Communications to SIMAI Congress, ISSN 1827-9015, Vol. 3 (2009) 196 (12pp)
DOI: 10.1685/CSC09196

On Volterra's sixth elementary distortion analyzed by Saint Venant's theory

Ivana Bochicchio[1], Ettore Laserra[1], Massimo Pecoraro[2]

[1] *Dipartimento di Matematica ed Informatica, Facoltà di Scienze MM.FF.NN.*
Università degli Studi di Salerno, Italy
ibochicchio@unisa.it
elaserra@unisa.it

[2] *DIIMA, Facoltà di Ingegneria*
Università degli Studi di Salerno, Italy
pecoraro1@libero.it

Abstract

We consider the sixth elementary Volterra's distortion for a circular hollow, homogeneous, elastic, isotropic cylinder, to analyze the load acting on the bases as a Saint Venant external stress. We prove the specific load connected to the sixth distortion is equivalent (in Saint Venant's theory) to a right combined compressive and bending stress and to a right combined tensile and bending stress. We have applied our results to a material made up if steel to compare the obtained numerical results with the Volterra's predictions. We can see that the values calculated through the Saint Venant's theory are more strictly related to those calculated by Volterra when the cylinder thickness tends to zero.

Keywords: `Volterra's distortions, Saint Venant theory.`

1. Introduction.

The theory of elastic distortions, proposed by Volterra about one hundred years ago, implied a deep revision of the mathematical theory of elasticity in the case of multi-connected bodies. More precisely, Volterra began with Weingarten's observations [11] to show that, in absence of external forces, equilibrium configurations for elastic bodies occupying a multi–connected domain and with with no null internal stress can exist.

The most general elastic distortion able to bring a right, circular, homogenous, hollow, isotropic cylinder to a state of spontaneous equilibrium, consists of six elementary distortions. For each, Volterra has tried to determine a field of displacements which fulfills the indefinite equations of elastic equilibrium and brings the body to a spontaneous equilibrium configura-

Received 14/01/2009, in final form 15/06/2009
Published 31/07/2009

tion. In the context of Volterra's distortions, our paper analyzes the forces induced by the sixth elementary distortion on the right circular, homogenous, hollow, isotropic cylinder exploiting Saint Venant's theory.
More precisely, we have underlined that, apart from a limited zone in the immediate vicinity of bases, the distribution of forces, considered as a specific load, can be replaced with one statically equivalent. This can be done without consequences on the effective distributions of stress and strain, and therefore, without the necessity to define the effective punctual distribution of this load acting on the bases of the cylinder.
Approaching the specific load as linear and constructing an auxiliary bar which has as longitudinal section the axial section of the cylinder, we have found the specific load connected to the sixth distortion is equivalent (in Saint Venant's theory) to a right combined compressive and bending stress and to a right combined tensile and bending stress.
Our results have been applied to a fixed material to compare the obtained numerical results with Volterra's predictions: the values calculated through Saint Venant's theory are more strictly related to those calculated by Volterra when the cylinder thickness tends to zero.

2. Volterra's distortions.

Now we consider a circular hollow (therefore doubly connected), homogeneous, elastic and isotropic cylinder C, which is, at a certain assigned temperature τ, in a natural state, that we assume as the reference configuration.

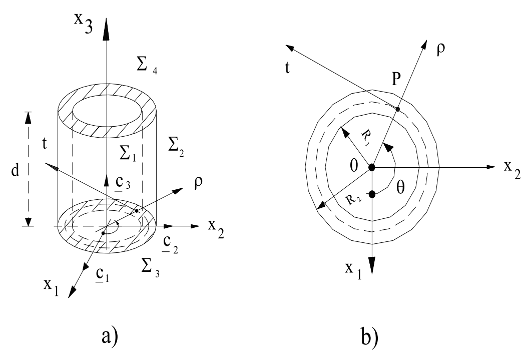

Fig. 2.1. Hollow cylinder in the natural state and the cross section $x_3 = 0$.

