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Direct numerical methods for integral equations in elasticity theory

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Abstract

We outline direct numerical methods for solving problems in elasticity theory, formulated via the Muskhelishvili integral equation.

1 Background

In the plane problem in elasticity theory the relationship between deformations u, v and constraints is described by,

$$\sigma_x = \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial u}{\partial x} \quad \sigma_y = \lambda(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + 2\mu\frac{\partial v}{\partial y} \quad \tau_{xy} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \quad (1)$$

where λ and μ are the Lamé constants, see [4] Sec. 15. The problem is the simultaneous resolution of the equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \quad \Delta(\sigma_x + \sigma_y) = 0, \quad (2)$$

Δ being the Laplace operator. Introducing the Airy function $U(x, y)$ it is possible to express the constraints components as

$$\sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = \frac{\partial^2 U}{\partial y \partial x}. \quad (3)$$

On substituting into (1) the first two equations are identically satisfied, while the last one becomes the biharmonic equation, $\Delta^2 U = 0$. To solve the latter, an analytic reformulation may be used. First of all, the Goursat formula allows to write any generic biharmonic function by means of two analytic functions ϕ and χ as follows

$$U(x,y) = Re[\bar{z}\phi(z) + \chi(z)]. \tag{4}$$

The Kolossov-Muskhelishvili formulae express the constraint tensor by means of the functions $\Phi(z) = \phi'(z)$ and $\Psi(z) = \psi'(z)$,

$$\sigma_x + \sigma_y = 4Re[\Phi(z)], \quad \sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)] \tag{5}$$

The components of the displacements are instead

$$2\mu(u + iv) = \kappa\phi(z) - z\,\overline{\phi'(z)} - \overline{\psi(z)}, \tag{6}$$

with $\kappa = \frac{\lambda+3\mu}{\lambda+\mu}$ for a plane deformation and $\kappa = \frac{\lambda^*+3\mu}{\lambda^*+\mu}$ for plane constraints, $\lambda^* = \frac{2\lambda\mu}{\lambda+2\mu}$. Let us assume now that the constraints X_n and Y_n are given along the arc $L = (a,b)$ of the boundary of the (finite) elastic body D^+. The resultant applied force along the normal to the boundary is then given by

$$X + iY = \int_L (X_n + iY_n)ds = -i[\phi(z) + z\,\overline{\phi'(z)} + \overline{\psi(z)}]_b^a. \tag{7}$$

Fixing now one of the arcends and letting the other one vary freely, we then find

$$\phi(t) + t\,\overline{\phi'(t)} + \overline{\psi(t)} = i\int_{t_0}^{t} (X_n + iY_n)ds \equiv f(t) + C, \tag{8}$$

C being an arbitrary constant, as the constraints are determined by formulae (5), i.e. upon differentiation. The problem (8) is known as the second problem of elasticity theory. After taking conjugates, rewrite it as follows

$$\psi(t) = \overline{f(t)} - \overline{\phi(t)} - \bar{t}\phi'(t) \equiv \Psi^+(t) - \Psi^-(t), \tag{9}$$

where the right hand side represents the limiting value of an analytic function in D^+, for $t \in L$. Using the Sokhotski-Plemelj formulae,

$$g(t) = G^+(t) - G^-(t), \qquad G(t) = \frac{1}{2}[G^+(t) + G^-(t)], \tag{10}$$

or also

$$G^+(t) = \frac{1}{2}g(t) + \frac{1}{2\pi i}\int_L \frac{g(t)}{\tau - t}d\tau, \quad G^-(t) = -\frac{1}{2}g(t) + \frac{1}{2\pi i}\int_L \frac{g(t)}{\tau - t}d\tau, \tag{11}$$

the function of (9) can be rewritten more generally for every point z outside D^+ as

$$\frac{1}{2\pi i}\int_L \frac{\overline{\phi(t)}+\bar{t}\phi'(t)}{t-z}dt = \frac{1}{2\pi i}\int_L \frac{f(t)}{t-z} \equiv F(z). \tag{12}$$

