DOI:10.1478/C1A0802007

Atti dell'Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
Vol. LXXXVI, C1A0802007 (2008)
Adunanza del 29 novembre 2007

STRUCTURES ON THE SPACE OF FINANCIAL EVENTS

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ABSTRACT. In this paper we endow the space of financial events with some structures, each of them represents basic facts of financial mathematics. We introduce a preordered structure, that we shall call the usual preorder of the financial events plane, and an algebraic structure, that we call the usual linearoid structure of the financial events plane. The algebraic structures introduced are not void of properties: the usual addition will confer to the space a gruppoid structure and the multiplication by scalars will be a law of action associative and distributive with respect to the addition. We shall prove that these structures are compatible among them and with the standard topology of the plane. Then we show the possibility of defining new (economically relevant) preorders by the use of capitalization factors and that there is a manner (the conjunction) to obtain the usual preorder from infinite continuous families of these new preorders induced by a capitalization factor.

1. Introduction

In this paper we formalize the fundamental idea of financial mathematics: "a capital is valuable if and only if it is referred to an instant of time". Many authors (see [1–4] and others similar) consider the concept of financial event, very few authors define it rigorously (as ordered pair time-capital) and none considers the set of all the financial events, that in our argumentation takes a principal role. There is a deep motivation (not simple to understand) of this lack in the literature: until now, this set was *void of structures*. The main original goal of this paper is to endow the set of all financial events with some canonical structures. In section 1 we define the natural preorders of financial events and prove that are induced by vector-valued functions. In section 2 we study the properties of the usual preorders of financial events and the set of upper and lower bounds of a financial event with respect to these preorders. In section 3 we study the compatibility of the preorders with the usual topology of the plane. In section 4 we prove that the preorders are transitive closure of a relation that we call the elementary confrontation of financial events (that sometimes appeared in the literature). In section 5 we show that the usual preorders are refined by each preorder induced by a separable capitalization factor. In section 6 we introduce what we call the usual operations on the space of financial events studying the principal properties of the associated algebraic structures.

2. Canonical preorders on the financial events plane

Let us formalize the concept of financial event.

Definition (of financial event). *The* ***plane of financial events*** *is the usual cartesian plane $\mathbb{R}^2$. It is interpreted as the cartesian product of a time-axis and of a capital-axis. Every pair (t, C) belonging to this plane is called a* ***financial event of time*** t ***and capital*** C*. Each event with positive (resp. non-negative) capital is said a* ***strict credit*** *(resp. a* ***weak credit****), every event with negative (resp. non-positive) capital is said a* ***strict debt*** *(resp. a* ***weak debt****), each event with capital equal 0 is said a* ***null event****.*

The first structure that we introduce in the plane of financial events is a preorder.

Definition (of the usual preorder of financial events). *We call* ***usual lower preorder of the financial events plane*** $\mathbb{R}^2$ *the binary relation $\leq_{\mathrm{fe}}$ such that, for each pair of events (e_0, e) of the plane, the relation $e_0 \leq_{\mathrm{fe}} e$ holds if and only if at least one of the following conditions holds true, where $e_0 = (t_0, c_0)$ and $e = (t, c)$,*
a) e_0 and e are strict credits with $t_0 \geq t$ and $c_0 \leq c$;
b) e_0 is a weak debt and e is a weak credit;
c) e_0 and e are strict debts with $t_0 \leq t$ and $c_0 \leq c$.
We call ***usual upper preorder of the financial events plane*** *the binary relation $\geq_{\mathrm{fe}}$ opposite of the usual lower preorder of the financial events plane.*

Actually the usual preorder of the financial events plane is induced by a vector-valued function from the plane into itself. We recall that, if X and X' are two non empty sets, if R' is a binary relation on X' and if $f : X \to X'$ is a function, the relation R on the set X for which xRy holds if and only if $f(x)R'f(y)$ is said the reciprocal image of the relation R' by the function f, or the binary relation induced by f on the set X with respect to R.

Theorem 1. *The usual lower preorder of the financial events is the reciprocal image of the usual lower order of the plane by means of the function*

$$f : \mathbb{R}^2 \to \mathbb{R}^2 : f(t, c) = sgn(c)(a^{-t}, a^{sgn(c)c}),$$

where a is any real number strictly greater then 1. In other terms, the relation $\leq_{fe}$ is induced by the vector-valued function f with respect to the usual lower order of the plane. Consequently the usual lower preorder is a preorder. Moreover $\leq_{fe}$ is an order on the parts of the plane in which f is injective.

