A NUMERICAL MODEL FOR SANDPILE GROWTH ON OPEN TABLES WITH WALLS

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Abstract.

We present a finite-difference scheme for the dynamics of a growing sandpile on an open flat table when (infinite) vertical walls are present on part of the boundary. This approach generalizes the one studied in Ref. 1 for the numerical resolution of the double-layers model of Hadeler and Kuttler[2] in the case of the totally open table problem. The presence of walls strongly affects the equilibrium solutions for this model, whose characterization has been studied in Ref. 3, introducing singularities which propagate from the extreme points of the walls. The experiments show that the scheme is sufficiently able to detect the development of such singularities.

Keywords: granular matter; hyperbolic systems; finite differences schemes.

1. The continuous model

In recent years the mathematical modelling of granular materials has become an attracting field of research: the *realistic* simulation of the complex behavior of those materials is in fact an hard task and a lot of models have been proposed and compared, mainly in the physical literature. From the mathematical point of view many of these models have an interest for their own, and strong connections with other subjects of pure research.

Here we concentrate ourselves on the numerical simulation of growing sandpiles, a study started in Ref. 1 for the so-called open table problem (we refer to that paper for a list of recent references on the subject). The typical situation is the following. On a bounded open domain Ω of $\mathbb{R}^2$ (the initially empty table), the sand is poured according to the value of a nonnegative function f (the vertical source) with support $D_f \subset \overline{\Omega}$. The pile grows in time with a slope which is always lower than a characteristic critical value α, which for simplicity we will assume equal to one. Then the distance function from the boundary, that is $d(x) = \inf\{|x - z| : z \in \partial\Omega\}$, describes the maximal admissible, and stationary, profile of the sandpile on the given table Ω. Its singular set S (that is the discontinuity set of ∇d) is usually called the *ridge* of Ω. The effective stationary profile reached by the pile in the growing process is in general a function of f and D_f.

In the model proposed by Hadeler and Kuttler,[2] an eikonal type equation for the standing layer u of the growing sandpile forming heaps and slopes is coupled to an advection type equation for the small rolling layer v running down the slope. The dynamics of the two layers is then described by the following system of nonlinear partial differential

equations (note that nonzero rolling layers are allowed during the evolution even before the corresponding standing layer becomes critical):

$$(1) \qquad v_t = \nabla \cdot (v\nabla u) - (1 - |\nabla u|)v + f, \quad \text{in } \Omega \times (0, T]$$

$$(2) \qquad u_t = (1 - |\nabla u|)v, \quad \text{in } \Omega \times (0, T]$$

$$(3) \qquad u(\cdot, 0) = 0 \quad \text{in } \Omega \ .$$

Existence and uniqueness results for this system are not known even when the table is completely open, *i.e.* when the sand can fall down from every point of the boundary $\partial\Omega$ and we have the homogeneous boundary condition

$$(4) \qquad u = 0 \quad \text{on } \partial\Omega \times (0, T];$$

no boundary condition is needed for v, since equation (1) shows that all the characteristic lines are leaving Ω through the boundary (in the opposite direction to ∇u). In this open table case, anyway, a good characterization of the stationary solutions u^* and v^* of (1)–(4) has been recently given by Cannarsa and Cardialiaguet[4] by means of viscosity solution techniques. By the above result, there exists a unique equilibrium solution v^* for any given source f, and an integral representation formula is proved for it outside the ridge set (where $v^* = 0$):

$$(5) \qquad v(x) = \int_0^{\tau(x)} f(x + t\nabla d(x)) \frac{1 - (d(x) + t)k(x)}{1 - d(x)k(x)} \ dt, \quad \forall x \in \Omega \backslash \overline{S} \ ,$$

where, for any point $x \in \Omega$, $k(x)$ denotes the curvature of the projection point $\Pi(x)$ of x on the boundary, whereas $\tau(x)$ indicates the so-called normal distance to $\overline{S}$, defined as

$$\tau(x) = \min\{t \geq 0 : x + t\nabla d(x) \in \overline{S}\} \ .$$

On the contrary, u^* is uniquely determined only in the set where v^* is positive, and it coincides there with the distance function d. The mathematical characterization of u^* in the complementary set (if not empty) is not known for our model, and we have only a numerical description.[1] It is interesting to remark that, formally, the same set of admissible equilibria is attained by a different model for growing sandpiles proposed by Prigozhin:[5] the dynamics of its variational model is anyway rather different, since it allows sand rolling only at the critical slope, and the real equilibria of the two models coincide only under particular hypotheses. See Ref. 6 for a detailed comparison of the two models.