We introduce into an ordinary space a Cartesian rectangular reference $0x_1x_2x_3$ with respective versors $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ and we choose the axis $0x_3$ coinciding with the symmetry axis of the cylinder and the coordinate plane

$0x_1x_2$ placed over the base. We indicate with $\rho(\mathbf{x}) = \sqrt{\mathbf{x_1^2}+\mathbf{x_2^2}}$ and $\theta(\mathbf{x}) = arctg\frac{x_2}{x_1}$ respectively, the distance of P from the axis of the cylinder and the anomaly.
Hereafter, we call Σ the surface of C (see Fig. 2.1), made from the two cylindrical coaxial surfaces Σ_1 (internal surface of radius R_1) and Σ_2 (external surface of radius R_2), and from the two bases Σ_3 (at height $x_3 = 0$) and Σ_4 (at the height $x_3 = d$).

Let $\mathbf{u}(\mathbf{x})$ be the displacement vector which is the solution of the elastic equilibrium problem for a body subjected to given external forces (without external constraints and mass forces); let's assume that $\mathbf{u}(\mathbf{x})$ includes a many–valued term [a] related to $\theta(\mathbf{x})$.
The many–valued field of displacement $\mathbf{u}(\mathbf{x})$ has been physically interpreted by Volterra [10] in terms of the following operations:
if the doubly connected cylinder is transformed into one which is simply connected by a transversal cut on an axial semi–plane having the x_3 axis as edge, the vector $\mathbf{u}(\mathbf{x})$ can be characterized by a discontinuity of the first type through the semi-plane of the cut. If a translatory and a rotatory displacement is imposed on one of the faces of the cut by the application, at constant temperature, of a system of external forces, a state of deformation, and therefore of stress due to the many–valued term including $\theta(\mathbf{x})$, is created into the cylinder. In order that the cylinder remains in a state of *spontaneous equilibrium* in the deformed configuration, i.e. with a regular internal stress but absent of superficial forces, it is enough to re-establish the continuity remaking the cylinder doubly connected by soldering the two faces of the cut. In this way the cylinder assume a helicoidal configuration absent of superficial forces and results in a state of regular internal stress.
So, if the cylinder C undergoes an isothermal many-valued displacement $\mathbf{u}(\mathbf{x})$ from its natural state to an equilibrium configuration, then the stress tensor field for all internal points of the equilibrium configuration and the vector field acting on the boundary of the cylinder are in equilibrium. To permit the boundary forces to vanish, we have to find an appropriate auxiliary displacement $\mathbf{u}'(\mathbf{x}) \neq \mathbf{u}(\mathbf{x})$, which brings the cylinder from the initial natural configuration into a new equilibrium configuration (see [2,9]). Then we can apply the displacement field

$$\mathbf{u}''(\mathbf{x}) = \mathbf{u}(\mathbf{x}) - \mathbf{u}'(\mathbf{x}) \tag{2.1}$$

to the natural configuration which changes into a new configuration. In this

[a] This term is physically significant in a doubly–connected region of space, as a body with hollow cylindrical symmetry.
Note that $\mathbf{u}(\mathbf{x})$ can eventually include another term whose singularity points belong to a locus that doesn't intersect domain C.

new configuration an internal stress tensor field is present, but the boundary forces vanish. The new configuration is called a "spontaneous equilibrium configuration" and the vector field (2.1) (a many-valued function with the characteristic axis coincident with the axis of the cylinder) represents what Volterra called a "distortion".
In addition, since the rigid displacement of a face of cut with respect to the other can be obtained through a rigid translation displacement and a rigid rotation displacement, a distortion can be described by six constant parameters l, m, n, p, q, r, called *characteristic coefficients of distortion*. They correspond to the three Cartesian components of translation and rigid rotation in respect to the axes x_1, x_2, x_3. The distortion that has only one of the six characteristic coefficients different from zero is defined *Elementary distortion* [6,4,7,10]. Analogously, the displacement induced by an elementary distortion has no null in only one of the following coefficients l, m, n, p, q, r. In particular, the 6^{th} elementary distortion is the distortion related to the coefficient r. It is realized by cutting the cylinder with an axial plane, rotating the face of the cut that faces the semi–plane $x_2 < 0$ and, after adding (when $r > 0$) or removing (when $r < 0$) a thin slide of matter, soldering the sides.

