_{ij}u = g(x) - \beta^i(x)d_i u - \beta_0(x)\partial_{\mathbf{n}}u \quad \text{a.e. on } \partial\Omega$$

and, as before Theorem **2.2** yields

$$\|u\|_{W^{2,p}(\partial\Omega)} \leq C_1\Big(\|u\|_{L^p(\partial\Omega)} + \|\beta^i(x)d_i u\|_{L^p(\partial\Omega)} + \|\beta_0(x)\partial_{\mathbf{n}}u\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}\Big)$$

with the same constant C_1 as in (15). The term $\|\beta^i(x)d_i u\|_{L^p(\partial\Omega)}$ estimates in the same manner as before, and using (17) we get

$$\|u\|_{W^{2,p}(\partial\Omega)} \leq C_1'\Big(\|u\|_{L^p(\partial\Omega)} + \|\beta_0(x)\partial_{\mathbf{n}}u\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}\Big) \tag{19}$$

Starting from $u|_{\partial\Omega} \in W^{2,p}(\partial\Omega)$ the operator constructed in Lemma **3.3** defines a function $\mathrm{E}\big(u|_{\partial\Omega}\big) \in W^{2,p}(\Omega)$ such that

$$\begin{aligned}\big\|\mathrm{E}\big(u|_{\partial\Omega}\big)\big\|_{W^{2,p}(\Omega)} &\leq C'\|u\|_{W^{2,p}(\partial\Omega)}\\ &\leq C'\Big(\|u\|_{L^p(\partial\Omega)} + \|\beta_0(x)\partial_{\mathbf{n}}u\|_{L^p(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}\Big)\end{aligned}$$

by means of (19) and (14). Moreover, $u-\mathrm{E}\big(u|_{\partial\Omega}\big)\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ is a strong solution to the Dirichlet problem

$$\begin{cases} a^{ij}(x)D_{ij}\Big(u-\mathrm{E}\big(u|_{\partial\Omega}\big)\Big)u=f(x)-a^{ij}(x)D_{ij}\mathrm{E}\big(u|_{\partial\Omega}\big) & \text{a.e. in } \Omega,\\ u-\mathrm{E}\big(u|_{\partial\Omega}\big)=0 & \text{on } \partial\Omega, \end{cases}$$

whence Theorem **2.2** yields

$$\big\|u-\mathrm{E}\big(u|_{\partial\Omega}\big)\big\|_{W^{2,p}(\Omega)}\le C\Big(\|u\|_{L^p(\Omega)}+\|f\|_{L^p(\Omega)}+\big\|a^{ij}(x)D_{ij}\mathrm{E}\big(u|_{\partial\Omega}\big)\big\|_{L^p(\Omega)}\Big)$$

and therefore

$$\|u\|_{W^{2,p}(\Omega)}\le C_2'\Big(\|u\|_{L^p(\Omega)}+\|u\|_{L^p(\partial\Omega)}+\|f\|_{L^p(\Omega)}+\|g\|_{L^p(\partial\Omega)} +\|\beta_0(x)\partial_{\mathbf{n}}u\|_{L^p(\partial\Omega)}\Big) \tag{20}$$

with C_2' depending on the same quantities as C_1 above and also on $\operatorname{diam}\Omega$ and on the *VMO*-moduli of the coefficients a^{ij}.

To get the desired bound (13), it remains to estimate the term $\|\beta_0(x)\partial_{\mathbf{n}}u\|_{L^p(\partial\Omega)}$ and this is done running the above procedure that led to (17). Precisely, $\partial_{\mathbf{n}}u$ is a $W^{1,p}$-smooth in a neighbourhood of $\partial\Omega$ and we use the embedding $W^{1,p}(\Omega)\hookrightarrow C^0(\overline{\Omega})$ when $p>n$, and the trace embeddings of $W_p^1(\Omega)$ into $L^{p^*(n-1)/n}(\partial\Omega)$ with $p^*=p(n-1)/(n-1-p)$ when $p<n$ and into the Orlicz space $L^{\psi}(\partial\Omega)$ with $\psi(t)=\mathrm{e}^{|t|^{n/(n-1)}}-1$ when $p=n$ (see Besov *et al.* 1978, Sections 10.5-10.6). Thus, in all the three cases we get

$$\|\beta_0\partial_{\mathbf{n}}u\|_{L^p(\partial\Omega)}\le\frac{1}{2C_2'}\|u\|_{W^{2,p}(\Omega)}+C_3'\|u\|_{L^p(\Omega)}, \tag{21}$$

where C_2' is the constant from (20), while C_3' depends on n, p, $\operatorname{diam}\Omega$, the regularity of $\partial\Omega$ and on the modulus of continuity of $|\beta_0|$ in the corresponding functional spaces appearing in (12).

The desired estimate (13) follows by employing (21) into (20) and (19) and this completes the proof of Theorem **3.1**.

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