In Ref. 1 we proposed a finite difference approximation scheme for the above system analyzing its properties and showing that it inherits several characteristics of the continuous model.

Here we want to extend this approach to the case of a growing pile on a table Ω whose boundary is the union of two distinct regions: the open boundary Γ_0, that is the subset of the boundary through which the sand can leave the table, and the closed boundary Γ_1, which is the subset of $\partial\Omega$ where the sand is detained by a vertical wall (assumed to be sufficiently high to avoid sand trespassing). In this situation (4) has to be replaced by mixed type boundary conditions, that is

$$(6) \qquad u = 0 \quad \text{on } \Gamma_0 \ , \qquad v\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1 \ , \qquad \forall t \in (0, T].$$

The second relation in (6) reflects a necessary condition for the equilibrium solutions. In the stationary regime, in fact, by adding equations (1) and (2) we get, by Green's formula,

$$\int_\Omega f \, dx + \int_{\Gamma_0} v \frac{\partial u}{\partial n} \, d\sigma + \int_{\Gamma_1} v \frac{\partial u}{\partial n} \, d\sigma = 0 \ .$$

The first two terms represent respectively the instantaneous incoming sand from the source and the sand leaving in the same time the table through Γ_0, and therefore they have to be in equilibrium; then, the third one has to vanish.

The asymptotic behavior of solutions in the unrealistic 1D case is simple to describe; now Γ_0 and Γ_1 coincide with the two extremal points of the interval $\Omega = (a,b)$: it is easy to derive an explicit integral formula for the stationary transport density v^*, and the pile profiles converge towards the distance function to the open extremum in regions where such density is positive and towards a subcritical profile elsewhere (see Ref. 7).

Things are considerably more difficult in $\mathbb{R}^2$. Even for regular convex domains Ω, singularities may naturally arise on the boundary at the wall extrema (that is where an infinite number of transport rays meet together), causing the rolling layer v^* to be discontinuous along the normal direction to the boundary at those points. In order to give a rigorous definition of stationary solutions, suitable assumptions on the wall structure are needed which allow an a priori decomposition of the table into regular subdomains. In such a way it is possible to state a piece-wise representation theorem for the equilibria analogous to that of Ref. 4 for the open table case (see Ref. 3 for the proof and more details):

Theorem 1.1. *Let Ω be a convex Lipschitz domain of $\mathbb{R}^2$, Γ_0 be a non-empty closed subset of $\partial\Omega$, union of a finite number of pairwise disjoint connected arcs of class C^2; then the pair (d_0, v^*) is a stationary solution for system (1)-(3),(6), where $d_0(x) = dist(x, \Gamma_0)$ and*

$$(7) \qquad v^*(x) = \begin{cases} \displaystyle\int_0^{\tau(x)} f(x + t \, \nabla d_0(x)) \, M_x(t) \, dt & \text{if } x \in \Omega \backslash \overline{S}, \\ 0 & \text{if } x \in S \, . \end{cases}$$

Moreover, any other solution pair (u, v) has to satisfy $v = v^*$ and, in regions where $v^* > 0$, $u = d_0$.

In (7) the weight function $M_x(t)$ has a different expression if the projection point $\Pi(x)$ of x on the boundary is an extremum of Γ_0 or not:

$$(8) \qquad M_x(t) = \begin{cases} \dfrac{d_0(x) + t}{d_0(x)}, & \text{if } \Pi(x) \in \partial\Gamma_0, \\ \dfrac{1 - (d_0(x) + t)k(x)}{1 - d_0(x)k(x)}, & \text{otherwise} \, , \end{cases}$$

where we have denoted by $\partial\Gamma_0$ the set of extremal points of the arcs in Γ_0. As already remarked, Theorem 1.1 implies that v^*, that is the rolling layer density at the equilibrium, is in general only a piece-wise continuous function in L^1, unbounded at the extremal points of the walls. The discontinuity lines are the transport rays normal to those points. For non convex tables the characterization is till an open problem: in that case the possible singularity regions for v^* are no longer made by isolated points, but they could be supported also by entire portions of the close boundary Γ_1.