Observe now the identities

$$-\frac{1}{2}\overline{\phi(t_0)} = \frac{1}{2\pi i}\int_L \frac{\overline{\phi(t)}\ \overline{dt}}{\bar{t}-t_0}dt, \qquad \frac{1}{2}\phi'(t_0) = \frac{1}{2\pi i}\int_L \frac{\phi'(t)}{t-t_0}dt.$$

Add to (12) the above identities:

$$\frac{\bar{t}_0}{2}\overline{\phi(t_0)} + \frac{1}{2\pi i}\int_L \frac{\overline{\phi(t)}}{t-z}dt - \frac{\bar{t}_0}{2\pi i}\int_L \frac{\overline{\phi(t)}\ \overline{dt}}{\bar{t}-t_0}dt$$
$$+\frac{1}{2}\phi'(t_0) + \frac{1}{2\pi i}\int_L \frac{\bar{t}\phi'(t)}{t-z}dt + \frac{1}{2\pi i}\int_L \frac{\phi'(t)}{t-t_0}dt = F(z).$$

Taking then the limiting values for $z \longrightarrow t_0 \in L$ and integrating by parts, we are then led to the Muskhelishvili equation,

$$-\overline{\phi(t_0)} - \frac{1}{2\pi i}\int_L \overline{\phi(t)}\ d\log\frac{\bar{t}-\bar{t}_0}{t-t_0} - \frac{1}{2\pi i}\int_L \phi(t)\ d\frac{\bar{t}-\bar{t}_0}{t-t_0} = F(t_0), \tag{13}$$

2 Results

The numerical techniques employed for the simulations have been presented and analized in [2]. Basically, they are a Galerkin and a collocation method applied to a suitable modification of (13), to render it invertible. Indeed as the operator M defining its left hand side is indeed not invertible, [3], [4], we introduce the modified operator $M^+ = M + M_0$ with M_0 given by

$$M_0\phi\,(t) \equiv \frac{1}{2\pi i}\int_\Gamma \frac{\phi\,(\tau)\,d\tau}{\tau} + \frac{1}{t}\frac{1}{2\pi i}\int_\Gamma \left(\frac{\phi\,(\tau)}{\tau^2}d\tau + \frac{\overline{\phi\,(\tau)}}{\bar{\tau}^2}d\bar{\tau}\right). \tag{14}$$

The solution is sought in the form

$$\phi_n\,(t) = \sum_{k=0}^{n-1} c_k S_{kn}\,(t)\,, \quad t \in L. \tag{15}$$

where the functions S_{kn} are spline functions of order $m \in \{2,3,4,5\}$ defined on a suitable partition of the boundary L. To determine their coefficients, a set of algebraic equations is obtained as follows. In the Galerkin method we require the residual to be orthogonal to the splines S_{kn},

$$\left(M^+\phi_n, S_{kn}\right) = (F, S_{kn}), \quad k = 0, 1, ..., n-1 \tag{16}$$

while in the collocation method the residual is zeroed out at a set of suitable knots

$$M^+\phi_n\left(t_j^{(n)}\right) = F\left(t_j^{(n)}\right), \quad t_j^{(n)} = \gamma\left(\frac{j+\epsilon}{n}\right), \quad j = 0, 1, ..., n-1, \quad 0 \le \epsilon < 1. \tag{17}$$

We show the results of the calculations on the following two examples, both defined over an elliptical contour L with increasing eccentricities.

Example 1. We consider the forcing functions $F(t_0) = \frac{\sin(t_0)+it_0^4}{\cos(t_0)}$. The results are plotted in Figure 1

Example 2. Here we take $F(t_0) = \frac{t_0^5 - i\cos(t_0)}{\exp(t_0)}$. Figure 2 shows the results.

Convergence of the schemes is shown by increasing n, the size of the partition of L, and comparing the results of the calculations. The fact that the same pictures are obtained by different methods also validates the results.