Proof. Let e_0 and e be two events. (a) If e_0 and e are both credits, then $f(e_0) = (a^{-t_0}, a^{c_0})$ and $f(e) = (a^{-t}, a^c)$; so the usual vectorial inequality $f(e_0) \leq f(e)$ is equivalent to the inequality $(-t_0, c_0) \leq (-t, c)$, and this one means, by definition, $e_0 \leq_{\mathrm{fe}} e$. (b) If e_0 is weak debt and e is a weak credit, then $f(e_0) = (0, 0)$ or $f(e_0) = -(a^{t_0}, a^{c_0})$ and $f(e) = (a^{-t}, a^c)$ or $f(e) = (0, 0)$; so the usual vectorial inequality $f(e_0) \leq f(e)$ is true, and this means, by definition, $e_0 \leq_{\mathrm{fe}} e$. (c) If e_0 and e are both strict debts, then $f(e_0) = -(a^{-t_0}, a^{-c_0})$ and $f(e) = -(a^{-t}, a^{-c})$; so the usual vectorial inequality $f(e_0) \leq f(e)$ is equivalent to the inequality $(-t_0, -c_0) \geq (-t, -c)$, and this one means, by definition, $e_0 \leq_{\mathrm{fe}} e$. We proved that the lower financial order is the reciprocal image of the lower usual order of the plane; since the reciprocal image of a preorder is a preorder we can conclude that $\leq_{fe}$ is a preorder. Recalling now that the reciprocal image of an order is an order when f is injective we complete the proof. ■

Remark. The function of the above theorem is not injective, in fact it sends all the null events into $(0, 0)$. Nevertheless, for instance, f is injective on the plane without the null events; more specifically, the maximal parts of the plane in which f is injective are the parts obtaining removing all the null events but one.

We conclude the section with another characterization of the usual financial preorders.

Theorem 2. *Let $\geq_{fe}$ the usual preorder of the financial events. Then the relation $(t_0, c_0) \geq_{fe} (t, c)$ is equivalent to the relation $(p(c_0, c)t_0, c_0) \geq (p(c_0, c)t, c)$, where p is the continuous function defined, for every bi-capital (c_0, c), by*

$$p(c_0, c) = -\left(c_0 \left|c\right| + \left|c_0\right| c\right).$$

Proof. If (t, c) and (t_0, c_0) are two strict credits, the inequality $(t_0, c_0) \geq_{fe} (t, c)$ is equivalent to the inequality $(-t_0, c_0) \geq (-t, c)$, that is equivalent to

$$(p(c_0, c)t_0, c_0) \geq (p(c_0, c)t, c),$$

since the weight $p(c_0, c)$ is the strictly negative number $-\left(c_0 \left|c\right| + \left|c_0\right| c\right)$. If (t, c) and (t_0, c_0) are two strict debts, the inequality $(t_0, c_0) \geq_{fe} (t, c)$ is equivalent to the inequality $(t_0, c_0) \geq (t, c)$, that is equivalent to $(p(c_0, c)t_0, c_0) \geq (p(c_0, c)t, c)$, since the weight $p(c_0, c)$ is the strictly positive number $-\left(c_0 \left|c\right| + \left|c_0\right| c\right)$. If (t, c) is a weak debt and (t_0, c_0) is a weak credit, the inequality $(t_0, c_0) \geq_{fe} (t, c)$ is always true as the following

$$(p(c_0, c)t_0, c_0) \geq (p(c_0, c)t, c),$$

since the weight $p(c_0, c)$ is zero. The claim is proved. ■

3. Properties of the preorder of financial events

In this section we study the basic properties of the preordered space $(\mathbb{R}^2, \leq_{fe})$.

Theorem 3. *Let $\geq_{fe}$ the usual preorder of the financial events. Then*
1) the preorder $\geq_{fe}$ is not a conic-preorder;
2) the preorder $\geq_{fe}$ is not an order;
3) the preorder $\geq_{fe}$ is not total;
4) the preordered set $(\mathbb{R}^2, \leq_{fe})$ is a lattice.