2. The approximation scheme

Using the forward Euler difference operator for the time derivatives, the fully explicit numerical scheme proposed in Ref. 1 for the solution of the totally open table problem (1)–(4) in the one-dimensional case reads as

$$(1) \qquad v_i^{n+1} = v_i^n + \Delta t \left[v_i^n D^2 u_i^n + \overline{D} v_i^n D u_i^n - (1 - |Du_i^n|) v_i^n + f_i \right]$$

$$(2) \qquad u_i^{n+1} = u_i^n + \Delta t (1 - |Du_i^n|) v_i^n$$

$$(3) \qquad u_i^0 = v_i^0 = 0 \quad \forall i , \quad u_1^n = u_N^n = 0 \quad \forall n$$

where u_i^n and v_i^n denote the discrete solutions computed at any node x_i $(i = 1, .., N)$ of a uniform mesh of the interval Ω at time $t_n = n\Delta t$. The first order space derivative Du_i is computed for any time iteration as the difference with maximal absolute value between the backward and the forward differences at x_i, whereas the corresponding derivative $\overline{D}v_i$ is chosen as the upwind (with respect to the sign of Du_i) finite difference, and $D^2 u_i$ denotes the usual second order central difference for u. This scheme easily extends to the more realistic two-dimensional case of a rectangular table on a uniform grid of nodes $x_{i,j}$, and it gives good results in the experiments, allowing also a numerical characterization of the effective stationary solutions for any given (time-independent) distributed source f (see Ref. 1 for more details).

From a numerical point of view, the extension of such a scheme to the wall problem requires only the implementation of the new boundary condition in (6). If Ω is for example the unit square of $\mathbb{R}^2$, this can be done as follows :

$$(4) \qquad \bullet \ x_{i,j} \in \overline{\Gamma}_0 \Rightarrow \ u_{i,j}^n = 0 \ \text{(and } v_{i,j}^n = 0 \text{ if } x_{i,j} \text{ is an open vertex)},$$

$$(5) \qquad \bullet \ x_{i,j} \in \Gamma_1 \Rightarrow \ \text{if } Du_{i,j}^n \cdot \nu_{i,j} > 0 \text{ then } \ v_{i,j}^n = 0, \ \text{else} \ \ u_{i,j}^n = u_{(i,j)-\nu_{i,j}}^n$$

where $\nu_{i,j}$ denotes the outward normal unit vector at $x_{i,j}$. In (4) the technical condition on v at the open vertices of Ω is necessary to avoid locking at those points. In practice, no boundary condition is needed at the points of Γ_1 where the transport direction to Γ_0 lies on the boundary itself (see next Example 2.2). Moreover, in order to prevent instability phenomena around the singular boundary points, we found essential to impose at every time iteration the explicit gradient constraint which is known to hold at the equilibrium:

$$(6) \qquad |Du_{i,j}^n| = \min(|Du_{i,j}^n|, 1) , \quad \forall (i, j) , \forall n .$$

2.1. Numerical tests

Here we present two numerical experiments in the case of a constant source term $f = 1$ distributed on the whole square table $\Omega = (0, 1) \times (0, 1)$, that is when D_f coincides with the whole domain. Under such an assumption Theorem 1.1 yields the uniqueness of the pair (d_0, v^*) of stationary solutions.

Example 2.1. Let Γ_1 just coincide with one whole side of the square Ω: in this case there are no singular points on the boundary, and v^* can be proved to be continuous. The scheme produces the correct dynamics and equilibria, even without the gradient constraint (6) (see Figure 1).