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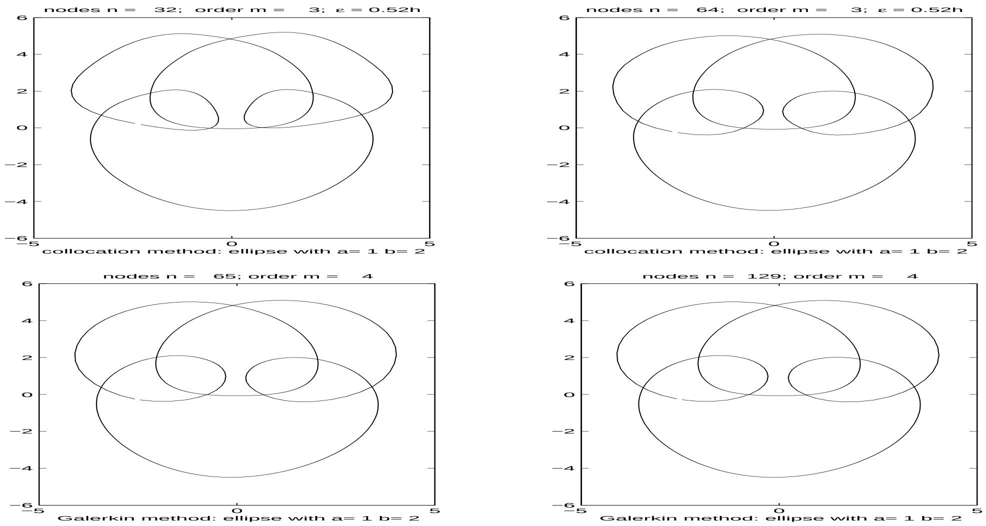

Figure 1: Example 1, L ellipse with $a = 1$, $b = 2$
Collocation solution (top) $m = 3$, $\epsilon = 0.52$, $n = 32$ (left), $n = 64$ (right);
Galerkin solution (bottom) $m = 4$, $n = 65$ (left), $n = 127$ (right).

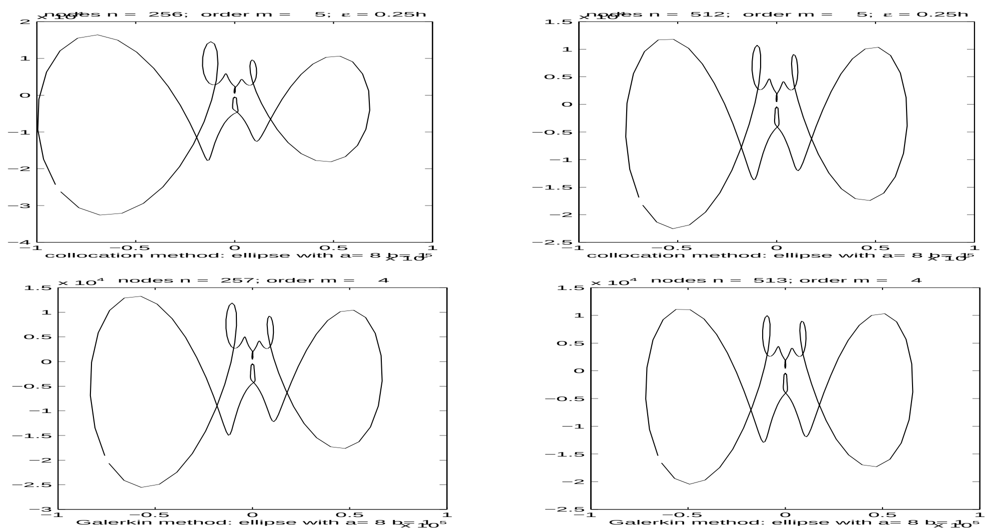

Figure 2: Example 1, L ellipse with $a = 8$, $b = 1$
Collocation solution (top) $m = 5$, $\epsilon = 0.25$, $n = 256$ (left), $n = 512$ (right);
Galerkin solution (bottom) $m = 4$, $n = 257$ (left), $n = 513$ (right).