Proof. (1). In fact, the set of upper bounds of a strict debt is never a cone. (2). In fact any two different null events are indifferent. (3). Indeed the two events $(1, 1)$ and $(2, 2)$ are not comparable. (4). Let $e_0 = (t_0, c_0)$ and $e = (t, c)$ be two strict credits then $\sup(e_0, e)$ is the event $(t_0 \wedge t, c_0 \vee c)$ and $\inf(e_0, e)$ is $(t_0 \vee t, c_0 \wedge c)$. Let, now, e_0 and e be two strict credits then $\sup(e_0, e)$ is the event $(t_0 \vee t, c_0 \vee c)$ and $\inf(e_0, e)$ is the event

$(t_0 \wedge t, c_0 \wedge c)$. Let, now, e_0 and e be two null events then $\sup(e_0, e)$ and $\inf(e_0, e)$ are both the entire line of null events. Let, e_0 be a strict credit (debt) and e be a weak debt (credit) then $\sup(e_0, e) = e_0$ and $\inf(e_0, e) = e$ ($\sup(e_0, e) = e$ and $\inf(e_0, e) = e_0$). ■

Theorem 4. *Let $\geq_{fe}$ the usual preorder of the financial events. Then*
1) the restrictions of $\leq_{fe}$ to the open half plane of strict credits is the costs-benefit order;
2) the restriction of $\leq_{fe}$ to the open half-plane of strict debts coincides with the usual order.

Proof. This follows immediately from the definition. ■

With not too work it is possible to prove the following.

Theorem 5. *The following assertions hold true:*
1) if e_0 is a credit $e_0 \leq_{\mathrm{fe}} e$ if and only if e is in the translation by e_0 of the convex cone generated by the pair $(-e_1, e_2)$, where (e_1, e_2) is the canonical basis;
2) if e_0 is a credit $e_0 \leq_{\mathrm{fe}} e$ if and only if e is in the translation by e_0 of the convex cone generated by the pair $(e_1, -e_2)$, where (e_1, e_2) is the canonical basis, or (inclusive) in the half-plane of debts;
3) if e_0 is a null-event $e_0 \leq_{\mathrm{fe}} e$ if and only if e is a weak credit and $e_0 \leq_{\mathrm{fe}} e$ if and only if e is a weak debt;
4) if e_0 is a debt $e_0 \leq_{\mathrm{fe}} e$ if and only if e is in the translation by e_0 of the convex cone generated by the pair $-(e_1, e_2)$, where (e_1, e_2) is the canonical basis;
5) if e_0 is a credit $e_0 \leq_{\mathrm{fe}} e$ if and only if e is in the translation by e_0 of the convex cone generated by the pair (e_1, e_2), where (e_1, e_2) is the canonical basis, or (inclusive) in the half-plane of credits.

Application. The canonical preorder of the financial events is the natural criterion to use when the financial market is endowed with a positive rate of interest, possibly a variable one but non-negative. In fact, in that case a capital gives a non-negative interest. We can reduce the decision problems, in the financial events plane, when we know that the rate of interest is non-negative, even if variable. For example, let S be the part of the plane convex envelope of the points $A = (-2, -1)$, $B = (2, 1)$ and $C = (1, 2)$, our aim is to find the best choices and the worst ones, note that the problem is an infinite problem and its dimension is 2. It is evident that the best choices must be in the part of S contained in the half-plane of credits and the worst ones in the part of S contained in the half-plane of debts. The problem has no upper solution, since the shadow maximum is the event $H = (-1, 2)$, not belonging to S, the upper Pareto boundary is the segment $]M, C]$, where M is the event $(-1, 0)$. When the non-negative rate shall be fixed in the market, the upper solution of the problem, with respect to the valuation induced by the rate, must belong to this upper boundary: the upper optimization problem is reduced to a problem of dimension 1. The worst decision is the event A, that is the infimum of S with respect to the canonical preorder: at every non-negative rate the worst solution shall be A.

4. Euclidean topology and the preorder of financial events

This section is devoted to the following theorem.

Theorem 6. *Let $\geq_{fe}$ the usual preorder of the financial events. Then, the preorder $\geq_{fe}$ is both upper and lower compatible with the usual topology of the plane.*

To prove the above theorem, first we formulate and prove the following general result.