Example 2.2. In order to see the effects of the boundary singular points, let us assume now Γ_0 to be only half of a square side, namely $\Gamma_0 = \{0 \leq \ x \leq \ 0.5, \ y = 0\}$. Then the
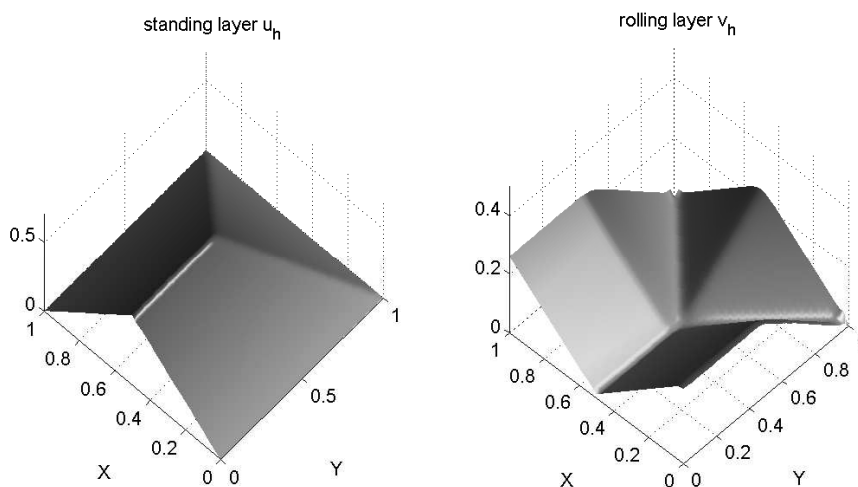
Fig. 1.   Example 2.1. $\Gamma_1 = \{0 < x < 1, \ y = 0\}$: numerical stationary solutions.

exact stationary solutions can be explicitly computed by decomposition (see Figure 2): sand flow is never crossing the line $x = 0.5$, so that a piecewise description of solutions is possible. Computation shows in particular that v^* is only an L^1 function, unbounded at the boundary point $P = (0.5, 0)$ and discontinuous along the normal direction to that point (the segment $\overline{PQ}$ in Figure 2, see also Figure 3).
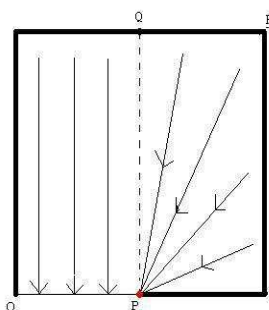
Fig. 2.   Example 2.2. Domain decomposition and transport rays.

In the implementation of our scheme, we have imposed condition (5) on the northern and eastern sides of the square. No condition is imposed on the western and southern sides of the walls, where the sand transport direction is parallel to the boundary. The experiments show that the time iterates u^n converge towards the correct solution (Figure 4, left), but the rolling layers v^n "feel" very much the developing singularity in a large region around the exact discontinuity line, and the final result is not completely satisfactory (Figure 4, right).

