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Generalized Nash Equilibrium Problems

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We consider the Generalized Nash Equilibrium Problem (GNEP for short). The GNEP extends the classical Nash Equilibrium Problem by assuming that each player's feasible set can depend on the rival players' strategies. There are N players, and each player ν controls the variables $x^\nu \in \Re^{n_\nu}$. We denote by $\boldsymbol{x}$ the vector formed by all these decision variables:

$$\boldsymbol{x} \equiv \begin{pmatrix} x^1 \\ \vdots \\ x^N \end{pmatrix},$$

which has dimension $n := \sum_{\nu=1}^N n_\nu$ and by $\boldsymbol{x}^{-\nu}$ the vector formed by all the players' decision variables except those of player ν. Each player's strategy must belong to a set $X_\nu(\boldsymbol{x}^{-\nu}) \subseteq \Re^{n_\nu}$ that depends on the rival players' strategies. The aim of player ν, given the other players' strategies $\boldsymbol{x}^{-\nu}$, is to choose a strategy x^ν that solves the minimization problem

$$\begin{array}{ll} \text{minimize}_{x^\nu} & \theta_\nu(x^\nu, \boldsymbol{x}^{-\nu}) \\ \text{subject to} & x^\nu \in X_\nu(\boldsymbol{x}^{-\nu}). \end{array} \tag{0.1}$$

For any $\boldsymbol{x}^{-\nu}$, the solution set of this problem is denoted by $\mathcal{S}_\nu(\boldsymbol{x}^{-\nu})$. The GNEP is the problem of finding a vector $\bar{\boldsymbol{x}}$ such that

$$\bar{x}^\nu \in \mathcal{S}_\nu(\bar{\boldsymbol{x}}^{-\nu}) \qquad \text{for all } \nu.$$

Such a point $\bar{\boldsymbol{x}}$ is called a (generalized) Nash equilibrium or, more simply, a solution of the GNEP. GNEPs have received an increasing amount of attention in recent times because GNEPs naturally arise in the modelling of complex competition situations, especially in the energy markets, see for example [1], [2], [3], [7] and references therein. However, probably due to the daunting difficulty of the problem, advancements on the algorithmic side have been rather scarce. The aim of our research is to develop a solution algorithm that requires reasonable assumptions to be applied. As a first step, we focus on local methods, and in particular we study three different Newton methods for the computation of generalized Nash equilibria. All these methods solve a nonsmooth structured system that derives from the KKT systems for the player's optimization problems. To this system we apply some semismooth methods requiring, at each iteration, the solution of a linear system of equations. The GNEP is a somewhat tricky object to study because, in spite of the many similarities to optimization/VI problems, it presents challenging peculiarities that make its analysis especially demanding. The natural extension of standard conditions and assumptions normally used in the optimization/VI theory may turn out to be inappropriate in many cases and care must be exercised so that realistic assumptions are made in dealing with GNEPs. For example, in large and interesting classes of GNEP local uniqueness of the solutions is not likely to be encountered. Therefore, classical conditions and techniques for the development of Newton methods must be abandoned in favor of more sophisticated approaches. The three Newton methods we propose require different assumptions, and each of them can be applied to a class of problems satisfying certain assumptions. The first Newton method we introduce, is a direct application of the classical semismooth Newton method to the system obtained by writing the KKT conditions for the player's optimization problems and reformulating the complementarity conditions as nonlinear equations by means of a complementarity function. This simple method can be applied if a regularity condition is satisfied by the system, and this happens for example if there are no active constraints shared by more than one player or if all the players have different constraints, or in the case of a (not generalized) Nash problem. A wide class of GNEPs is the one where all (or some of) the players share some constraints. This situation occurs often in practice, most notably when the players have some common resource available in limited amount (e.g. the capacity of a transmission or shipping channel). It can be proved that when there are active shared constraints the generalized Nash equilibria are not isolated in general. An important case of GNEP is that explicitly introduced by Rosen in his fundamental paper [11]. Rosen's setting corresponds to the fact that those constraints that depend on the other players' variables are shared by all the players and are convex with respect to all the variables. In the seminal paper [8], Harker singled out a class of GNEPs that can be solved by finding a solution of a Variational Inequality (VI); the latter being a much more tractable problem for which a very substantial set of results is available, see [1] for a review. We show that Rosen's class of GNEPs can be solved by finding a solution of a single VI, thus extending to this very significant class of problems all the benefits of the VI reduction illustrated in [8]. By exploiting this result, we can define a different local solution method that is essentially the semismooth Newton method applied to the KKT conditions of the VI that is equivalent to the GNEP. If the GNEP does not satisfy the Rosen's setting conditions, then it is necessary to look at local methods that can deal with non isolated solutions. In particular, we show that a recently developed Levemberg-Marquardt method ([5], [6]) can be applied under reasonable assumptions to the system arising from the KKT conditions of the players' problem. All the three proposed methods converge quadratically in a neighborhood of the solution under reasonable smoothness assumptions on the functions involved in the GNEP. Moreover, we show that the assumptions required by the methods are reasonable by analyzing a model for internet switching and showing that this practical problem satisfies them all. The next step is to define a globally convergent method for the solution of the GNEP. In the literature there are only few globally convergent algorithms for the solution of GNEPs. Basically, if one restricts to algorithms that are guaranteed to find a solution by performing well defined tasks and by solving subproblems that are surely solvable, they boil down to two: the relaxation algorithm (see for example [10], [12]) and the reduction to an MPEC problem (see [9] and references therein). In both these approaches Rosen's setting is assumed to begin with. Furthermore, some other (strong) assumptions are made. By using the reduction to the VI we can use a host of algorithms (old and new) that are guaranteed to find a solution (if one exists) by making much simpler assumptions on the problem. Finally, some open questions will be pointed out for future research.

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