Theorem 7. *Let (X', τ') be a topological space and let $(X, \tau, \leq)$ be an upper (lower) topological preordered space. Let f be a continuous function from X' to X, with respect to the pair of topologies (τ', τ), and let $\leq'$ be the reciprocal image of $\leq$. Then, the preorder $\leq'$ is upper (lower) compatible with the topology τ'.*

Proof. It follows immediately from the fact that the preorder $\leq'$ is induced by a (τ', τ)-continuous function f and that the preorder $\leq$ is compatible with the topology τ. Indeed, for instance, the set of upper-bounds of an event e, i.e. the interval $[e, \rightarrow[_{\leq'}$ is the reciprocal image by f of the usual interval $[f(e), \rightarrow[_{\leq}$, by definition of induced preorder. Now, the usual interval $[f(e), \rightarrow[_{\leq}$ is closed, since the order $\leq$ is upper compatible with the topology τ, and then its reciprocal image is closed too. ■

The above theorem has an immediate consequence, that is a sub case of theorem 6.

Corollary. *The preorder $\leq_{fe}$ is compatible with the topology induced by the standard topology of the plane on the complement of the set of null events.*

Remark. We can not deduce theorem 6 from theorem 7, since the function inducing the usual preorder is not continuous on the entire plane. This is the interest of theorem 6: although it is induced by a discontinuous function the usual preorder is both lower and upper compatible with the usual topology of the plane.

Proof of theorem 6. Let e be an event of the plane, we shall prove that the set of its upper bounds is closed. If e is a strict credit, it follows from the preceding corollary. If e is a null event the set of upper bounds of e is the half-plane of weak credits that is closed. If e is a strict debt then the set of upper bounds of e is the union of the cone of upper bounds of e in the half-plane of strict credits (that is not closed with respect to the usual topology) with the half plane of weak credits. This last union is closed since the second set is closed and contains the part of the boundary of the first set that is not included in the first itself. ■

5. The preorder of financial events as transitive closure

We can consider on the financial events plane two binary relations more "elementary" than the usual preorders, as the following definition shows.

Definition (of elementary confrontation of financial events). *We call **elementary lower confrontation of the financial events plane** the binary relation $\leq_{\mathrm{fe}}^{el}$ on $\mathbb{R}^2$ such that, for each pair of events (e_0, e) of the plane, the relation $e_0 \leq_{\mathrm{fe}}^{el} e$ holds if and only if at least one of the following conditions holds true, where $e_0 = (t_0, c_0)$ and $e = (t, c)$,*
a) e_0 and e are weak credits with ($t_0 \geq t$ and $c_0 = c$);
b) e_0 is a weak debt and e is a weak credit with $t_0 = t$;
c) e_0 and e are weak debts with ($t_0 \leq t$ and $c_0 = c$);
d) ($t_0 = t$ and $c_0 \leq c$).
*We call **elementary upper confrontation of the financial events plane** the binary relation $\geq_{\mathrm{fe}}$ opposite of the elementary lower confrontation of the financial events plane.*

Remark. The graph of this relation is contained in that of $\leq_{fe}$ but the elementary confrontation (that is not a preorder) can generate the preorder $\leq_{fe}$, as we will show in this section.

First we recall the concept of transitive closure of a relation.

Definition (of transitive closure). *The transitive closure of a binary relation R on a set X is the smallest transitive relation on X containing R.*

The above definition is justified by the following proposition.

Proposition (existence of the smallest transitive relation containing a given relation). *For any relation R on a non empty set X, there exists the smallest transitive relation on X containing R.*

Proof. Note that the intersection of any family of transitive relations is transitive. Furthermore, there exists at least one transitive relation containing R, namely the total one, whose graph is $X \times X$. The smallest transitive relation containing R is then given by the intersection of all transitive relations containing R. ■

The central result of this section is the following theorem.

Theorem 8. *Let $\geq_{\mathrm{fe}}$ the usual preorder of the financial events. Then the preorder $\leq_{fe}$ is the transitive closure of the elementary confrontation among financial events.*

To prove this result we have to introduce a new binary relation: the *connection by R*.

Terminology. Recall that a pair (x, y) is said R-related, or related by R, if and only if the relation xRy holds; in this case, we also say that x is related with y by R.