Remark 2.1. After a general distortion, the body is in equilibrium without external forces; hence in the body are been created compressed and stretched fibres, distributed in such a way that the volume remains unchanged [4,7,9,10]

If we consider the field of displacement relative to the sixth elementary distortion it generates on the bases a distribution of surface forces which has the following components in the cylindric reference frame (P, ρ^*, t^*, x_3^*) obtained by translating in a generic point of the cylinder the axes ρ, t, x_3:

$$\begin{cases} f_\rho(\rho,0) = 0 \\ f_t(\rho,0) = 0 \\ f_{x_3}(\rho,0) = \dfrac{r}{2\pi}\dfrac{\lambda\mu}{\lambda+2\mu}\left(1+\log\rho^2 - \dfrac{R_2^2\log R_2^2 - R_1^2\log R_1^2}{R_2^2-R_1^2}\right) \;=\; -a[b+\log\rho^2]\ ; \end{cases}$$

(2.2)

$$\begin{cases} f_\rho(\rho,d) = 0 \\ f_t(\rho,d) = 0 \\ f_{x_3}(\rho,d) = -\dfrac{r}{2\pi}\dfrac{\lambda\mu}{\lambda+2\mu}\left(1+\log\rho^2 - \dfrac{R_2^2\log R_2^2 - R_1^2\log R_1^2}{R_2^2-R_1^2}\right) \;=\; a[b+\log\rho^2] \end{cases}$$

where μ and λ are the two Lamé constants,

$$a = -\frac{r}{2\pi}\frac{\lambda\mu}{\lambda+2\mu}$$

and

$$b = 1 - \frac{R_2^2 \log R_2^2 - R_1^2 \log R_1^2}{R_2^2 - R_1^2}.$$

3. Saint's Venant's theory to analyze the sixth elementary distortion.

This section is devoted to analyze the specific load induced by the sixth elementary distortion in Saint Venant's theory (see [1,8]).
In Saint Venant's theory one can replace the specific load with an equivalent one. In this way, apart from a thin zone near the bases, called *extinction zone*, we have no consequences on the effective distribution of stress and strain. So, every solution of the problem of the elastic equilibrium can be considered as a solution of an infinity of cases which are pertinent to an infinity of load models, distributed with different laws, but having the same resultant. This resultant can be replaced, as we know from static, by a force through a generic point P' belonging to the base section, and by a couple that has, in respect to P', the same moment of the resultant. [b]
Since the force and the couple can be decomposed with respect to the three axes of the reference system, the six *characteristics of the external solicitation*, i.e. the three components of the force and of the couple, are individuated.
Hence, since these characteristics completely define every system of external loads acting on the bases of the solid, it is unnecessary to define their effective punctual distributions. As a consequence, the more general case can be solved through a linear combination of six elementary cases: normal stress, shear stress along x_2, shear stress along x_1, bending moment around x_1, bending moment around x_2 and torsional moment.

Hereafter, we will assume that the hollow cylinder is thin [c] and we will consider just the vertical component of the load, i.e. $f_{x_3}(\rho, d)$, acting on the base $x_3 = d$. This component can be simply denoted with $f(\rho)$, since, once x_3 is fixed, it is a function of ρ only.

Moreover, since $f(\rho)$ is monotone in $[R_1, R_2]$, the equation $f(\rho) = 0$ has

[b] This resultant is applied in a suitable point, generally different from P'.
[c] This means that its thickness $\Delta\rho = R_2 - R_1$ is small with respect to the radius R_1.

in (R_1, R_2) one real root:

$$\rho_n = \sqrt{e^{\frac{R_2^2 \log R_2^2 - R_1^2 \log R_1^2}{R_2^2 - R_1^2} - 1}}. \tag{3.1}$$

In other words, ρ_n is the value or the radius of the cylindrical neutral surface of the hollow body with respect to the specific load.

Now, let's consider a simply connected auxiliary rectangular beam. We

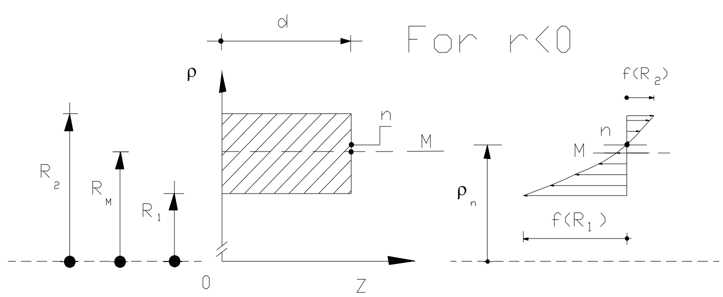

Fig. 3.1. Specific load distribution.

suppose that it has height d (i.e. the same height of the cylinder) and cross section with unitary base for convenience. Moreover, it is subjected to the load $f(\rho)$ on the bases. [d]

Now we would like to analyze the two zones delimitated by ρ_n (see Fig.3.1) and separately study the distribution of load.