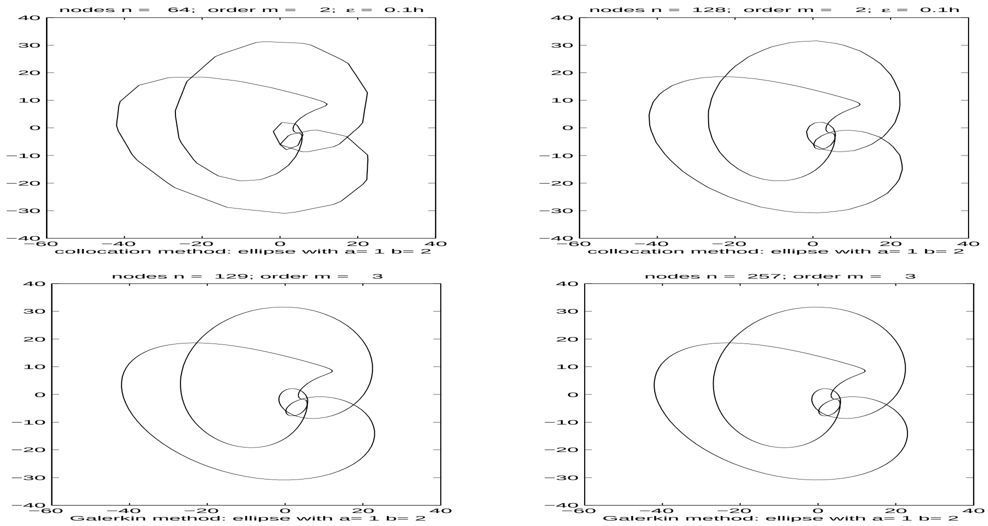

Figure 3: Example 2, L ellipse with $a = 1$, $b = 2$
Collocation solution (top) $m = 2$, $\epsilon = 0.1$, $n = 64$ (left), $n = 128$ (right);
Galerkin solution (bottom) $m = 3$, $n = 129$ (left), $n = 257$ (right).

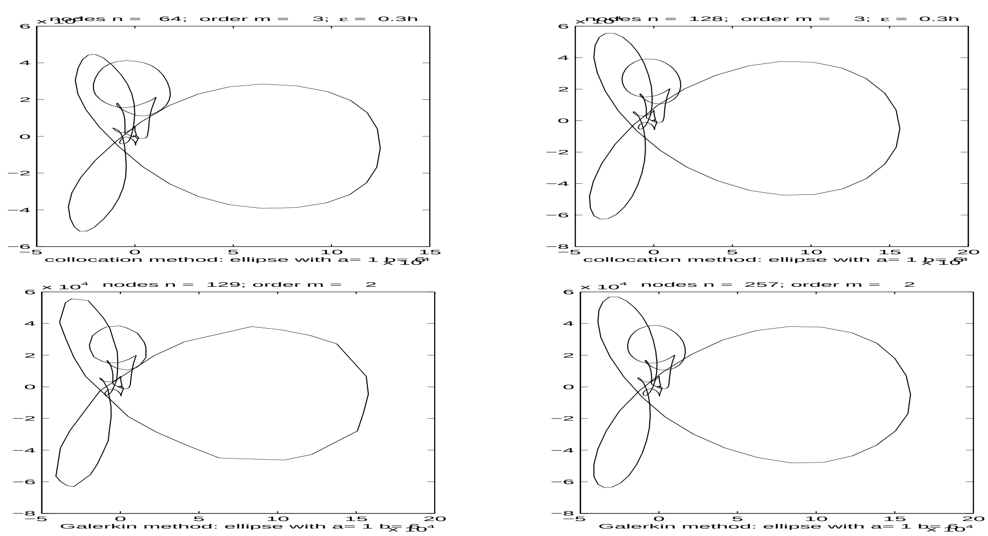

Figure 4: Example 2, L ellipse with $a = 1$, $b = 6$
Collocation solution (top) $m = 3$, $\epsilon = 0.3$, $n = 64$ (left), $n = 128$ (right);
Galerkin solution (bottom) $m = 2$, $n = 129$ (left), $n = 257$ (right).