Definition (of connected pairs). *Let X be a set and let R be a relation on X. We say that a pair (x, y) of elements in X is R-**connected**, or **connected by** R, if and only* *if there exists a finite (ordered) sequence of elements of X, containing x and y, in which every component is related by R with the successive one. In this case, we also say that x is connected with y by R.*

Remark. It is clear that an element x of X is connected with another element y by R if and only if there exists a finite (ordered) sequence of elements of X whose first element is x, whose last element is y and such that every component of the sequence is related by R with the successive one. A finite sequence in X satisfying the last property is said a path R-connecting x with y. It is clear that, with respect to a relation, a related pair is a connected one, the conversely is not true in general. Nevertheless, if R is a transitive relation then the pair (x, y) is R-related if and only if it is R-connected.

Definition (the relation of connection). *Let R be a relation on a non-empty set X. The relation C defined on X, for every x, y in X, by writing xCy if and only if x is connected with y by R is called **the relation of connection by** R.*

Let c and d be two finite sequences in a set X, of m and n members respectively. If the last member of c is not the first member of d, the composite sequence cd is the sequence of $m + n$ elements coinciding with c from 1 to m and with d from $m + 1$ to $m + n$. If the last member of c is the first member of d then cd is the sequence of $m + n - 1$ elements coinciding with c from 1 to m and with d from m to $m + n - 1$.

Now we can prove the basic lemma.

Lemma. *The relation of connection C by a relation R is the transitive closure of R.*

Proof. The relation C is transitive. In fact, let x, y, z elements of X such that the relations xCy and yCz hold, then there exist two paths, say c and d, R-connecting x with y and y with z respectively; consequently, the compose path cd R-connects x with z, so the relation xCz holds. The relation C contains R. In fact, if xRy then x and y are R-connected by the path (x, y) and then the relation xCy holds true. The relation C is the smallest transitive relation containing R. Let C' be a transitive relation containing R and let (x, y) be a C-related pair; by definition of C, there is a path, say c, R-connecting x with y; since C' contains R , the path c C'-connects x with y, and being C' transitive, this implies that the pair (x, y) is C'-related, and so that C' includes C. ■

Proof of theorem 8. We have to prove that the preorder $\leq_{fe}$ includes the relation C of connection by $\leq_{ef}^{el}$ and viceversa. The preorder $\leq_{fe}$ includes the relation of connection by $\leq_{ef}^{el}$ since $\leq_{fe}$ is transitive and contains $\leq_{fe}^{el}$. Conversely. Let $(t_0, c_0) \leq_{fe} (t, c)$, then at least one of the following holds: (a) e_0 and e are strict credits with $t_0 \geq t$ and $c_0 \leq c$; (b) e_0 is a weak debt and e is a weak credit; (c) e_0 and e are strict debts with $t_0 \leq t$ and $c_0 \leq c$. In all the cases the path (e_0, e', e) where $e' = (t_0, c)$ connects e_0 with e with respect to the elementary confrontation. ■

6. Relation with the preorders generated by separable capitalizations

To give another characterization of the usual preorder of the financial events we give the following definition.

Definition (the preorder induced by a separable capitalization factor). *On the financial events plane $\mathbb{R}^2$, fixed $i > -1$, we define a relation $\leq_i$ for which the inequality $(t_0, c_0) \leq_i (t, c)$ is equivalent to*

$$c_0(1+i)^{t-t_0} \leq c.$$

*This relation $\leq_i$ is called the **preorder induced on $\mathbb{R}^2$ by the separable capitalization factor of rate** i, that is the function $f : h \mapsto (1+i)^h$.*

Remark. The preorder $\leq_i$ is induced by the functional defined on the plane, for every event (t, c), by the equality $f_i(t, c) = c(1+i)^{-t}$.