More precisely, since in Saint Venant's theory it is unnecessary to define the effective punctual distribution of the load on the bases of the body, we will appropriately reduce the load induced in each section by the sixth elementary distortion to a normal stress and to a couple.

The normal stress and the momentum of the couple, both applied in the barycenter of the section, will have a fundamental role in our analysis; more precisely, they allow us to prove the specific load connected to the sixth distortion is equivalent (in Saint Venant's theory) to a right combined compressive and bending stress and to a right combined tensile and bending stress. Now we separately study the sections delimited by ρ_n.

In the upper section, where $\rho \in (\rho_n, R_2)$, let ρ_e be the value of ρ where we have to translate the diagram of $f(\rho)$ to divide the upper section in two with, in modulus, the same area (see Fig. 3.2). Its explicit value is:

[d] If we consider an axial section of the cylinder of height d, then it can be assimilated to a longitudinal section of the auxiliary beam.

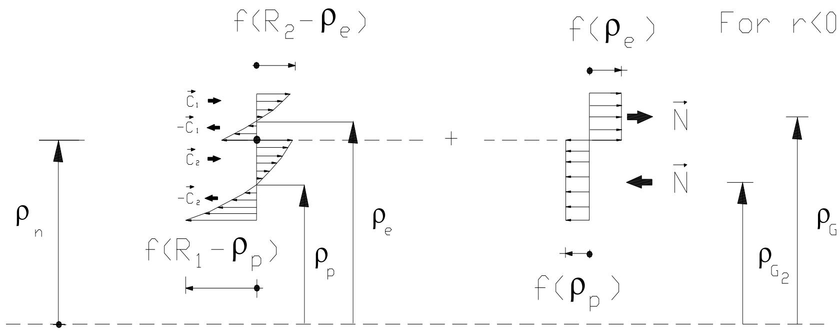

Fig. 3.2. Decomposition of the specific load.

$$\rho_e = \sqrt{e^{-\dfrac{2R_2 - 2\rho_n - R_2 \log R_2^2 + \rho_n \log \rho_n^2}{R_2 - \rho_n}}} \ . \tag{3.2}$$

As already underlined, the specific load acting on the section can be represented by a normal stress $\mathbf{N}$ applied on the barycenter G_1 of the section whose modulus is

$$N \ = \ f(\rho_e)(R_2 - \rho_n) = a(R_2 - \rho_n)(b + \log \rho_e^2) \ , \tag{3.3}$$

and by a couple $(\mathbf{C}_1, -\mathbf{C}_1)$. The modulus of the vector $\mathbf{C}_1$ is

$$C_1 \ = \ a\left[2(R_2 - \rho_e) - R_2(\log R_2^2 - \log \rho_e^2)\right] \ . \tag{3.4}$$

Moreover, the modulus of the total momentum respect to G_1 of $\mathbf{N}$ applied in the center of stress is

$$M_{G_1} = -a\left(\rho_{G_1^{(1)}} - \rho_{G_1^{(2)}}\right)\left[2R_2 - 2\rho_e - R_2\left(\log\, R_2^2 - \log\, \rho_e^2\right)\right] \ , \tag{3.5}$$

where $\rho_{G_1^{(1)}}$ and $\rho_{G_1^{(2)}}$ are the positions of the barycenters $G_1^{(1)}$ and $G_1^{(2)}$ of the two sections having the same area.
Then, in agreement with Saint Venant's theory, for all $z \in [0, d]$ in the section there is the action of the following linear $\sigma_{x_3}^{(1)}(\rho)$ (right combined tensile and bending stress):

$$\begin{aligned} &\sigma_{x_3}^{(1)}(\rho) = a(b + \log\, \rho_e^2\,)C + \\ &-\frac{12a(\rho_{G_1^{(1)}} - \rho_{G_1^{(2)}})\left[2R_2 - 2\rho_e - R_2\left(\log\, R_2^2\, - \log\, \rho_e^2\right)\right]}{(R_2 - \rho_n)^3}\left(\rho - \frac{R_2 + \rho_n}{2}\right) \ . \end{aligned} \tag{3.6}$$