Due to the particular a-priori known structure of this example, we have also tested the separate application of the scheme to the two half-tables, with no boundary conditions imposed along the central cut. We see the results in Figure 5: the solutions are now correctly detected in the left-hand side of the table, where the singular point has no effects, but in the other side there is not a great improvement.
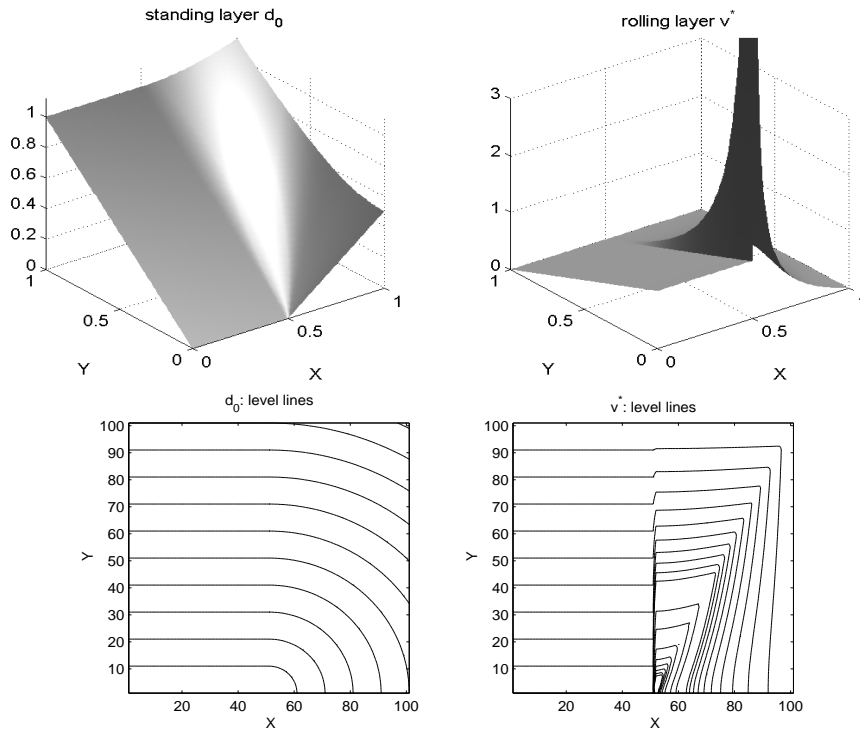
Fig. 3. Example 2.2. $\Gamma_0 = \{0 \le x \le 0.5, \ y = 0\}$: exact stationary solutions and their level lines.
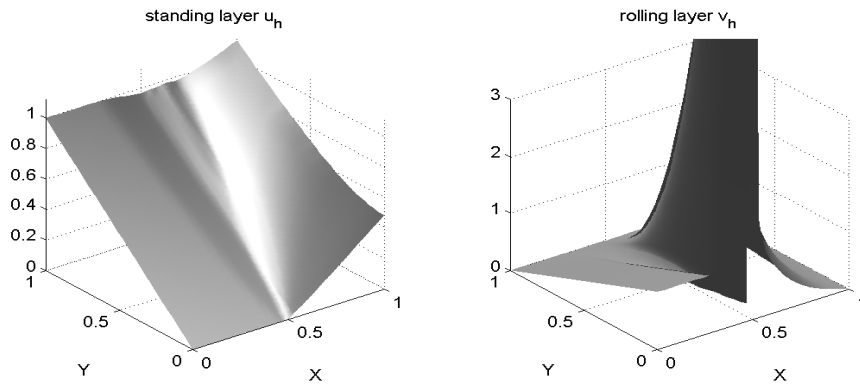

Fig. 4. Example 2.2. Numerical stationary solutions.

Summing up, the experiments show that our scheme is able to give a correct qualitative description of the sandpile growth, even in presence of the wall induced singularities. It also allows a sufficiently good approximation of the stationary standing layer and of the ridge set (as the zero level set of v^*). However, a more careful study is necessary for an accurate reconstruction of the discontinuity region for v^* and better results could be obtained by using adaptive techniques on unstructured grids.

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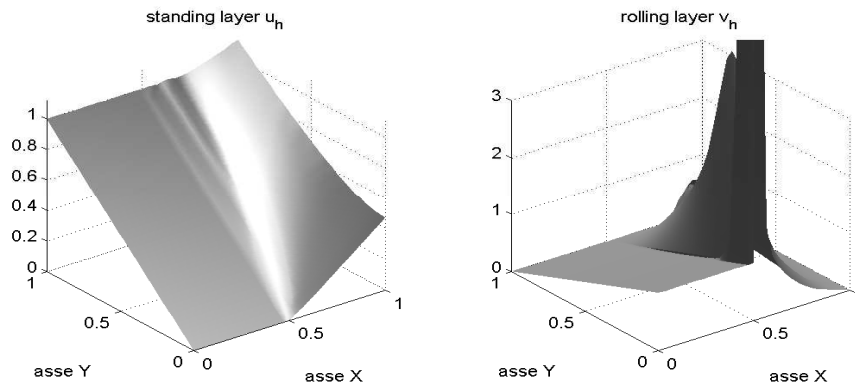2. K. P. Hadeler and C. Kuttler, *Granular Matter* **2**, 9 (1999).

Fig. 5. Example 2.2. Numerical stationary solutions by domain decomposition.

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