Theorem 9. *Let $\geq_{\mathrm{fe}}$ the usual preorder of the financial events. Then the preorder $\leq_{fe}$ is refined by each preorder $\leq_i$ with non-negative rate.*

Proof. We shall give two proofs: algebraic and geometrical. *Geometrical proof.* Recall that a preorder $\leq_2$ on a set X is a refinement of another preorder $\leq_1$ on X if and only if, for each y in X, the set of lower bounds of y with respect to $\leq_1$ (resp. the set of his upper bounds) is contained in the set of the lower bounds of (resp. of his upper bounds) with respect to the preorder $\leq_2$. Let e be a credit, the set of lower bounds of e, i.e., the interval $]\leftarrow, e]_{\leq_i}$ is the ipograph of a strict increasing real function and hence it must include the strip of strict credits lower bound (with respect to the usual preorder of financial events) of e; the half-plane of debts is obviously contained in the ipograph, since the evolution of a credit is a strongly positive function. Let e be a strict debt, since its financial evolution is a strict decreasing function, its ipograph must contain the cone of lower bounds of e with respect to the preorder $\leq_{fe}$. And the theorem is proved. If you prefer it is possible to use the sets of upper bounds. *Algebraic proof.* We work in the half-plane of credits (the other case is analogous). Suppose an event (t_0, c_0) be weakly dominated by another element (t, c) with respect to the usual preorder $\leq_{ef}$, in symbols $(t_0, c_0) \leq_{\mathrm{fe}} (t, c)$, that is, suppose the conjunction ($t_0 \geq t$ et $c_0 \leq c$) hold. We must prove that $c_0\left(1+i\right)^{t-t_0} \leq c$. We have $c_0\left(1+i\right)^{t-t_0} \leq c_0$, indeed $t_0 \geq t$ and hence $t - t_0 \leq 0$, consequently $\left(1+i\right)^{t-t_0} < 1$; but $c_0 \leq c$, and thus $c_0\left(1+i\right)^{t-t_0} \leq c_0 \leq c$. That is $e_0 \leq_i e$. Note that if also when one and only one of the two inequality of the conjunction "$t_0 \geq t$ and $c_0 \leq c$" is strict then we shall have $e_0 <_i e$, and thus the preorder induced by the exponential capitalization refine strictly the usual preorder $\leq_{\mathrm{fe}}$. ■

Theorem 10. *Let $\geq_{fe}$ the usual preorder of the financial events. Then the preorder $\leq_{fe}$ is the intersection, for $i \geq 0$, of the preorders $\leq_i$.*

Proof. We shall prove, for instance that the interval $[e, \rightarrow[_{fe}$, where e is the event $(0, 1)$ is the intersection of the family of intervals $([e, \rightarrow[_i)_{i\geq 0}$ (note that for every strict credit e' the interval $[e', \rightarrow[_{fe}$ is the translation by the vector $e' - e$ of the interval $[e, \rightarrow[_{fe}$. To prove that

$$[e, \rightarrow[_{fe} = \bigcap_{i\geq 0} [e, \rightarrow[_i \,,$$

note that above we saw that

$$[e, \rightarrow[_{fe} \subseteq \bigcap_{i\geq 0} [e, \rightarrow[_i \,,$$

since $\leq_{fe}$ is a refinement of each preorder $\leq_i$; so we have only to prove that

$$\bigcap_{i\geq 0} [e, \rightarrow[_i \subseteq [e, \rightarrow[_{fe} \,.$$

To do this, note that if (t', c') belongs to the intersection $\bigcap_{i\geq 0} [e, \rightarrow[_i$ then it must belong to the interval $[e, \rightarrow[_0$, that is the half plane of all events with capital greater or equal to c; then $c' \geq c$. Suppose, arguing by contradiction, that $t' > 0$, then there is a real $a > 1$ such that $a^{t'} = c'$ so $c' < (a+1)^{t'}$ but so (t', c') does not belong to the interval $[e, \rightarrow[_{a-1}$ against the assumption. ■

7. Canonical algebraic structures on the financial events plane

We give the following definition of standard addition on the financial events plan.

Definition (of standard addition in the financial events plane). *In the plane of financial events we call* ***standard addition*** *the not-everywhere defined binary internal operation defined by* $(t, c) + (t', c') = (t, c + c')$*, when* $t = t'$ *and only in this case. We define, moreover, the everywhere defined external binary operation of multiplication by real scalars as follows* $a(t, c) = (t, ac)$*, we call it the* ***standard multiplication by scalars of the financial plane.***

Remark (the standard addition for financial events as functions-addition). The standard addition on the financial events plane should not seem too strange. The plane of financial events can be immersed in, what we call, the space of *financial projects*, we define it as the space $\mathcal{F}(\mathbb{R}, \mathbb{R})$ of functions of the time-axis into the capital-axis; the canonical immersion is the injective function $j : \mathbb{R} \times \mathbb{R} \to \mathcal{F}(\mathbb{R}, \mathbb{R}) : (t, c) \mapsto c\chi_t$, associating to each financial event (t, c) the characteristic function centered in t and multiplied by the scalar c. In this space of functions is defined the standard operation of addition; well, if $c\chi_t$ and $c'\chi_{t'}$ are (corresponding to) financial events, then their sum with respect to the standard addition for functions is another financial event if and only if $t = t'$ and, in this case, we have $c\chi_t + c'\chi_t = (c + c')\chi_t$, and this sum corresponds to the events $(t, c + c')$.