In the lower section, where $\rho \in (R_1, \rho_n)$, let ρ_p be the value of ρ where we have to translate the diagram of $f(\rho)$ to divide the upper section in two with, in modulus, the same area (see Fig. 3.2). Its explicit value is:

$$\rho_p = \sqrt{e^{-\dfrac{2\rho_n - 2R_1 - \rho_n \log \rho_n^2 + R_1 \log R_1^2}{\rho_n - R_1}}} \, . \tag{3.7}$$

As already underlined, the specific load acting on the section can be represented by a normal stress $\mathbf{N}$ applied on the barycenter G_2 of the section whose modulus is

$$N \;=\; f(\rho_p)(\rho_n - R_1) \;=\; a(\rho_n - R_1)(b + log\rho_p^2) \;, \tag{3.8}$$

and by a couple $(\mathbf{C}_2, -\mathbf{C}_2)$. The modulus of the vector $\mathbf{C}_2$ is

$$C_2 \;=\; a\left[2(\rho_n - \rho_p) - \rho_n(\log \rho_n^2 - \log \rho_p^2)\right] \;. \tag{3.9}$$

Moreover, the modulus of the total momentum respect to G_2 of $\mathbf{N}$, applied in the center of stress is

$$M_{G_2} = -a\left(\rho_{G_2^{(1)}} - \rho_{G_2^{(2)}}\right)\left[2\rho_n - 2\rho_p - \rho_n\left(\log\,\rho_n^2\; - \log\,\rho_p^2\right)\right] \;,$$

where $\rho_{G_2^{(1)}}$ and $\rho_{G_2^{(2)}}$ are the positions of the barycenters $G_2^{(1)}$ and $G_2^{(2)}$ of the two sections having the same area.
Thus, in agreement with Saint Venant's theory, for all $z \in [0, d]$ in the section there is the action of the following linear $\sigma_{x_3}^{(2)}(\rho)$ (right combined compressive and bending stress):

$$\begin{aligned} &\sigma_{x_3}^{(2)}(\rho) = a(b + \log\,\rho_p^2\,) + \\ &-\frac{12a(\rho_{G_2^{(1)}} - \rho_{G_2^{(2)}})\left[2\rho_n - 2\rho_p - \rho_n\left(\log\,\rho_n^2\; - \log\,\rho_p^2\right)\right]}{(\rho_n - R_1)^3}\left(\rho - \frac{R_1 + \rho_n}{2}\right) . \end{aligned} \tag{3.10}$$

The importance of Saint Venant's theory applied to the sixth elementary distortions is mainly based on the information content of the Eq. (3.6) and (3.10). More precisely, they underline what kind of load is induced (in Saint Venant's theory) by the sixth elementary distortion: it is a right combined tensile and bending stress and a right combined compressive and bending stress. Hence, for every axial section it is possible to evaluate the tensional state with the well–known Saint Venant's formulas [8, pp. 144–145].
However, in order to apply Saint Venant's theory, our analysis has required

some assumptions: we have considered a suitable auxiliary beam and we have assumed that the load on the bases has a linear diagram. So, to evaluate the deviation of our results from Volterra's predictions, in this section we compare Eq. (3.6) and (3.10) with $f_{x_3}(\rho)$ computed by Volterra.
More precisely, let's consider the cylinder made of steel, and let's subject the side of the cut to this rotation $r = -1.62 * 10^{-5}\ rad.$ [e] Moreover, we fixed $R_2 = 4$ cm and then we examined the following two cases:

$$\beta = \frac{R_1}{R_2} = 0.5\ ;\ \beta = \frac{R_1}{R_2} = 0.9\ .$$

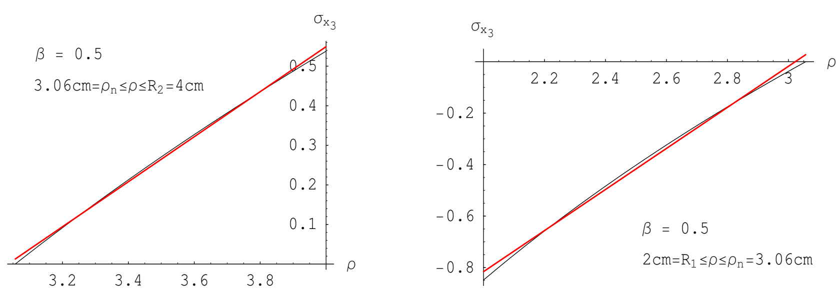

Fig. 3.3. Load in Volterra 's theory (black) and load in our results (red) for $\beta = 0.5$. The picture on the left refers to the upper section, i.e. 3.06 cm$= \rho_n \leq \rho \leq R_2 =$4 cm; while the picture on the right to the lower section, i.e. 2 cm$= R_1 \leq \rho \leq \rho_n =$3.06 cm.