To understand what kind of structure is the pair $(\mathbb{R}^2, +)$ we recall the following definition of Bourbaki (see [5] and [6]).

Definition (of grupoid). *An algebraic structure endowed with a not necessarily everywhere defined internal binary operation* $(X, .)$ *is said to be a* ***grupoid*** *if it satisfies the following axioms:*

(1) for every triple (x, y, z) *of elements of* X*, if one of the compositions* $x(yz)$*,* $(xy)z$ *is defined, so is the other and they are equal;*

(2_r*) if the compositions* xy *and* $x'y$ *are defined and equal then* $x = x'$*;*

(2_l*) if the compositions* yx *and* yx' *are defined and equal then* $x = x'$*;*

(3) for all x *there are* e_x *and* e'_x *such that* $\mathrm{e}_x x = x\mathrm{e}'_x = x$*;*

(4) for all indempotents e *and* e' *there exists* x *such that* $\mathrm{e}_x x = x e'_x = x$*;*

(5) for every x *in* E *there is an* x' *in* E *such that* $xx' = \mathrm{e}_x$*.*

Now we can characterized the structure $(\mathbb{R}^2, +)$.

Theorem 11. *The structure* $(\mathbb{R}^2, +)$*, where* $+$ *is the usual addition of financial events is a commutative grupoid, consequently, each idempotent element of this grupoid is both left and right and unit.*

Proof. Let x, y, z be three financial events. If the composition $(x + y) + z$ is defined then the events must have the same time, consequently, also the composition $x + (y + z)$ is defined and in both cases the capital of the composition is the sum of the three capitals, so the two compositions coincide. Note that the addition is obviously commutative, then 2_r and 2_l are equivalent. Let us prove (2_r) if $x + y$ and $x + y'$ are defined (and this implies that the four events and the two sums are with the same time) we have $c_x + c_y = c_x + c_{y'}$ and then it follows $c_y = c_{y'}$ and $y = y'$. Let us see (3), for every (t, c) the event $(t, 0)$ satisfies the condition as left and right unit. (4) Suppose $(t, c) + (t', c) = (t, c)$, then $t = t'$ and $2c = c$, then $c = 0$, so an idempotent element must be a null-event, now, every event of the form (t, c') with c' real verifies the property. (5) For every event (t, c), we have $(t, c) + (t, -c) = (t, 0)$. ■

We conclude with two theorems whose proofs are straightforward.

Theorem 12. *The usual multiplication by scalars . on the financial events plane is a distributive external operation of the standard multiplicative monoid* $(\mathbb{R}, \cdot)$ *into the grupoid* $(\mathbb{R}^2, +)$*.*

To give the second theorem we introduce a new algebraic structure, that is a generalization of the structure of vector space, as the grupoid is a generalization of the group.

Definition (of linearoid space). *We say that a structure* $(X, +, .)$ *is a* ***real linearoid space*** *when the structure* $(X, +)$ *is a commutative grupoid and when the operation . is an external operation with domain of operators the real line, left distributive with respect to the standard addition on the real line and such that the corresponding action is an associative and distributive action of the multiplicative monoid of real numbers.*

Theorem 13. *The structure* $(\mathbb{R}^2, +, .)$ *is a linearoid space. Moreover the operations of this structure are compatible with respect to the usual preorders of financial events and with respect to the usual topology of the plane.*

Theorem 14. *The structure* $(\mathbb{R}^2, +, .)$ *is isomorph to the tangent bundle of the real line (with its standard differentiable structure) endowed with its natural (not-everywhere defined) operations.*

8. Some applications of the formalization

Some applications of the general methodology introduced in this paper are presented in [7]. In any case, we present here possible uses of the structures introduced.

Application 1 (confrontation among two events at every rate of interest). Let $A = (t_A, c_A)$ and $B = (t_B, c_B)$ be two credits such that $t_A < t_B$ and $c_B > c_A$. The two events are incomparable with respect to the usual preorder of financial events plane. There are three possibilities: a) the evolution-curve through A remains under B; b) the evolution-curve through A passes through B too; c) the evolution curve through A remains upper B. Let i_0 be the rate of interest such that the case (b) happens (the so called equivalence-rate for A and B), then A is better than B if the rate i of the market is greater than i_0 (case a) and B is better that A when the rate i is lower than i_0 (case c). The problem of decision is solved at every rate of interest $i > 0$. When the rate is negative the problem has, obviously, B as solution.

Application 2. Suppose that a decision maker D must receive a unit of money today (time 0). Assume that the debtor proposes to the decision-maker a change: the decision-maker can choose 1 today or $1 + mt$ at a time $t > 0$, with $m > 0$ and $t \in [a, b]$, where a and b are times beyond 0. What is the best choice for D?

The preceeding problem is contained in the following one.

Application 3. Let S be the convex envelope, in the financial events plane, of the events $C = (a, 0)$, $D = (b, 0)$, $A = (a, 1 + am)$ and $B = (b, 1 + bm)$, where a, b, m are three positive real numbers. S is the set of possible choices of a decision-maker, the problem is: what are the best choices and what are the worst ones, when the financial market offers a positive rate of interest i?

Since the rate is positive, we must find the solutions in the segment $[A, B]$. In fact: (i) that segment is the Pareto maximal boundary of S, with respect to the usual preorder of the financial events plane; (ii) the function

$$F : \mathbb{R}^2 \to \mathbb{R} : F(t, C) = C(1 + i)^{-t},$$

which is the criterion-function of our problem, is widely increasing on S and strictly increasing on $S \setminus [C, D]$, with respect to the usual preorder of the financial events plane, and it vanishes on the segment $[C, D]$. Our decision problem is, then, restricted to the segment

$$[A, B] = \left\{(t, C) \in \mathbb{R}^2 : t \in [a, b] \text{ et } C = 1 + mt\right\}.$$

We must maximize on $[A, B]$ the function F (it is the function "value at time 0 with interest rate i"). The section of the functional F on the straight-line passing through A and B is the function g defined by

$$g(t) = F(t, 1 + mt) = (1 + mt)\,(1 + i)^{-t}.$$

The derivative of g in t is

$$g'(t) = [m - (1 + mt)\,\delta]\,(1 + i)^{-t},$$

where δ is the positive number $\ln(1 + i)$ (it is called the instant rate of interest associated with the exponential capitalization). Concerning the sign of the derivative $g'(t)$, it is non negative if and only if

$$t \leq \tau := \delta^{-1} - m^{-1}.$$

The time τ depends upon the rate i and upon the slope m, we call it the characteristic time of the problem. Fixed the slope m, there is a bijective correspondence associating with every rate $i > 0$ a characteristic time. Calculating the rate corresponding to the times a and b it is possible (straightforwardly) to determine the solutions of the problem at every rate of interest i. Specifically, the above correspondence can be extended to the mapping

$$T : [0, +\infty] \to [-1/m, +\infty]\,,$$

associating with every real positive rate i the characteristic time

$$T(i) = 1/\ln(1 + i) - 1/m,$$

associating with the rate $i = 0$ the time $+\infty$ and with the rate $i = +\infty$ the time $-1/m$.

Concerning the inverse of the application T, we have

$$T^{-}(t) = \exp(\frac{m}{mt + 1}) - 1,$$

for every t belonging to $[-1/m, +\infty]$, note that $T^{-}(0) = e^{m} - 1$.

Remark. Note that for negative rates of interest the solution is B. Note, moreover, that there is a bijective correspondence among the set of rates $[-1, 0]$ and the set of times $[-\infty, -1/m]$ associating with the rate -100% the time $-1/m$, with the rate 0% the time $-\infty$ and with the rate $i \in \,]-1, 0[$ the time

$$T(i) = 1/\ln(1 + i) - 1/m,$$

this last time is the time minimizing the function value at 0, on the straight-line through A and B, upon the time interval $[-\infty, -1/m]$.

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Presented: November 29, 2007
Published on line: September 2, 2008