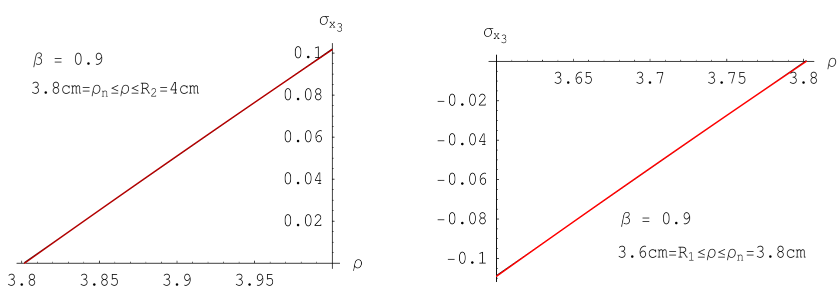

Fig. 3.4. Load in Volterra 's theory (black) and load in our results (red) for $\beta = 0.9$. The picture on the left refers to the upper section, i.e. 3.8 cm$= \rho_n \leq \rho \leq R_2 =$4 cm; while the picture on the right to the lower section, i.e. 3.6 cm$= R_1 \leq \rho \leq \rho_n =$3.8 cm. Note that the two lines are indistinguishable.

These pictures clearly demonstrate that the red line (expression of load in

[e] The smallness of the chosen angle is justified by the required thickness of the cylinder, by the material it is made of and by the hypothesis that Saint Venant's theory is valid for small displacements.

our results) is a good approximation of Volterra's prediction, rather, indistinguishable when $\beta = 0.9$, i.e. for a very thin cylinder. We can conclude by seeing that the values calculated through Saint Venant's theory are more strictly related to those calculated by Volterra when the cylinder thickness tends to zero.

REFERENCES

1. I. Bochicchio, E. Laserra and M Pecoraro, Sulla sesta distorsione elementare di Volterra per un cilindro cavo omogeneo e isotropo di altezza finita con carico alla Saint Venant, *Atti dell' Accademia Peloritana dei Pericolanti, Classe di Scienze Fisiche, Matematiche e Naturali*, **86** (2008).
2. G. Caricato, On the Volterra's distortions theory, *Meccanica*, **35** (2000), pp. 411-420.
3. C. Cattani and J. Rushchitsky, Volterra's distortions in nonlinear hyperelastic media, *Int. J. of Appl. Math. and Mech.*, **3** (2005), pp. 14–34.
4. G. Grioli, Le distorsioni elastiche e l'opera di Vito Volterra, *Atti dei Convegni Lincei*, **92** (1992), pp. 271-289.
5. E. Laserra and M. Pecoraro, Volterra's Theory of Elastic Dislocations For A Transversally Isotropic Homogeneous Hollow Cylinder, *Nonlinear Oscillations*, **6** (2003), pp.(2003)56-73.
6. A.E. H. Love, *The Mathematical Theory of Elasticity.* Cambridge Univerity Press, 4th edn., 1952.
7. A. Signorini. Lezioni di Fisica Matematica. Libreria Eredi Virgilio Veschi, Roma, 1953.
8. R. Sparacio. *Appunti delle lezioni del corso di scienza delle costruzioni.* Istituto di Scienza delle Costruzioni, Facoltà di Ingegneria, Napoli, 1975.
9. V. Volterra, Sulle distorsioni dei corpi elastici simmetrici, *Rend. Accad. Lincei*, **14** (1905).
10. V. Volterra. L'equilibre des corps élastique multiplement connexes. Imprimeur-libraire, Gauthier-Villars, Paris, 1907.
11. M. Weingarten, Sur les surfaces de discontinuité dans la théorie des corps solids, *Rend. Accad. Lincei*, **10** (1901), pp. 57-60.